3.5 Connectivity in Directed Graphs
Directed Graphs

Directed graph.  $G = (V, E)$

Edge $(u, v)$ goes from node $u$ to node $v$.

Ex.  Web graph - hyperlink points from one web page to another.  Directedness of graph is crucial.  Modern web search engines exploit hyperlink structure to rank web pages by importance.
Food web graph.

Node = species.
Edge = from prey to predator.

(Directed) Graph Search

Problems in directed graphs...

Directed reachability. Given a node $s$, find all nodes reachable from $s$.

Directed $s$-$t$ shortest path problem. Given two node $s$ and $t$, what is the length of the shortest path between $s$ and $t$?

(Directed) Graph search. BFS extends naturally to directed graphs.

Web crawler. Start from web page $s$. Find all web pages linked from $s$, either directly or indirectly.
Strong Connectivity

Def. Node u and v are **mutually reachable** if there is a path from u to v and also a path from v to u.

Def. A graph is **strongly connected** if every pair of nodes is mutually reachable.

Q. Which graph is strongly connected?

![Graph G1](image1)

![Graph G2](image2)
Q. Is this graph strongly connected?

Ex. Web of trust (e.g., PGP-key ring):
I trust some friends by signing their keys.
If web of trust is strongly connected → I can trust everyone and everyone trusts me!
Q. How to determine if $G$ is strongly connected, in $O(m + n)$ time? (1 min)
Strong Connectivity

**Q.** How to determine if $G$ is strongly connected, in $O(m + n)$ time?

**Lemma.** Let $s$ be any node.

$G$ is strongly connected $\iff$ every node is reachable from $s$, and $s$ is reachable from every node.
Q. How to determine if $G$ is strongly connected, in $O(m + n)$ time?

Lemma. Let $s$ be any node. 
$G$ is strongly connected $\iff$ every node is reachable from $s$, and $s$ is reachable from every node.

Pf.
Strong Connectivity

Q. How to determine if G is strongly connected, in O(m + n) time?

Lemma. Let s be any node.
G is strongly connected ⇔ every node is reachable from s, and s is reachable from every node.

Pf. ⇒
Pf. ⇐
Strong Connectivity

Q. How to determine if G is strongly connected, in $O(m + n)$ time?

Lemma. Let $s$ be any node.

$G$ is strongly connected $\iff$ every node is reachable from $s$, and $s$ is reachable from every node.

Pf. $\Rightarrow$: Suppose strongly connected.

To prove: every node reachable from $s$, and $s$ reachable from every node.

Q. Why does this hold?
Q. How to determine if G is strongly connected, in O(m + n) time?

Lemma. Let s be any node.
G is strongly connected ⇔ every node is reachable from s, and s is reachable from every node.

Pf. ⇒: Suppose strongly connected.
To prove: every node reachable from s, and s reachable from every node.
Q. Why does this hold?
A. Follows from definition of strongly connected graph (every pair of nodes is mutually reachable).
Q. How to determine if G is strongly connected, in $O(m + n)$ time?

**Lemma.** Let s be any node.
G is strongly connected $\iff$ every node is reachable from s, and s is reachable from every node.

**Pf.** $\Rightarrow$ Follows from definition (every pair of nodes is mutually reachable).
**Pf.** $\Leftarrow$ Suppose every node reachable from s, s reachable from every node.
To prove: G is strongly connected.
To prove: Every two nodes are mutually reachable.
Let two nodes u and v be given.
Q. Why is u reachable from v? And v reachable from u?
Q. How to determine if G is strongly connected, in $O(m + n)$ time?

Lemma. Let $s$ be any node.

G is strongly connected $\iff$ every node is reachable from $s$, and $s$ is reachable from every node.

Pf. $\Rightarrow$ Follows from definition (every pair of nodes is mutually reachable).

Pf. $\Leftarrow$ Path from $u$ to $v$: concatenate $u$-$s$ path with $s$-$v$ path.

Path from $v$ to $u$: concatenate $v$-$s$ path with $s$-$u$ path. □

ok if paths overlap
**Strong Connectivity**

**Lemma.** Let $s$ be any node.

G is strongly connected $\iff$ every node is reachable from $s$, and $s$ is reachable from every node.

**Q.** How to determine if $G$ is strongly connected, in $O(m + n)$ time?
Theorem. Can determine if G is strongly connected in $O(m + n)$ time.

Pf.
Pick any node $s$.
Run BFS from $s$ in $G$.
Run BFS from $s$ in $G^{rev}$.
Return true iff all nodes reached in both BFS executions.
Correctness follows immediately from lemma. □