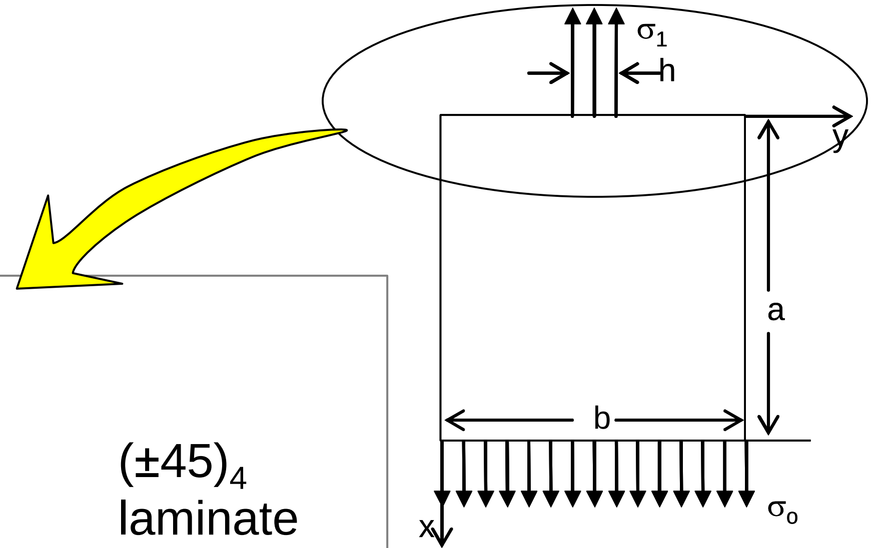
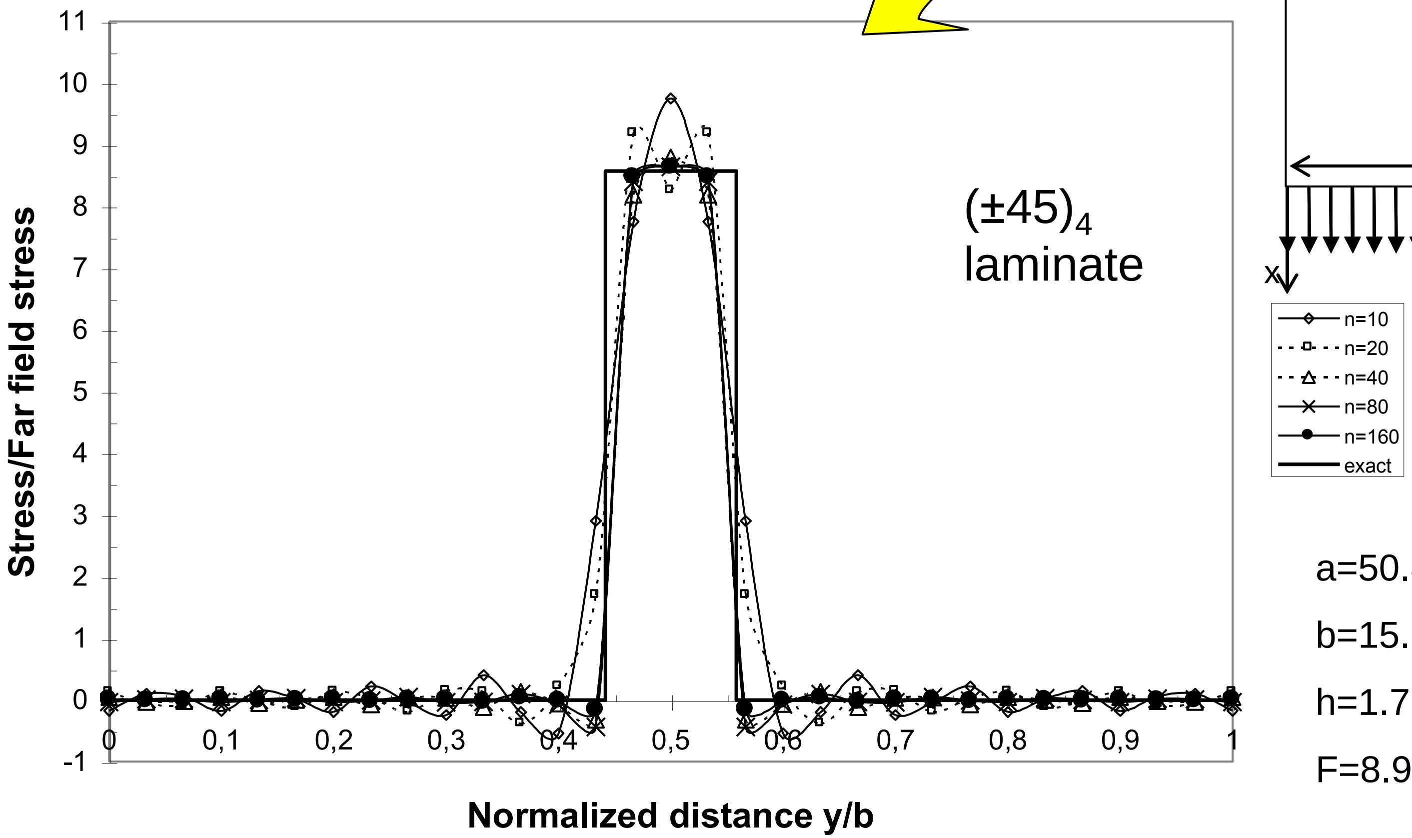
Solution – Number of terms
(series convergence)

Property	Value
E_{11}	73.0 GPa (10.6 msi)
E_{22}	73.0 GPa (10.6 msi)
G_{12}	5.30 GPa (0.77 msi)
ν_{12}	0.05
ply thickness	0.1905 mm (0.0075 in)





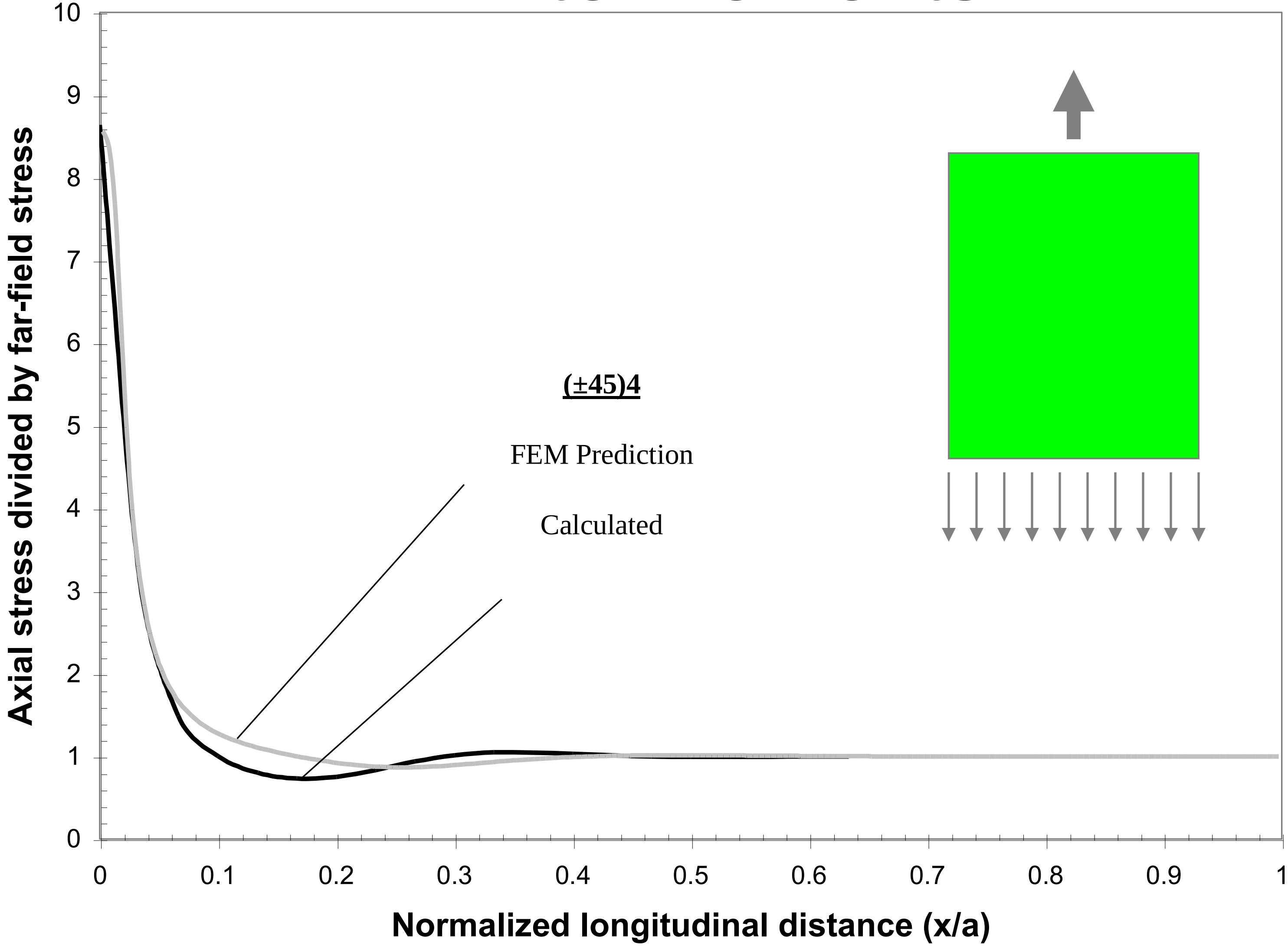
a=50.8 cm

b=15.24 cm

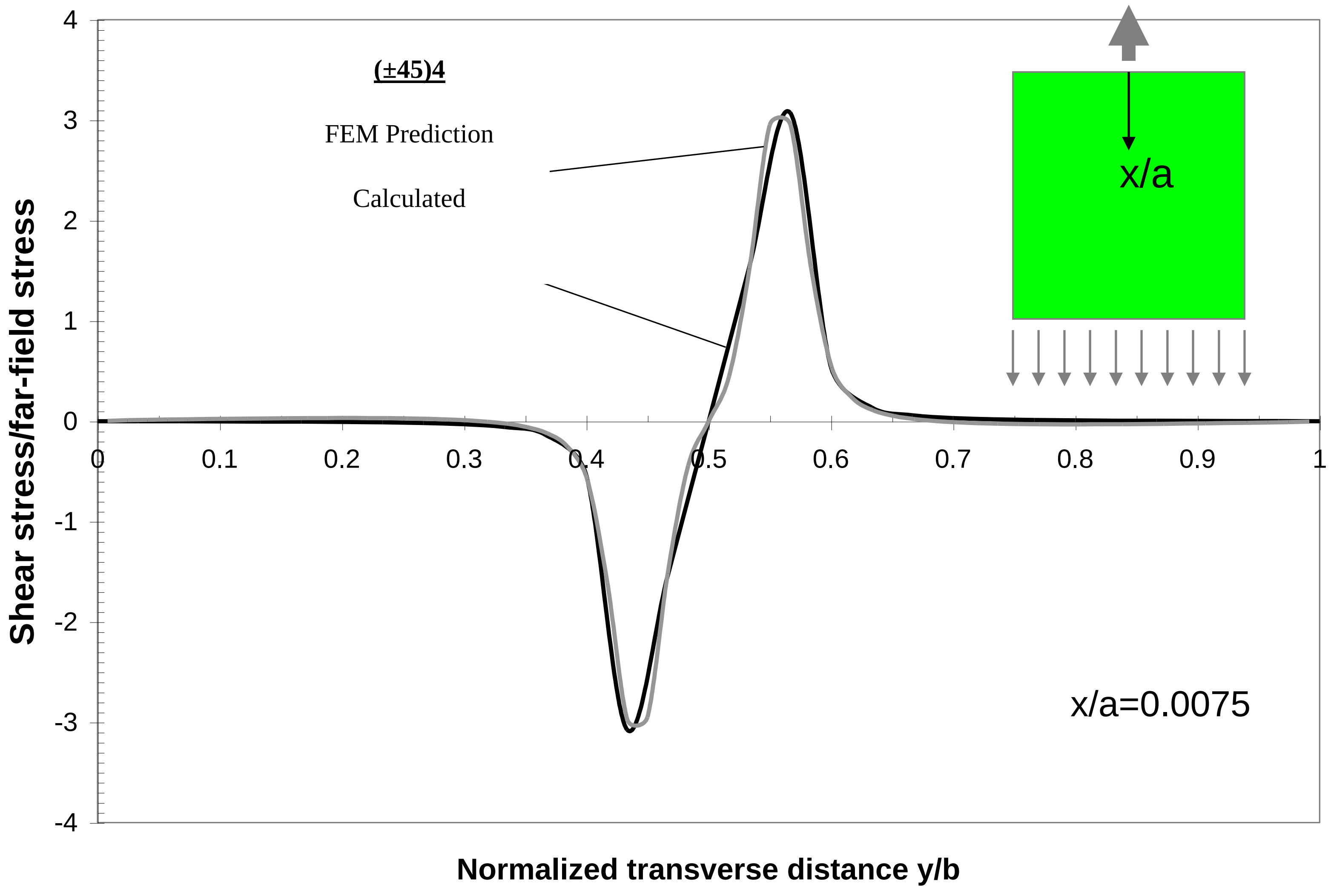
h=1.778 cm

F=8.9kN

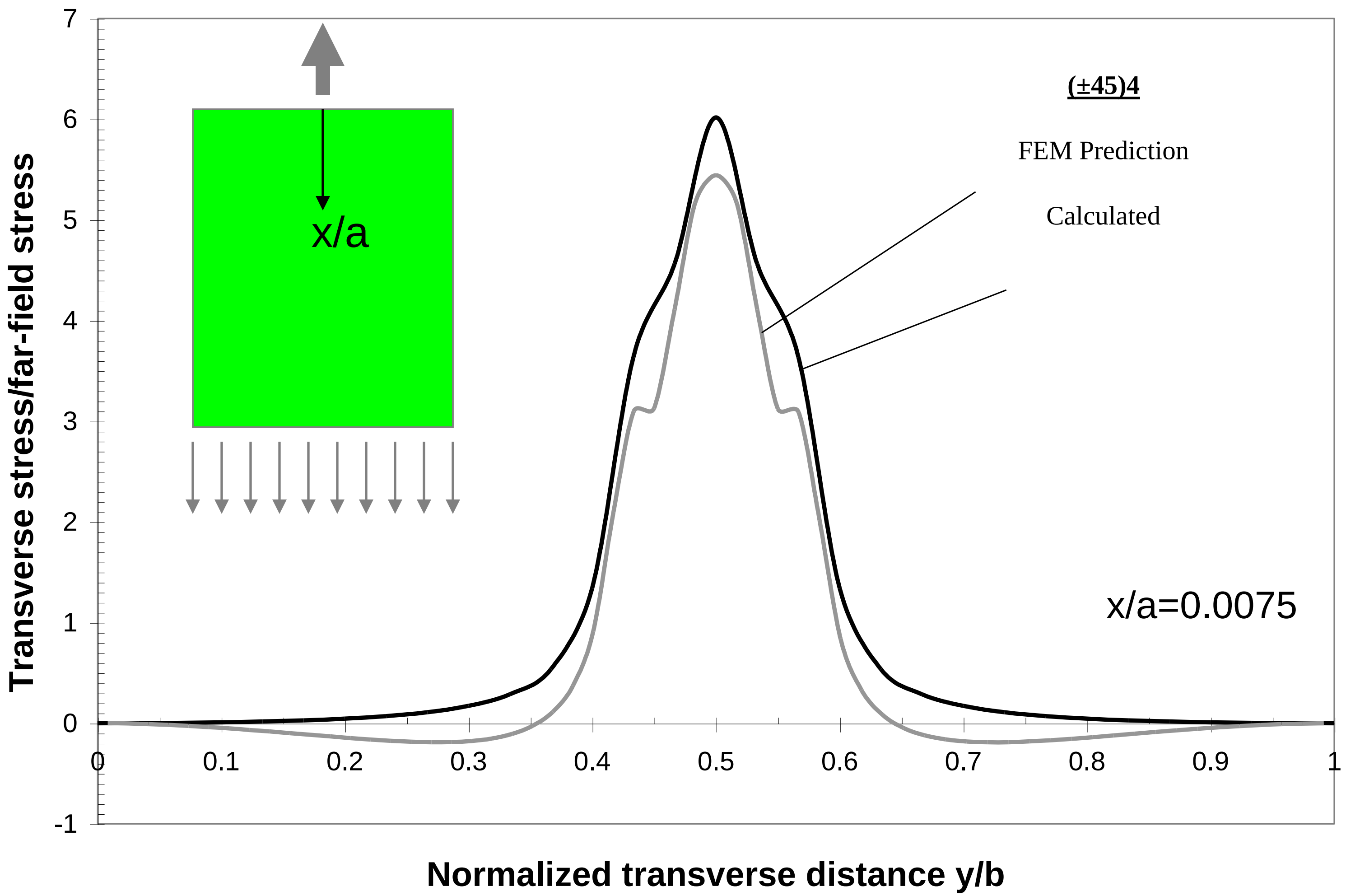
Solution accuracy – Comparison with Finite Elements



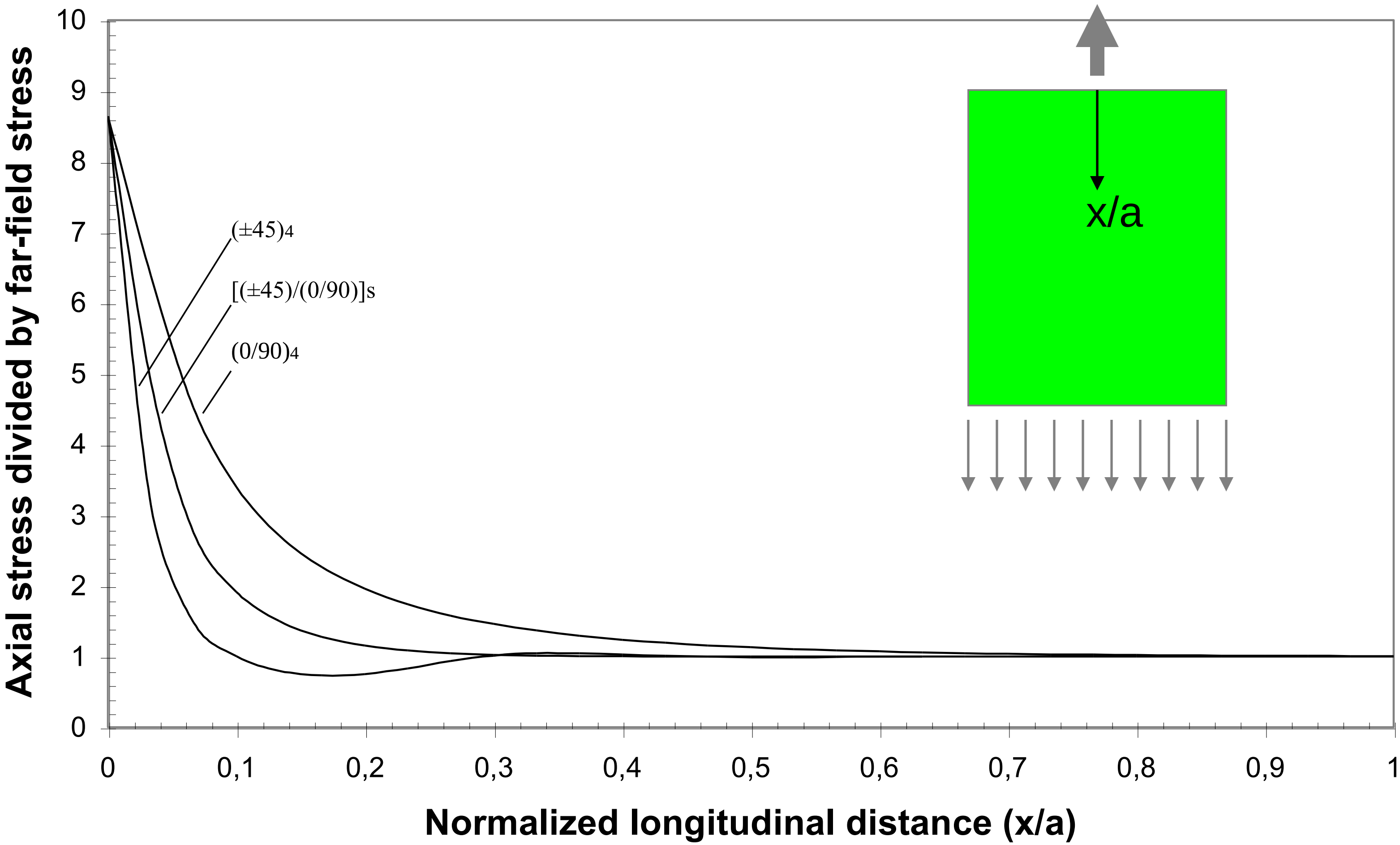
Solution accuracy-Comparison with Finite Elements



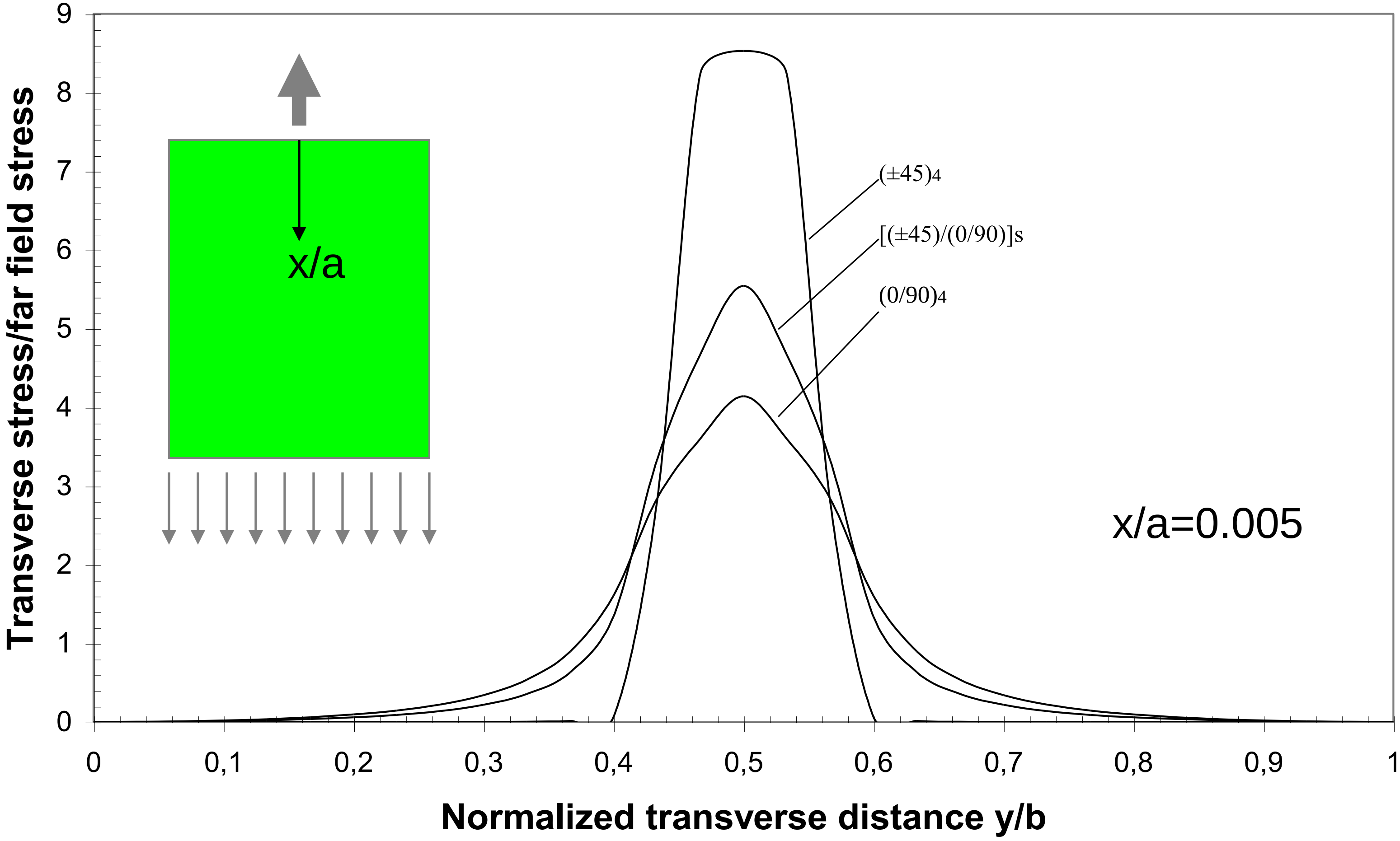
Solution accuracy – Comparison with Finite Elements



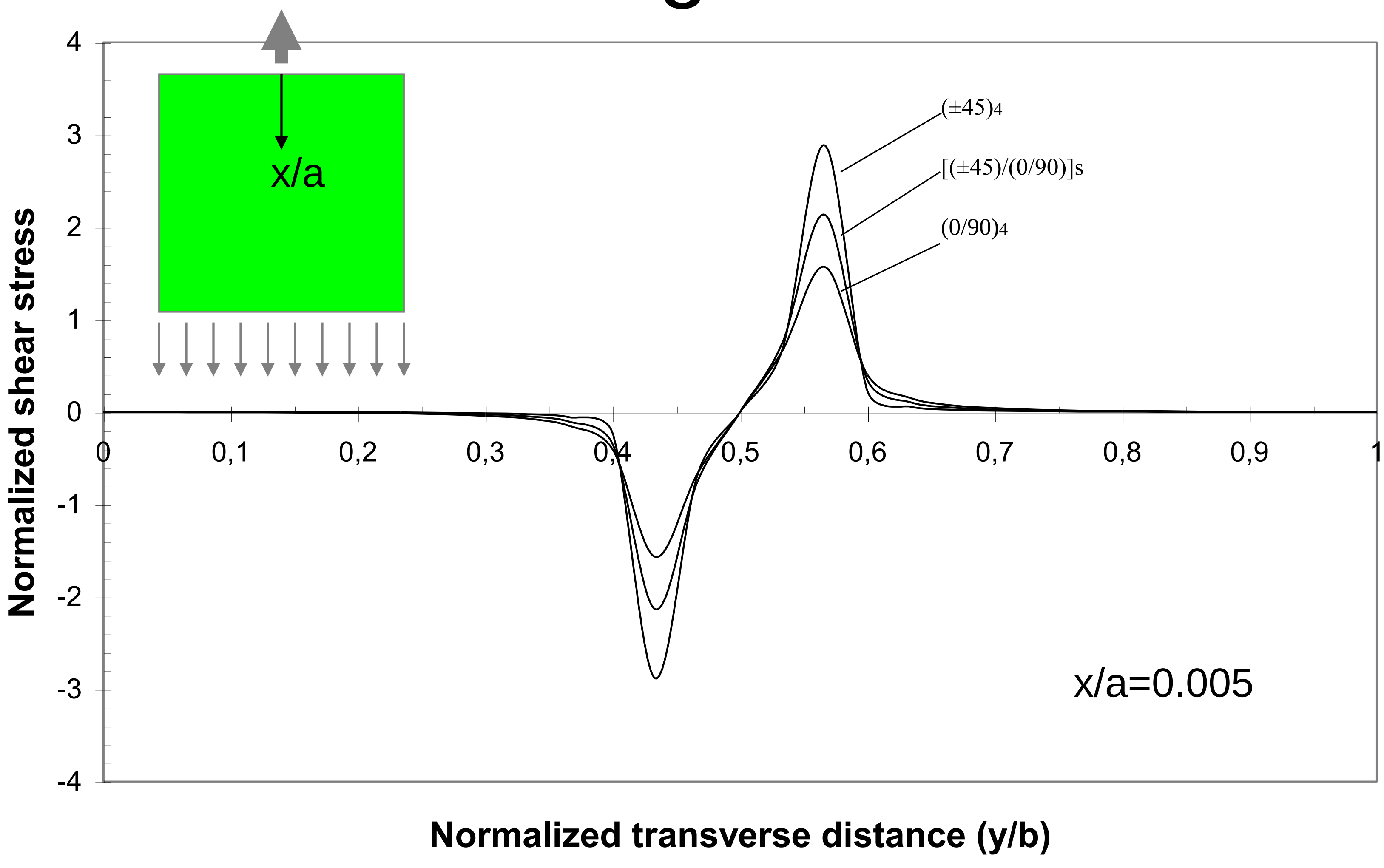
Results-Implications for the transition region



Results – Implications for the transition region



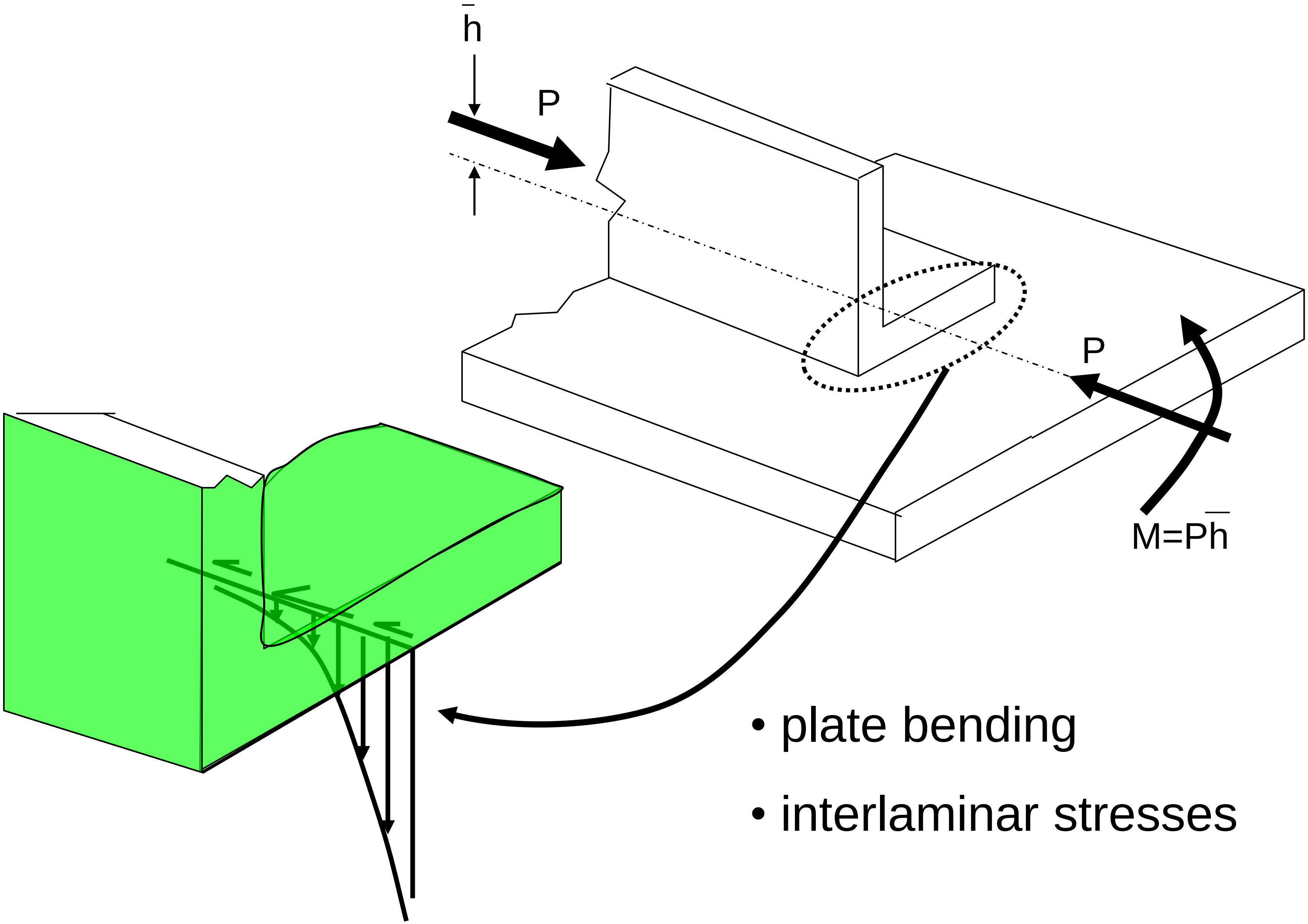
Results – Implications for transition region



Conclusions based on results so far

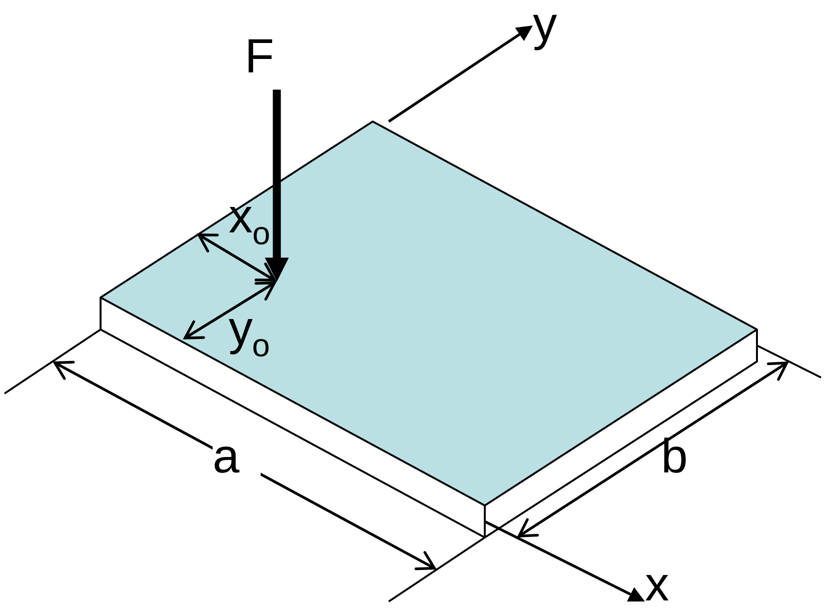
- Axial stresses in $(0/90)_4$ panels decay more slowly (require longer doublers) than $(\pm45)_4$ or $[(\pm45)/(0/90)]s$ panels
- On the other hand, transverse and shear stresses in $(\pm45)_4$ or $[(\pm45)/(0/90)]s$ panels are more critical than in $(0/90)_4$ panels
- Preliminary doubler (reinforcement dimensions): $\ell=0.5a$ for $(0/90)_4$ and 0.3a for $(\pm45)_4$ or $[(\pm45)/(0/90)]s$ panels and w=0.3b for all panels

Still needed



- plate bending
- interlaminar stresses

Anisotropic Plate Under Transverse Point Load



Simply-supported edges all-around

- Symmetric layup => $B_{ij=0}$
- D_{16}=D_{26}=0

Motivation

- Plates under point loads (e.g. step loads) are not uncommon
- More importantly, this can be a first step towards simple modeling of impact event

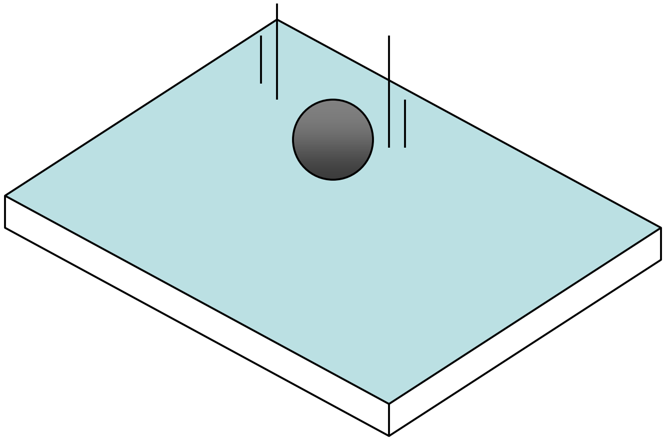


Plate Under Point Load – Governing PDE

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q_z$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = 0$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}$$

$$\kappa_y = -\frac{\partial^2 w}{\partial y^2}$$

$$\kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

$$\underbrace{D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4}}_{\text{common to many problems!}} = \overbrace{F\delta(x - x_o)\delta(y - y_o)}^{-q_z}$$

Dirac delta functions

Solution to PDE

- Since simply supported, solution for w is in the form:

$$w = \sum \sum A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

which is a double Fourier series

- Expand right hand side in a Fourier series and determine unknown coefficients A_{mn}

$$F\delta(x - x_o)\delta(y - y_o) = \sum \sum B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Solution to PDE cont'd

$$\underbrace{\iint F\delta(x-x_o)\delta(y-y_o)\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\,dxdy}_{=F\sin\frac{m\pi x_o}{a}\sin\frac{n\pi y_o}{b}} = \underbrace{\iint\sum\sum B_{mn}\sin\frac{m\pi x}{a}\sin\frac{p\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{q\pi y}{b}\,dxdy}_{=B_{mn}\frac{ab}{4}}$$

- Substituting in governing PDE and matching terms:

$$A_{mn} = \frac{\frac{4F}{ab}\sin\frac{m\pi x_o}{a}\sin\frac{n\pi y_o}{b}}{D_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12}+2D_{66})\frac{m^2n^2\pi^4}{a^2b^2} + D_{22}\left(\frac{n\pi}{b}\right)^4}$$

- Finally, plate deflections are given by

$$w = \sum\sum\frac{\frac{4F}{ab}\sin\frac{m\pi x_o}{a}\sin\frac{n\pi y_o}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{D_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12}+2D_{66})\frac{m^2n^2\pi^4}{a^2b^2} + D_{22}\left(\frac{n\pi}{b}\right)^4}$$

Special Case

- Force F acting at plate center: Max deflection δ at location where F acts is,

$$\delta = w_{\max} = \sum\sum \frac{\frac{4F}{ab}\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}}{D_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12}+2D_{66})\frac{m^2n^2\pi^4}{a^2b^2} + D_{22}\left(\frac{n\pi}{b}\right)^4}$$

Then…

- Once w is determined, strains ε_x, ε_y, γ_{xy} can be obtained from strain-displacement equations
- Strains can be substituted in a failure criterion (e.g. max strain) to check for failure and modify design accordingly

Energy Methods

- Solving the governing PDE is usually very difficult. Other methods such as energy methods are used in such cases
- **Min Potential Energy:** "Of all geometrically compatible displacement states, those which also satisfy the force balance conditions give stationary values to the potential energy"*
- **Min Complementary Energy:** "Of all self-balancing force states, those which also satisfy the requirements of geometric compatibility give stationary values to the complementary energy"*

* Crandall, S.H. *Engineering Analysis*, McGraw-Hill, 1956

Energy Methods (cont'd)

- In other words (methods of solution):

Equilibrium equations	**Strain Compatibility condition**	**Force Boundary Conditions**	**Displa-cement Boundary Conditions**	**Energy**	**Solution is**
Exactly	Exactly	Exactly	Exactly	Minimized	Exact
Approximately	Exactly	(on an average sense)	Exactly	Minimize potential (displ. -based)	Approximate
Exactly	Approximately	Exactly	(on an average sense)	Minimize comple-mentary	Approximate

Energy expressions for anisotropic plate

- Potential energy

$$U=\frac{1}{2}\iint\left\{A_{11}\left(\frac{\partial u}{\partial x}\right)^2+2A_{12}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+A_{22}\left(\frac{\partial v}{\partial y}\right)^2+A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+2\left(A_{16}\frac{\partial u}{\partial x}+A_{26}\frac{\partial v}{\partial y}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}dxdy+$$

$$\iint\left\{\begin{aligned}&-B_{11}\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}-B_{12}\left(\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial y^2}\right)-B_{22}\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y^2}-2B_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial^2 w}{\partial x\partial y}\\&-B_{16}\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}+2\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\right]-B_{26}\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial^2 w}{\partial y^2}+2\frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\right]\end{aligned}\right\}dxdy+$$

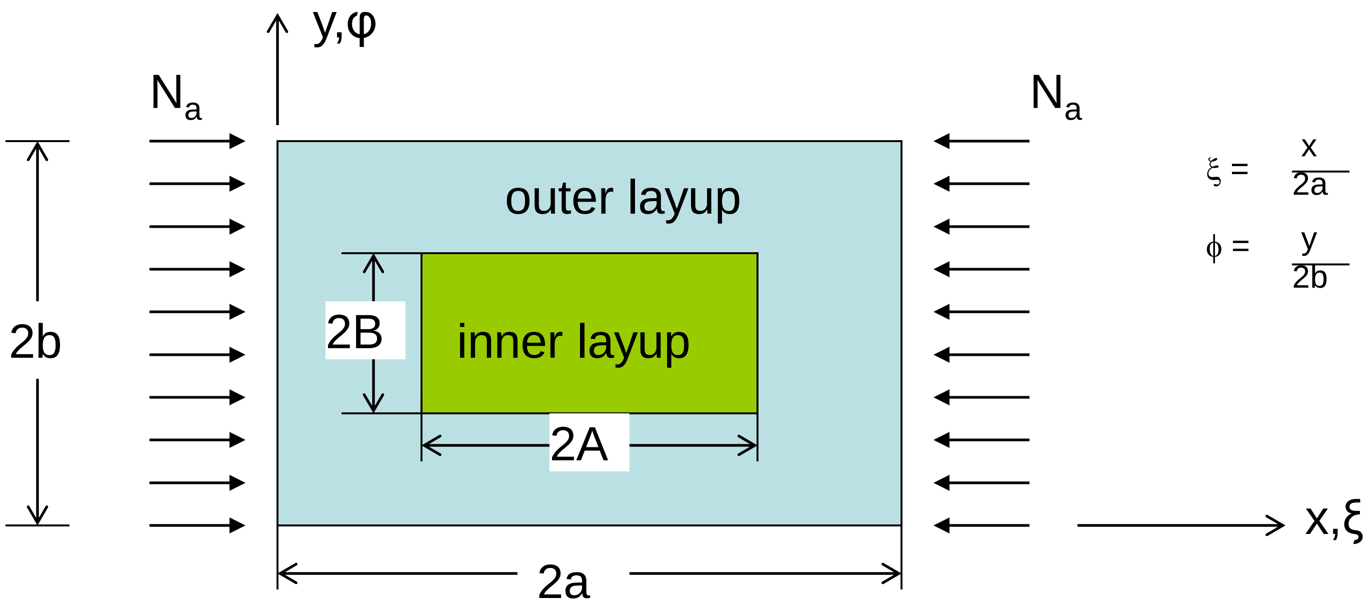
$$\frac{1}{2}\iint\left\{D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2+4D_{66}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+4D_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\partial y}+4D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x\partial y}\right\}dxdy$$

Pure membrane

Bending-stretching coupling

Pure bending

Example: Rectangular composite plate with two concentric layups[1]

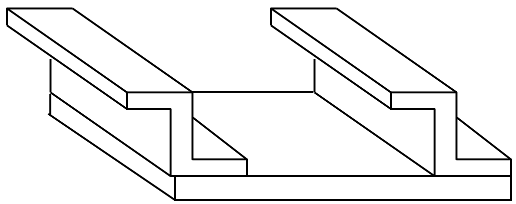


- simply supported outer perimeter
- applied <u>constant force</u> $N_a = F/(2b)$

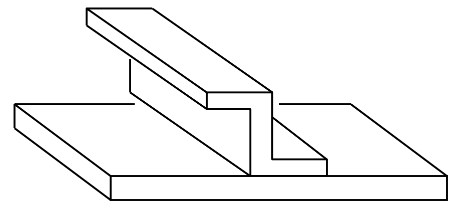
(1) Kassapoglou, C., "Composite Plates With two Concentric Layups Under Compression", Composites Part A: Applied Science and Manufacturing, 39, 2008, pp 104-112.

Motivation

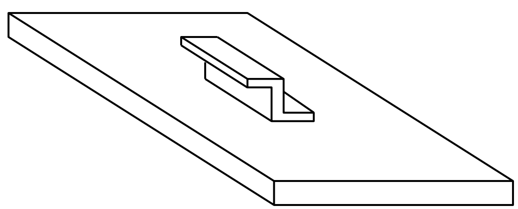
- special cases of stiffened panels
- terminated stiffeners
- panel with rectangular cutout



Two edge stiffeners



One center stiffener or panel breaker condition



One terminated stiffener at the center

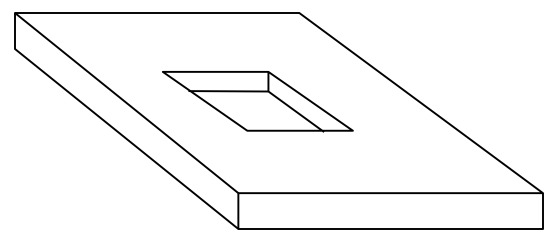


Plate with rectangular cutout

- Increased buckling load (same weight) over single layup case

Solution

- Assume inner and outer layups are symmetric
- Energy expression decouples to stretching and bending energy
- To determine in-plane forces need only the stretching part of the complementary energy:

$$U_c = \frac{1}{2}\iint \left[a_{11} N_x{}^2 + a_{22} N_y{}^2 + 2a_{12} N_x N_y + a_{66} N_{xy}{}^2 \right] dxdy$$

Solution: Minimization of complementary energy

- Select force resultants that satisfy equilibrium,

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

- force boundary conditions,

$$N_x(x=0) = N_x(x=2a) = \frac{F_a}{2b}$$

$$N_y(y=0) = N_y(y=2b) = 0$$

$$N_{xy}(x=0) = N_{xy}(x=2a) = N_{xy}(y=0) = N_{xy}(y=2b) = 0$$

- Selected Nx, Ny, Nxy:

$$N_x = \frac{F_a}{2b} + \sum_{m=1}^{M}\sum_{n=1}^{N} H_{mn}(\cos 2m\pi\xi - 1)\cos 2n\pi\phi$$

$$N_y = \frac{b^2}{a^2}\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{m^2}{n^2} H_{mn}\cos 2m\pi\xi(\cos 2n\pi\phi - 1)$$

$$N_{xy} = \frac{b}{a}\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{m}{n} H_{mn}\sin 2m\pi\xi\sin 2n\pi\phi$$

$$\xi = \frac{x}{2a}$$

$$\phi = \frac{y}{2b}$$

Minimize Total Energy

$$\Pi = \underbrace{\frac{1}{2}\iint\left[a_{11}N_x^{\ 2} + a_{22}N_y^{\ 2} + 2a_{12}N_xN_y + a_{66}N_{xy}^{\ 2}\right]dxdy}_{\text{complementary potential energy}} - \underbrace{\int \overbrace{N_x(x=2a)}^{N_a}u(x=2a)dy}_{\text{Work done by external forces}}$$

- determine H_{mn} by setting:

$$\frac{\partial \Pi}{\partial H_{mn}} = 0$$

a_{ij}=ij entry of A^{-1} matrix

Solution (cont'd)

- Determine u(x=2a)
  - Calculate axial strain

$$\varepsilon_x = \frac{N_x A_{22} - N_y A_{12}}{A_{11} A_{22} - A_{12}{}^2}$$

  - Substitute for Nx, Ny and integrate w.r.t x. This gives u to within an arbitrary function of y. Apply boundary condition u(x=0)=0 to obtain:

$$u(x=2a) = u(\xi=1) = \frac{2a}{A_{11}A_{22} - A_{12}{}^2}\left(\frac{A_{22}F_a}{2b} - A_{22}\sum_m \sum_n H_{mn}\cos 2n\pi\phi\right)$$

Solution (cont'd)

- Substitute in the expression for the total energy and minimize with respect to H_{mn}:

$$[E]\{H\} = \{R\}$$

from the complementary energy

unknowns H_{mn}

from the work done by applied force

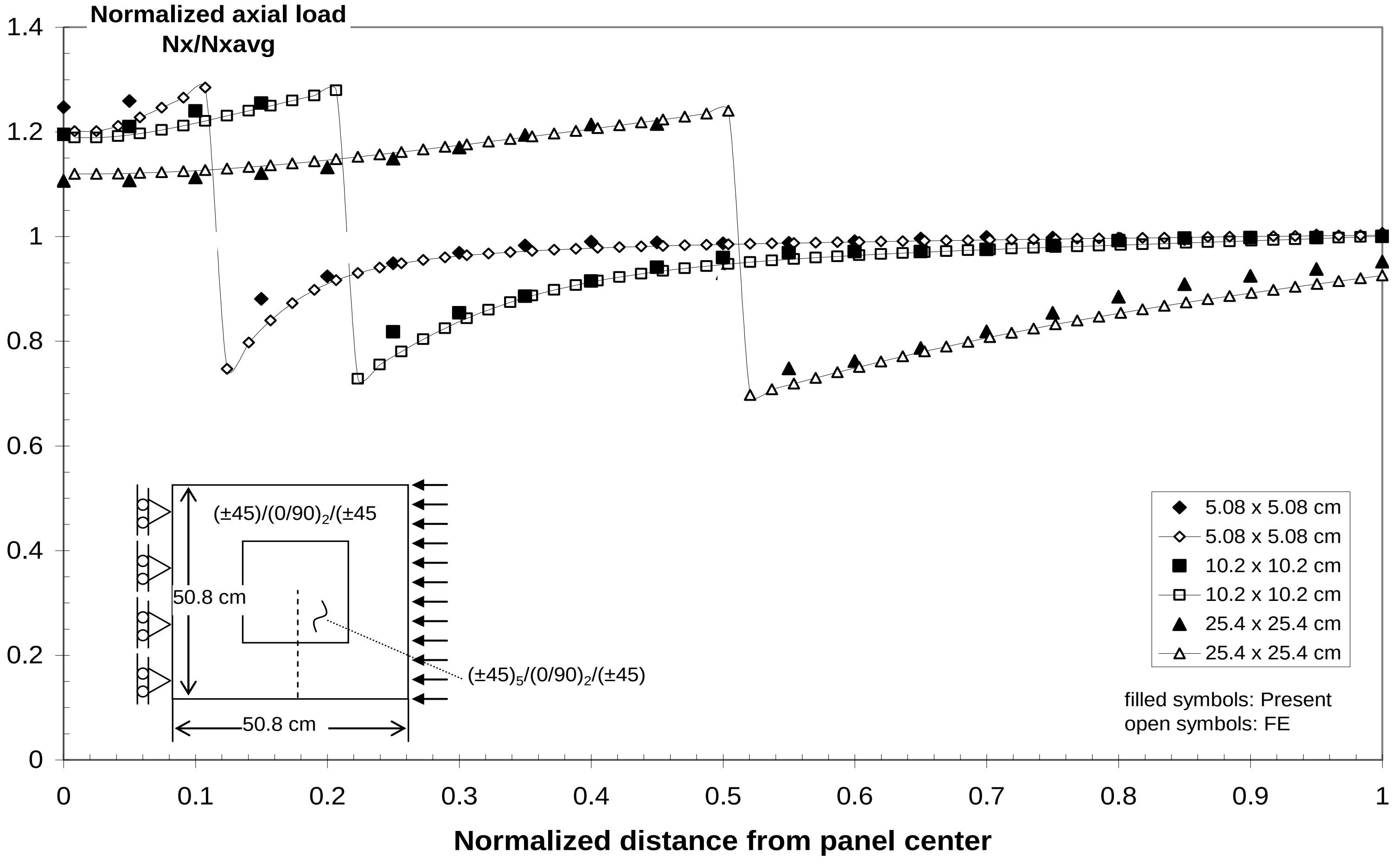
- System of mxn* equations by mxn unknowns solved by Gaussian elimination

* typically set m=n=30

Results – Comparison with FE

- Center layup: $(\pm 45)_5/(0/90)_2/(\pm 45)_5$
- Perimeter layup $(\pm 45)/(0/90)_2/(\pm 45)$
- Plain weave fabric
- 50.8cm x 50.8 cm plate with center layup dimensions:
  - 5.08 cm x 5.08 cm
  - 10.16 cm x 10.16 cm
  - 25.4 cm x 25.4 cm

Comparison to FE results



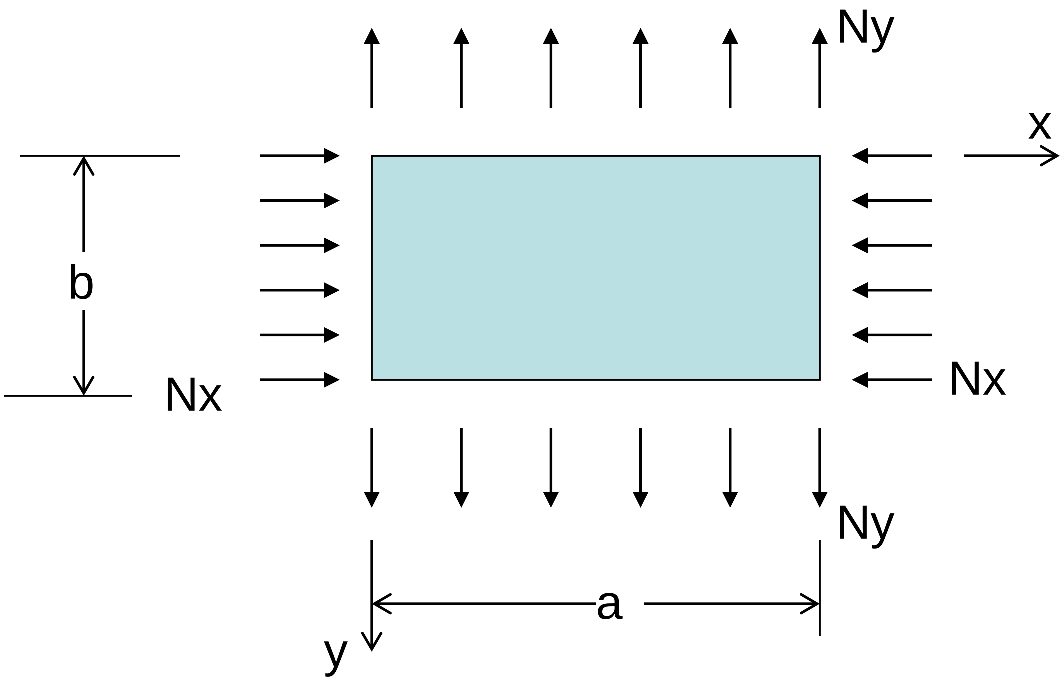
Buckling

Buckling under own weight

Buckling under own weight

Buckling

- Simply supported composite plate under bi-axial loading*



$D_{16}=D_{26}=B_{ij}=0$

* Whitney, J.M., *Structural Analysis of Laminated Anisotropic Plates*, Technomic Publishing, 1987, section 5.5

Buckling of Plate under Biaxial Load (cont'd)

- Governing equation (from before with N_{xy}=0)

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}$$

- BC's:

$$w=M_x=-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2}=0$$ at x=0 and x=a

$$w=M_y=-D_{12}\frac{\partial^2 w}{\partial x^2}-D_{22}\frac{\partial^2 w}{\partial y^2}=0$$ at y=0 and y=b

Buckling under Biaxial Load (cont'd)

- Solution:

$$w = \sum\sum A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

- Satisfies all BC's; Substitute in governing eqn and satisfy it term-by-term:

$$\pi^2 A_{mn}\left[D_{11}m^4 + 2(D_{12}+2D_{66})m^2n^2(AR)^2 + D_{22}n^4(AR)^4\right] = -A_{mn}a^2\left[N_x m^2 + N_y n^2 (AR)^2\right]$$

(AR)=aspect ratio=a/b

- For $A_{mn} \neq 0$ and Nx = -No, Ny = -k No

(N_o is special value of Nx at which out-of-plane deflections of the plate are possible=> buckling load)

Buckling under Biaxial Load (cont'd)

- Buckling Load

$$N_o = \frac{\pi^2\left[D_{11}m^4 + 2(D_{12} + 2D_{66})m^2n^2(AR)^2 + D_{22}n^4(AR)^4\right]}{a^2(m^2 + kn^2(AR)^2)}$$

find m,n that minimize N_o

- Uniaxial compression (Ny=0=>k=0)

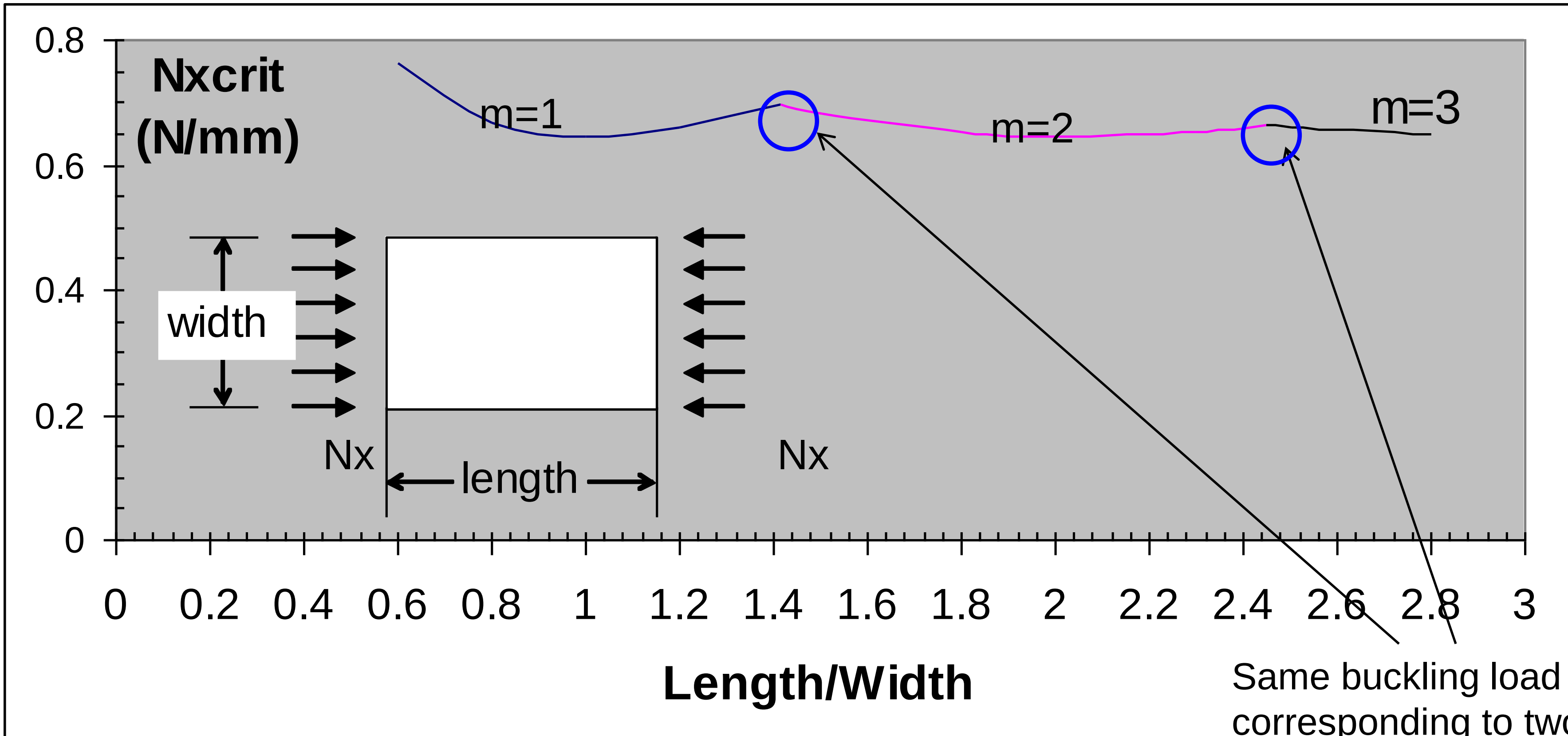
$$N_o = \frac{\pi^2\left[D_{11}m^4 + 2(D_{12} + 2D_{66})m^2n^2(AR)^2 + D_{22}n^4(AR)^4\right]}{a^2m^2}$$

which is minimized for n=1; determine m that minimizes N_o

$$N_o = \frac{\pi^2\left[D_{11}m^4 + 2(D_{12} + 2D_{66})m^2(AR)^2 + D_{22}(AR)^4\right]}{a^2m^2} \qquad (5.2.3.1)$$

Uniaxial compression (cont'd)

- m is number of halfwaves in x direction (function of AR)



D_{11}	0.66
D_{12}	0.47
D_{22}	0.66
D_{66}	0.49

Nm