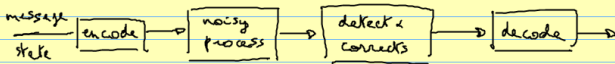


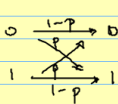
## QUANTUM ERROR CORRECTION

Reliable q. comp. with unreliable components

How: encoding, correcting, decoding



## Classical example



encoding  
 $0_L \rightarrow 000$   
 $1_L \rightarrow 111$

decoding  
 majority  
 vote

$$P(\text{failure}) = p \quad \text{w/o encoding}$$

$$= 3p^2(1-p) + p^3 \quad \text{with encoding}$$

$$\text{Encoding helps if } 3p^2(1-p) + p^3 < p$$

$$p < 1/2$$

Is QEC possible? Why (maybe) not?

- ① no cloning-theorem
- ② errors are continuous
- ③ measuring destroys state

1995: Shor, Steane

- ① entanglement
- ② digitize errors
- ③ measure error

## 3 qubits bit-flip errors

$$|\psi\rangle \xrightarrow{1-p} |\psi\rangle$$

$$p \rightarrow X|\psi\rangle$$

- encode  $|0\rangle \rightarrow |000\rangle$   $a|0\rangle + b|1\rangle \rightarrow a|000\rangle + b|111\rangle$  NOT a copy  
 $|1\rangle \rightarrow |111\rangle$   
 copy:  $(a|0\rangle + b|1\rangle) \otimes ( \quad ) \otimes ( \quad )$

- measure error (syndrome)

- comparison of  $q_1 \neq 2$  :  $z_1, z_2 = (|00\rangle\langle 00| + |11\rangle\langle 11| - |01\rangle\langle 01| - |10\rangle\langle 10|) \otimes I$

outcome 1  $\rightarrow$  equal  
 0  $\rightarrow$  not equal

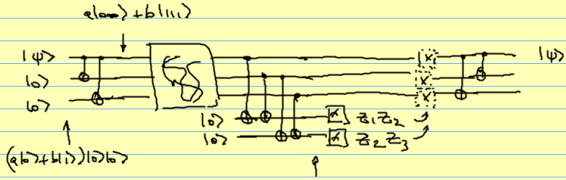
$$a|010\rangle + b|101\rangle$$

- comparison of  $q_1, z_2 \neq 3$  :  $z_2 z_3 = \dots$

correct	$z_1 z_2$	$z_2 z_3$	action
	1	1	I
	1	0	$X_3$
	0	1	$X_1$
	0	0	$X_2$

$$\begin{array}{l} a|010\rangle + b|101\rangle \xrightarrow{z_1 z_2} -a|010\rangle - b|101\rangle \\ \text{outcome } -1 \\ \xrightarrow{z_2 z_3} a|010\rangle + b|101\rangle \\ \text{outcome } +1 \end{array}$$

How do we do this?



## 3 qubit phase flip code

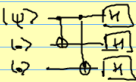
$$|\psi\rangle \xrightarrow{P} |\psi\rangle$$

$$|\psi\rangle \xrightarrow{P} Z|\psi\rangle$$

Note:  $HZH = X$

Method ① insert  $H$  before & after error process  
use bit flip code

② encode



$$|0\rangle_L \rightarrow (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$$

$$|1\rangle_L \rightarrow - \quad - \quad -$$

$$|0\rangle_L = |+\rangle|+\rangle|+\rangle$$

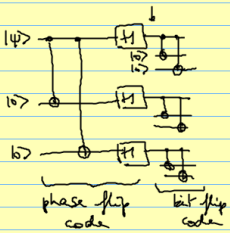
$$|1\rangle_L = |-\rangle|-\rangle|-\rangle$$

detecting  $X_1, X_2$  etc

correcting  $Z_1, Z_2, Z_3$

decoding: inverse of encoding

9 qubit code for arbitrary errors



$$|0\rangle_L = (|1000\rangle + |1111\rangle) \begin{pmatrix} + \\ + \\ + \\ + \end{pmatrix} \begin{pmatrix} + \\ + \\ + \\ + \end{pmatrix}$$

$$|1\rangle_L = (|1000\rangle - |1111\rangle) \begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix} \begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix}$$

bit flip  
 error det.:  $z_1 z_2, z_2 z_3$   
 $z_4 z_5, z_5 z_6$   
 $z_7 z_8, z_8 z_9$

phase flip:  $x_1 x_2 x_3 x_4 x_5 x_6,$   
 $x_4 x_5 x_6 x_7 x_8 x_9$

$$\therefore Y = XZ$$

## Arbitrary errors

$$E_i = e_{i0} I + e_{i1} X + e_{i2} Y + e_{i3} Z$$

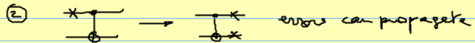
$$\rho \rightarrow \sum_i E_i \rho E_i^\dagger$$

measure syndrome collapses state into one of  $E_i \rho E_i^\dagger$   
outcome gives corresponding information

$$\rho + \underbrace{x_1 \rho X_1}_{\uparrow} + x_2 \rho X_2 + x_3 \rho X_3$$

## Fault-tolerant QC - QC with encoded states

Problems: ① errors can happen in state prep., gates, meas., ...



Idea: avoid  $\geq 2$  errors within 1 code block  
(1 error is ok, correctable)

→ count all poss. errors

$$P(\geq 2 \text{ errors in 1 block}) = c p^2$$

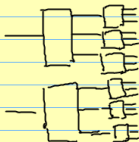
improvement

$$c p^2 < p$$

$$p \leq p_H = c$$

$\approx 10^{-4}$

## Concatenation



...

	For 1 actual comp. step	$P(\text{failure})$	# gates
no encoding	$p$	$p^2$	$d$
1 layer	$cp$	$(cp)^2$	$d^2$
2 layers	$c^2p$	$(c^2p)^2$	$d^4$
$\vdots$			
$k$	$\frac{(cp)^{2k}}{c}$		$d^{2k}$

expon small error with poly effort