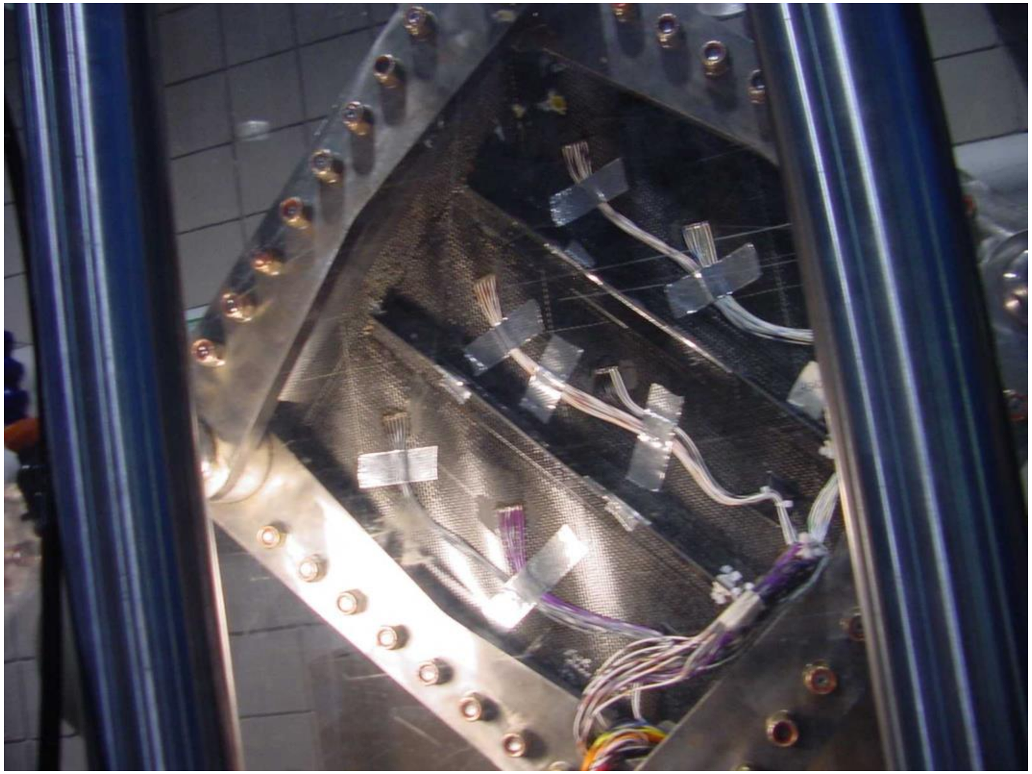
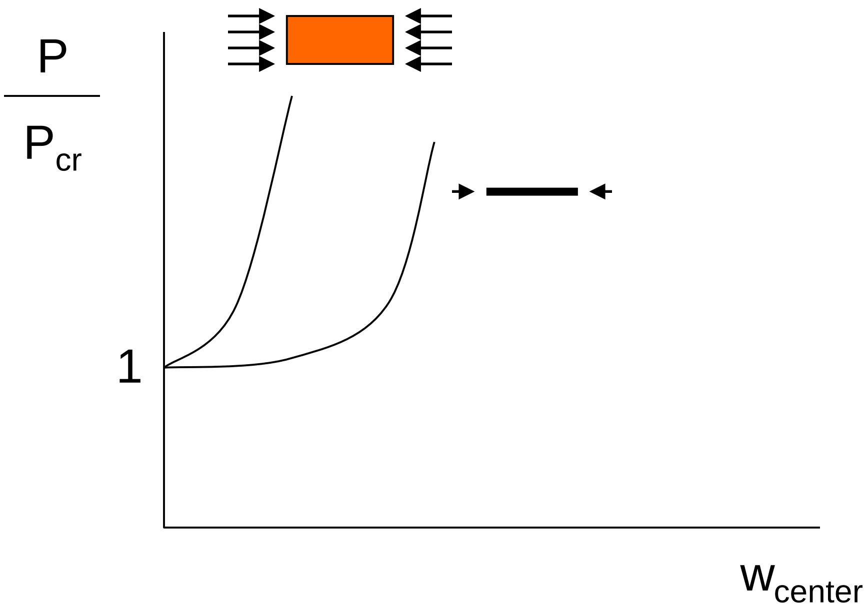
Post-Buckling



Post-buckled composite stiffened panel under shear

Buckling vs Post-buckling

- in general buckling does not imply failure especially for plates



beam post-buckling curve is flat and failure strains are reached at low post-buckling loads

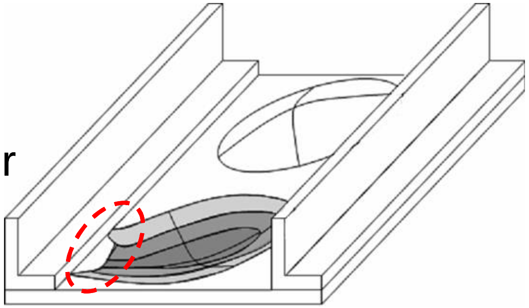
Buckling vs Post-buckling

• post-buckled (stiffened) skins are significantly lighter but have additional challenges:

– susceptible to skin/stiffener separation failure where only resin holds the structure (unless fasteners or other means are used to hold it together)

– susceptible to the creation and growth of delamination under fatigue loading especially at high post-buckling factors

skin/stiffener
separation



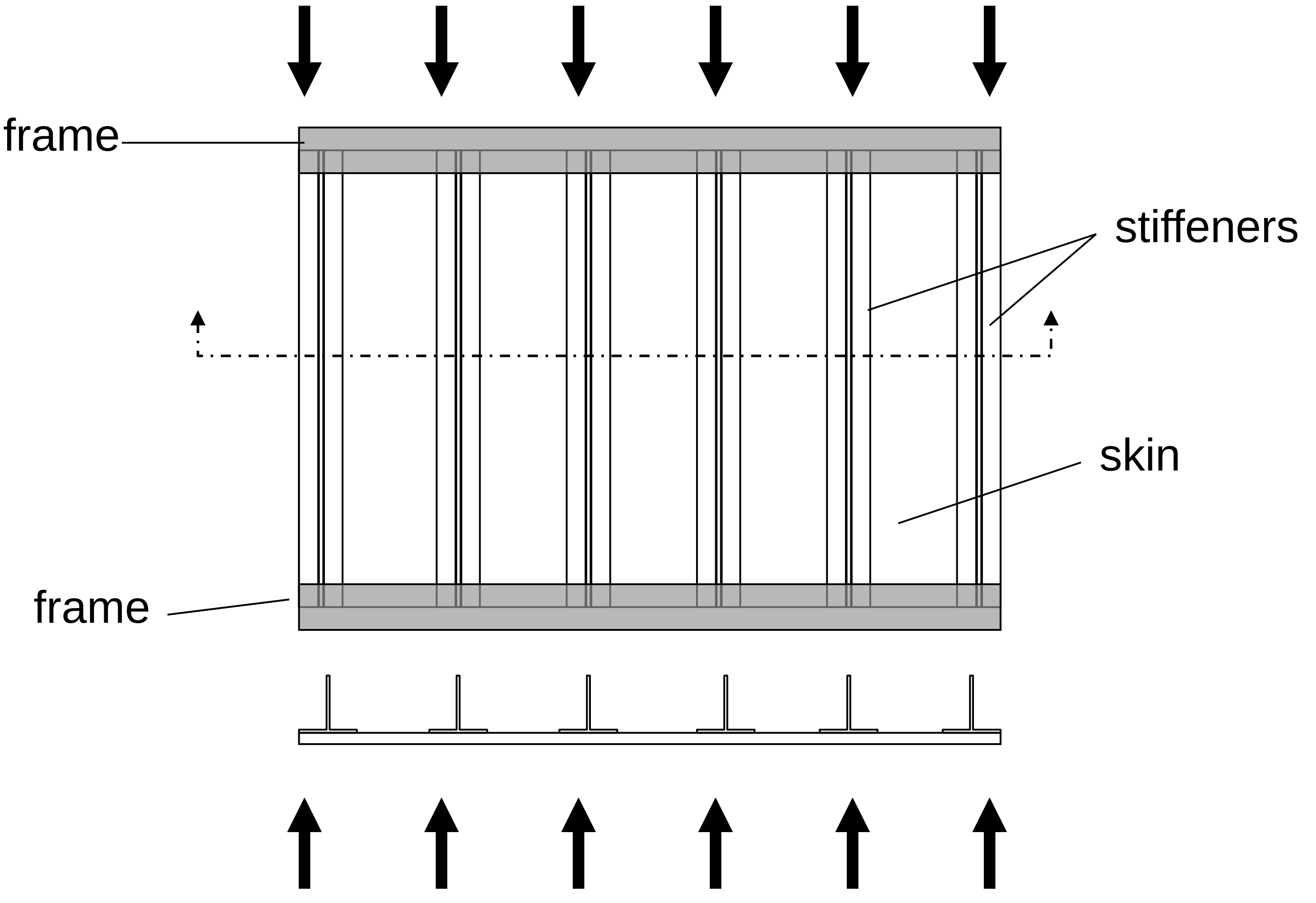
Buckling versus Post-Buckling

- If a composite structure is allowed to post-buckle, the ratio of final failure load to buckling load (post buckling ratio PB) must be chosen carefully (especially for shear loads)

- Conservative approach (but not very efficient): Do not allow buckling below limit load (i.e. PB=1.5)

- PB>5 great design challenge both under static and fatigue loads

Post-buckling mode



What buckles and when?

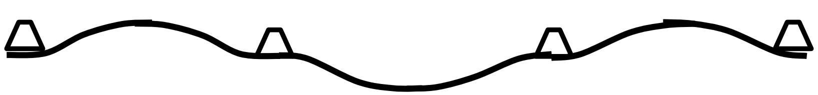
What BC's are the different components supposed to simulate?

Post-Buckling Scenarios

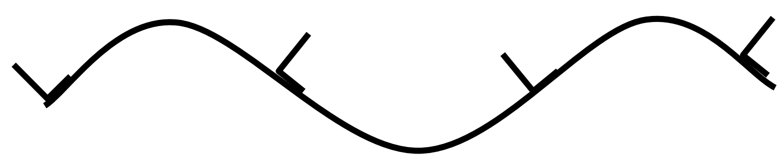
- Skin buckling as a whole (stiffeners only increase the bending stiffness of the skin)
- Skin buckling between the bays (stiffeners act as panel breakers)
- Stiffeners buckle as columns or locally (crippling)
- Frames do not buckle!

Post-buckling scenarios

- Most efficient:
  - – Skin between stiffeners and frames buckles first
  - – Stiffeners do not buckle and do not move out of plane (depending on cross-section they may rotate => BC implications)
  - – When required PB value is reached, skin fails in compression and/or stiffeners fail by crippling



stiffeners impose zero deflection and slope



stiffeners impose zero deflection only

Post-Buckling Analysis

- governing equations: Large deflection von Karman equations
- place moment equilibrium eqn into Fz equilibrium eqn:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - p_x \frac{\partial w}{\partial x} - p_y \frac{\partial w}{\partial y} + p_z = 0$$

p_x, p_y, p_z distributed loads (force/area)

- use the moment-curvature relations to substitute for the moments

$$M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2}$$

$$M_y = -D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2}$$

$$M_{xy} = -2D_{66}\frac{\partial^2 w}{\partial x \partial y}$$

$B_{ij}=0$

$D_{16}=D_{26}=0$

Derivation of von Karman equations (large deflections)

- to obtain:

$$\boxed{D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_y\frac{\partial^2 w}{\partial y^2}-p_x\frac{\partial w}{\partial x}-p_y\frac{\partial w}{\partial y}+p_z}$$

1st von Karman equation: Predominantly bending

Strain Compatibility

$$\varepsilon_{xo} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\varepsilon_{yo} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\gamma_{xyo} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)$$

$$\frac{\partial^2 \varepsilon_{xo}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{2\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right) = \frac{\partial^3 u}{\partial x \partial y^2} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x \partial y^2}$$

$$\frac{\partial^2 \varepsilon_{yo}}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{2\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right) = \frac{\partial^3 v}{\partial x^2 \partial y} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2 \partial y}$$

$$\frac{\partial^2 \gamma_{xyo}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x \partial y^2} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

$$\frac{\partial^2 \varepsilon_{xo}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yo}}{\partial x^2} - \frac{\partial^2 \gamma_{xyo}}{\partial x \partial y} = \ldots$$

Derivation of von Karman equations (large deflections)

- use (non-linear) strain compatibility

$$\frac{\partial^2 \varepsilon_{xo}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yo}}{\partial x^2} - \frac{\partial^2 \gamma_{xyo}}{\partial x \partial y} = \underbrace{\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}}_{\text{non-linear terms}}$$

ε_{xo}, ε_{yo}, γ_{xyo} mid-plane strains

- invert stress-strain eqns to express strains in terms of N_x, N_y, N_{xy}

$$N_x = A_{11}\varepsilon_{xo} + A_{12}\varepsilon_{yo}$$
$$N_y = A_{12}\varepsilon_{xo} + A_{22}\varepsilon_{yo}$$
$$N_{xy} = A_{66}\gamma_{xyo}$$

$$\Rightarrow$$

$$\varepsilon_{xo} = \frac{A_{22}}{A_{11}A_{22} - A_{12}^2} N_x - \frac{A_{12}}{A_{11}A_{22} - A_{12}^2} N_y$$
$$\varepsilon_{yo} = -\frac{A_{12}}{A_{11}A_{22} - A_{12}^2} N_x + \frac{A_{11}}{A_{11}A_{22} - A_{12}^2} N_y$$
$$\gamma_{xyo} = \frac{1}{A_{66}} N_{xy}$$

$A_{16}=A_{26}=0$

Derivation of von Karman equations (large deflections)

- substitute in strain compatibility to obtain,

$$\frac{1}{A_{11}A_{22}-A_{12}^{\;2}}\left(A_{22}\frac{\partial^2 N_x}{\partial y^2}-A_{12}\frac{\partial^2 N_y}{\partial y^2}+A_{11}\frac{\partial^2 N_y}{\partial x^2}-A_{12}\frac{\partial^2 N_x}{\partial x^2}\right)-\frac{1}{A_{66}}\frac{\partial^2 N_{xy}}{\partial x\partial y}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

- introduce Airy stress function and potential V that identically satisfy stress equilibrium:

$$N_x=\frac{\partial^2 F}{\partial y^2}+V$$

$$N_y=\frac{\partial^2 F}{\partial x^2}+V$$

$$N_{xy}=-\frac{\partial^2 F}{\partial x\partial y}$$

potential of distributed loads (=0 in our post-buckling problem)

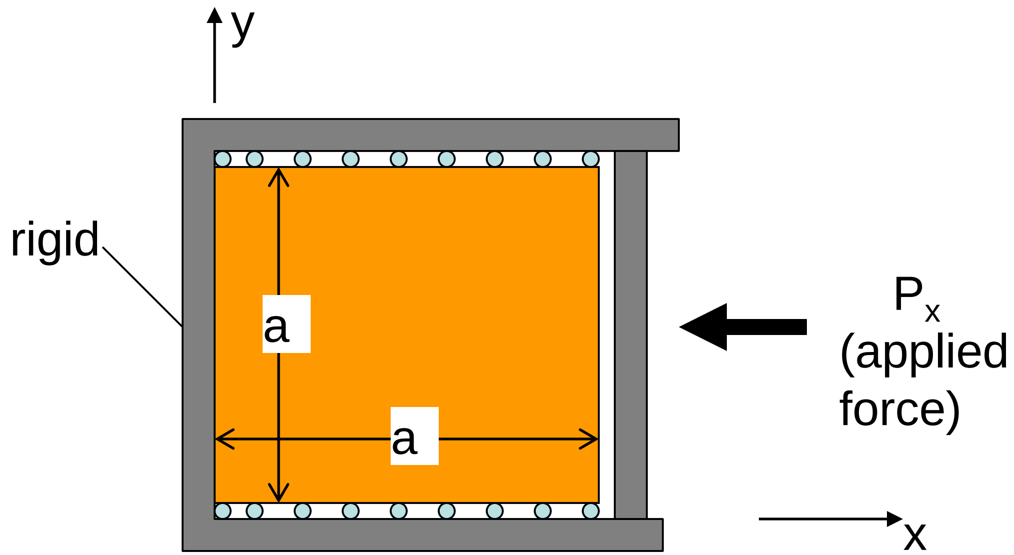
Derivation of von Karman equations (large deflections)

- to obtain

$$\frac{1}{A_{11}A_{22}-A_{12}{}^{2}}\left(A_{22}\frac{\partial^4 F}{\partial y^4}-2A_{12}\frac{\partial^4 F}{\partial x^2\partial y^2}+A_{11}\frac{\partial^4 F}{\partial x^4}\right)+\frac{1}{A_{66}}\frac{\partial^4 F}{\partial x^2\partial y^2}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

2nd von Karman equation: Predominantly stretching (and non-linear)

Post-buckling of a square anisotropic plate under compression



simply-supported with three immovable edges:

w=0 at x=y=0 and x=y=a

u=0 at x=0

v=0 at y=0 and y=a

u=-C at x=a (constant compressive displacement)

Solve the two von Karman equations approximately

- assume

$$w = w_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

$$F = -\frac{P_x}{a}\frac{y^2}{2} - \frac{P_y}{a}\frac{x^2}{2} + K_{20}\cos\frac{2\pi x}{a} + K_{02}\cos\frac{2\pi y}{a}$$

w_{11}, K_{02}, K_{20} unknown coefficients

- substitute in the 2nd von Karman equation

$$\frac{A_{22}}{A_{11}A_{22}-A_{12}{}^2}K_{02}\frac{16\pi^4}{a^4}\cos\frac{2\pi y}{a} + \frac{A_{11}}{A_{11}A_{22}-A_{12}{}^2}K_{20}\frac{16\pi^4}{a^4}\cos\frac{2\pi x}{a} = w_{11}{}^2\frac{\pi^4}{2a^4}\cos\frac{2\pi y}{a} + w_{11}{}^2\frac{\pi^4}{2a^4}\cos\frac{2\pi x}{a}$$

matching coefficients of cos(2πy/a) and cos(2πx/a)

$$K_{02} = \frac{A_{11}A_{22}-A_{12}{}^2}{A_{22}}\frac{w_{11}{}^2}{32}$$

$$K_{20} = \frac{A_{11}A_{22}-A_{12}{}^2}{A_{11}}\frac{w_{11}{}^2}{32}$$

and the 2nd von Karman eqn is identically satisfied

Solution (cont'd)

- P_y (as a function of P_x) and deflection C are determined by integrating stress-strain eqns and using average BC's on u and v:

$$\varepsilon_{xo} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

non-linear strain-displacement eq

also

$$\varepsilon_{xo} = \frac{A_{22}}{A_{11}A_{22} - A_{12}^{\,2}} N_x - \frac{A_{12}}{A_{11}A_{22} - A_{12}^{\,2}} N_y$$

inverted stress-strain eq

$$N_x = \frac{\partial^2 F}{\partial y^2} + \cancel{V}$$

$$N_y = \frac{\partial^2 F}{\partial x^2} + \cancel{V}$$

Airy stress function F

$$N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

$$\int_0^a\int_0^a \frac{\partial u}{\partial x} dxdy = \frac{A_{22}}{A_{11}A_{22} - A_{12}^{\,2}} \int_0^a\int_0^a \frac{\partial^2 F}{\partial y^2} dxdy - \frac{A_{12}}{A_{11}A_{22} - A_{12}^{\,2}} \int_0^a\int_0^a \frac{\partial^2 F}{\partial x^2} dxdy - \frac{1}{2}\int_0^a\int_0^a \left(\frac{\partial w}{\partial x}\right)^2 dxdy$$

Solution (cont'd)

$$a\big(u(a,y)-u(0,y)\big)=\frac{A_{22}a^2}{A_{11}A_{22}-A_{12}{}^2}\left(-\frac{P_x}{a}\right)-\frac{A_{12}a^2}{A_{11}A_{22}-A_{12}{}^2}\left(-\frac{P_y}{a}\right)-\frac{1}{2}\underbrace{\iint (w_{11}\frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a})^2\,dxdy}_{w_{11}{}^2\frac{\pi^2}{4}}$$

=-C =0

from which:

$$C=\frac{aA_{22}}{A_{11}A_{22}-A_{12}{}^2}\frac{P_x}{a}-\frac{aA_{12}}{A_{11}A_{22}-A_{12}{}^2}\frac{P_y}{a}+w_{11}{}^2\frac{\pi^2}{8a}$$

similarly, from $\varepsilon_{yo}=\dfrac{\partial v}{\partial y}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2$

$$P_y=P_x\frac{A_{12}}{A_{11}}-w_{11}{}^2\frac{\pi^2}{8a}\frac{A_{11}A_{22}-A_{12}{}^2}{A_{11}}$$

= Poisson's ratio v for isotropic

=0 for in-plane problems

Solution (cont'd)

- substitute now in the 1st von Karman equation

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

- use the following:

$$\frac{\partial^4 w}{\partial x^4}=\frac{\partial^4 w}{\partial x^2\partial y^2}=\frac{\partial^4 w}{\partial y^4}=w_{11}\left(\frac{\pi}{a}\right)^4\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w}{\partial y^2}=-w_{11}\left(\frac{\pi}{a}\right)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

$$\frac{\partial^2 F}{\partial x^2}=-\frac{P_y}{a}-\left(\frac{2\pi}{a}\right)^2\frac{A_{11}A_{22}-A_{12}{}^2}{A_{11}}\frac{w_{11}{}^2}{32}\cos\frac{2\pi x}{a}$$

$$\frac{\partial^2 F}{\partial y^2}=-\frac{P_x}{a}-\left(\frac{2\pi}{a}\right)^2\frac{A_{11}A_{22}-A_{12}{}^2}{A_{22}}\frac{w_{11}{}^2}{32}\cos\frac{2\pi y}{a}$$

Solution (cont'd)

- to substitute, and match coefficients of $\sin(\pi x/a)\sin(\pi y/a)$ to obtain the governing eq for w_{11}

$$\frac{\pi^2}{a}\frac{\left(A_{11}A_{22}-A_{12}{}^2\right)A_{11}+3A_{22}}{16A_{11}A_{22}}w_{11}{}^3+\left(\frac{\pi^2}{a}\left(D_{11}+2(D_{12}+2D_{66})+D_{22}\right)-P_x\left(1+\frac{A_{12}}{A_{11}}\right)\right)w_{11}=0$$

- which for $w_{11}\neq 0$ leads to

2nd von Karman eq is satisfied approximately!

$$w_{11}=\sqrt{\frac{16A_{11}A_{22}\left(D_{11}+2\left(D_{12}+2D_{66}\right)+D_{22}\right)}{\left(A_{11}A_{22}-A_{12}{}^2\right)\left(A_{11}+3A_{22}\right)}\left[\frac{P_x}{\frac{\pi^2}{a}\frac{\left(D_{11}+2\left(D_{12}+2D_{66}\right)+D_{22}\right)}{\left(1+\frac{A_{12}}{A_{11}}\right)}}-1\right]}$$

$=P_{cr}$, buckling load (units of force), exact for square plate

Solution (end)

- For out-of-plane deflections to be possible, must have

$$\frac{P_x}{P_{cr}} > 1$$

- the applied load P_x must exceed the plate buckling load

Results-Implications

- use as example,

(±45)/(0/90)/(±45) square plate of side 25.4 cm

Material is plain weave fabric with properties:

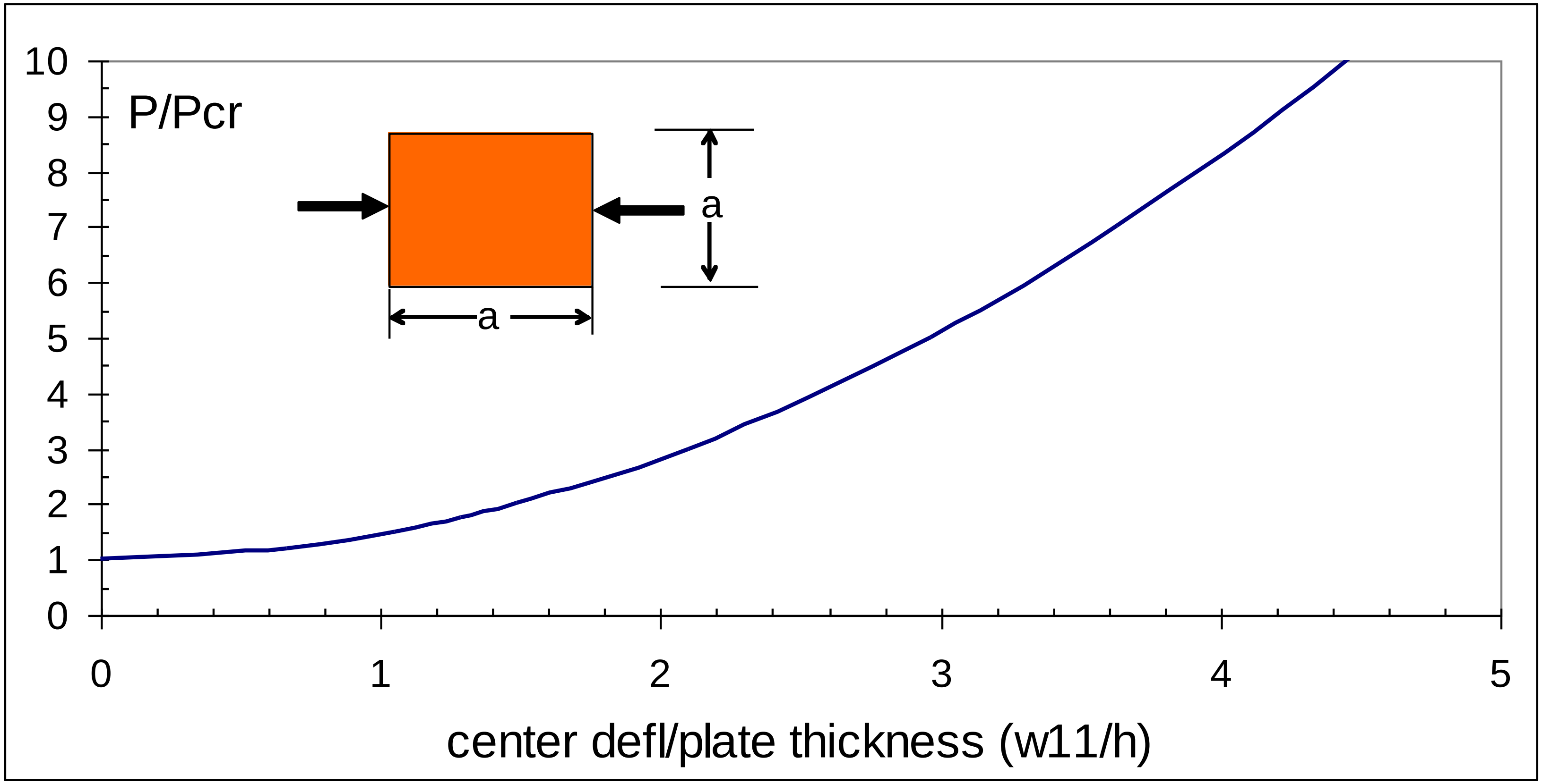
$E_x=E_y=68.94$ GPa

$v_{xy}=0.05$

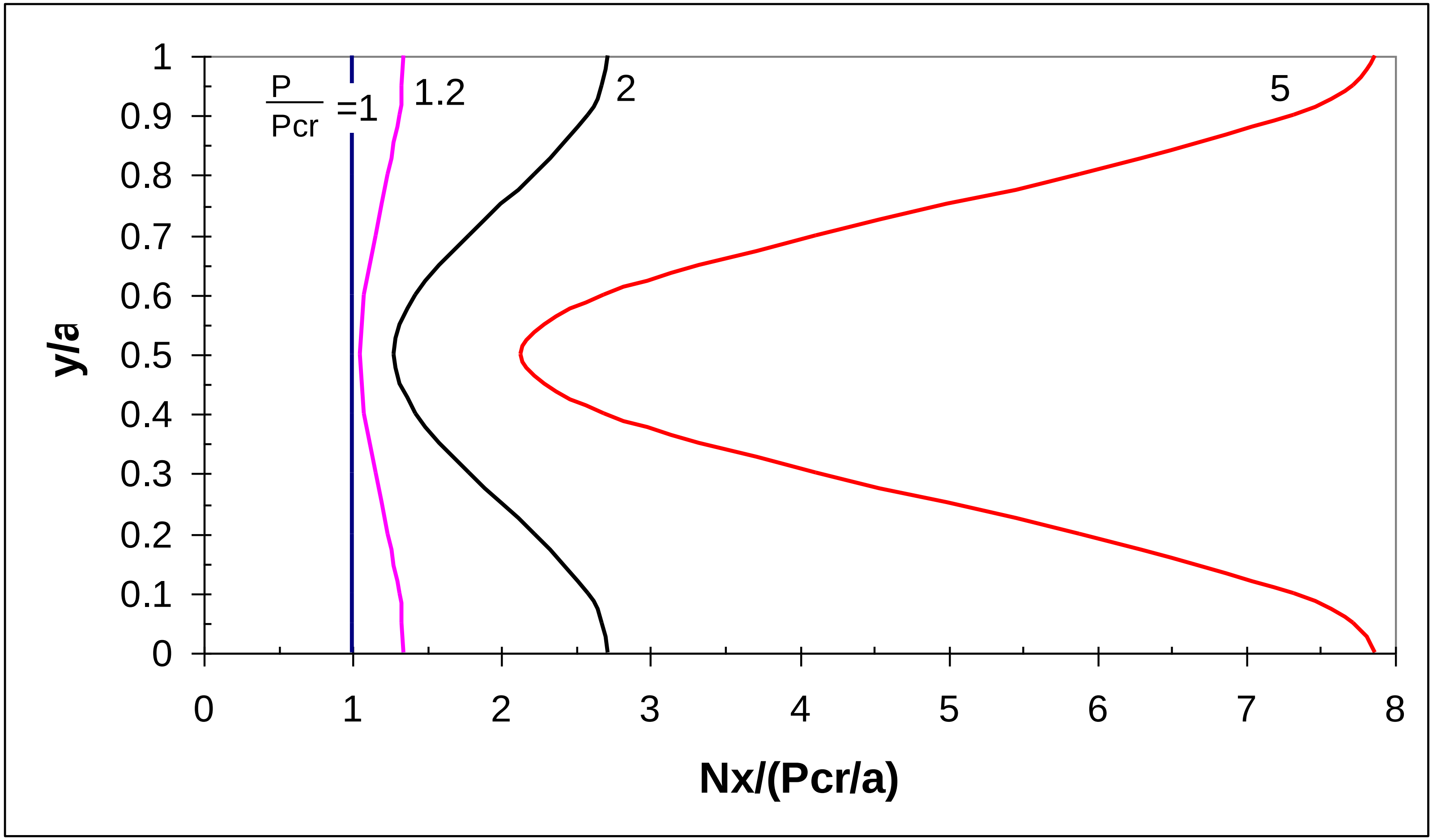
$G_{xy}=5.17$ GPa

ply thickness = 0.19 mm

Load versus center deflection



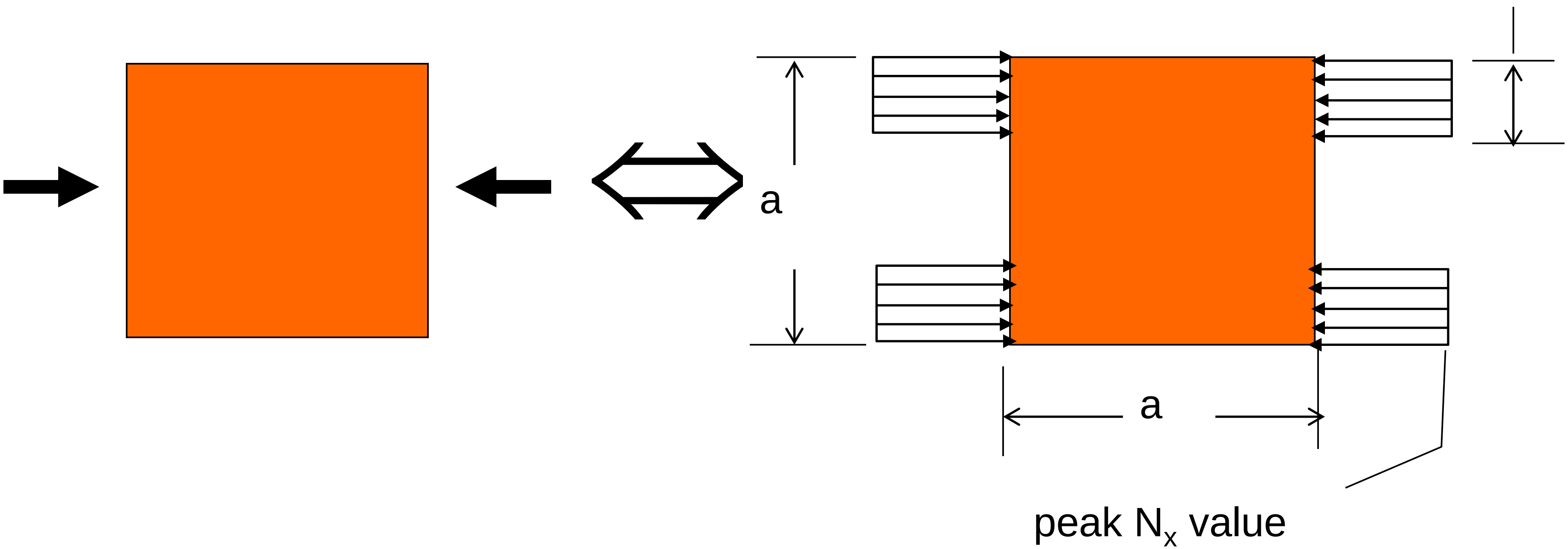
In-plane compression load



- **the center of the plate sees very little load!**

Implication

• after buckling approximate in-plane load



• **there is an effective width b_{eff} at the edges of the panel over which the load is concentrated**

Determination of effective width

• total applied force must equal the force created by the load applied over the effective width:

$$\int N_x dy = 2\left(N_{x\max}\right)b_{eff}$$

$$N_x = \frac{\partial^2 F}{\partial y^2} = -\frac{P_x}{a} - \frac{A_{11}A_{22} - A_{12}^2}{A_{22}} \frac{w_{11}^2}{32}\left(\frac{2\pi}{a}\right)^2 \cos\frac{2\pi y}{a}$$

$$N_{x\max} = -\frac{P_x}{a} - \frac{A_{11}A_{22} - A_{12}^2}{A_{22}} \frac{w_{11}^2}{32}\left(\frac{2\pi}{a}\right)^2$$

$$\int N_x dy = -P_x$$

Solving for b_{eff}

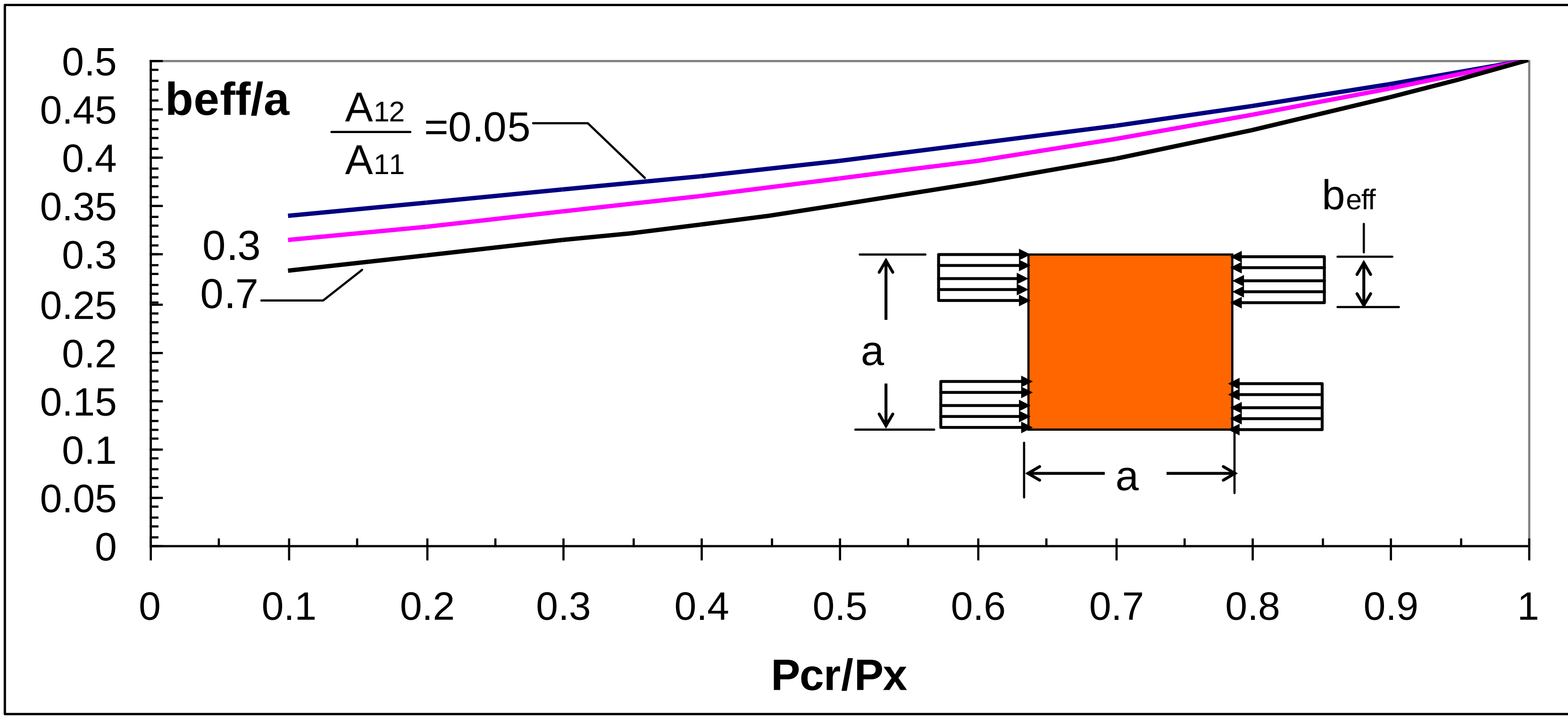
$$b_{eff} = a\frac{1}{2\left(1+2\left(1+\frac{A_{12}}{A_{11}}\right)\left(1-\frac{P_{cr}}{P_x}\right)\frac{A_{11}}{A_{11}+3A_{22}}\right)}$$

- Note that for quasi-isotropic layup,

$$\left.\begin{array}{l}\dfrac{A_{12}}{A_{11}} = \nu_{12} \\[2ex] \dfrac{A_{11}}{A_{11}+3A_{22}} = \dfrac{1}{4} \\[2ex] \nu_{12} \approx 0.3\end{array}\right\} \quad b_{eff} = a\frac{1}{2+1.3\left(1-\frac{P_{cr}}{P_x}\right)}$$

b_{eff}=0.303a for P_x>>P_{cr} and QI layup

Effective width b_{eff}



- weak dependence on A_{12}/A_{11}

Significance for design

- b_{eff} provides an idea of where failure is expected under compression and where reinforcement may be needed
- For (conservative) failure predictions obtain the maximum compressive stress

$$\sigma_{\max} = \frac{N_{x\max}}{h} \qquad \text{(h is plate thickness)}$$

- and compare to allowable ultimate compression stress

$$\sigma_{\max} \leq \sigma_{ult} \qquad \text{for no failure}$$

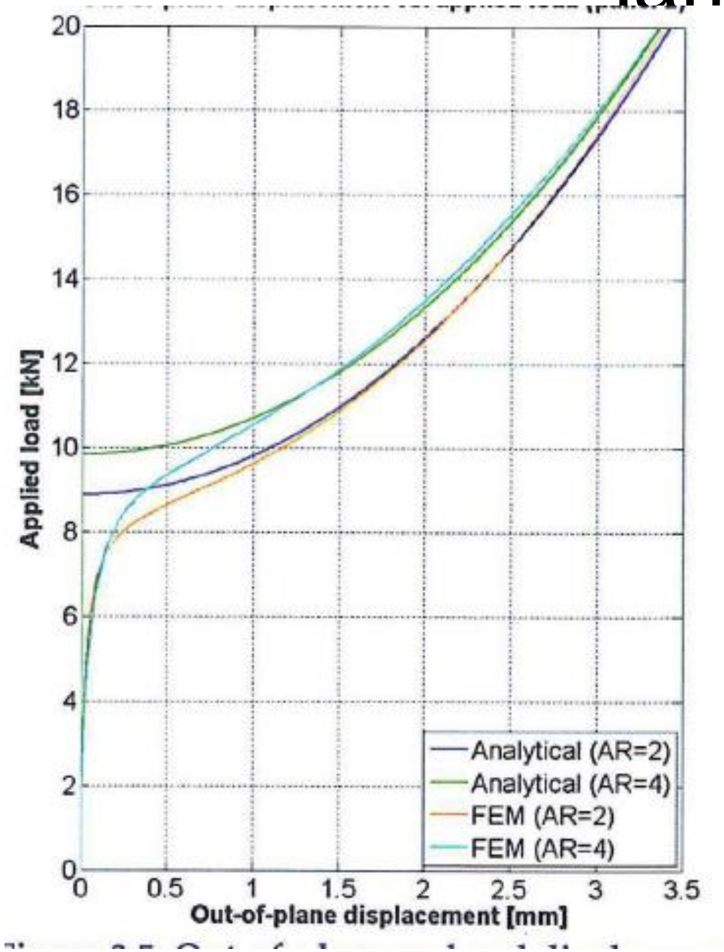
- plus one more use in crippling of stiffener flanges

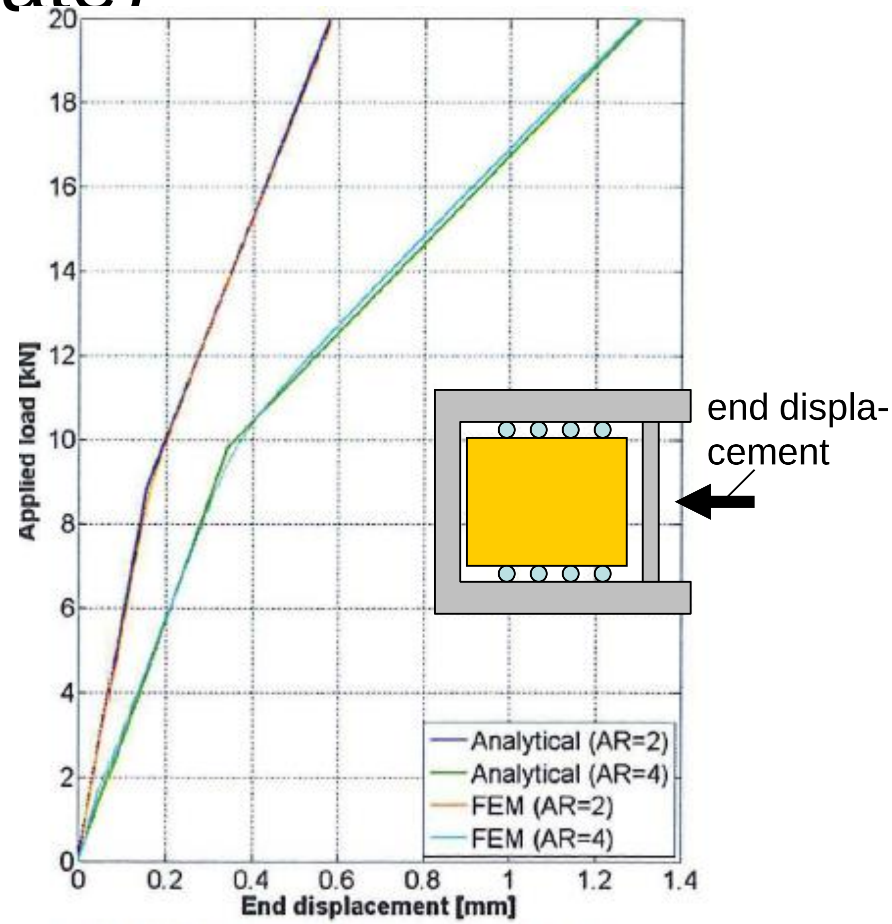
How good is the analysis compared to reality?

- For a meaningful comparison, two conditions must be met:
  - Boundary conditions in the "test case" must be the same as in the model
  - Sufficient number of terms must be included in the solution (as opposed to only one term used here for simplicity)
- As it is hard to find test results with exactly these boundary conditions, FE results are used to validate the analysis
- Two different laminates: (a) 45 degree dominated

  (b) Quasi-isotropic

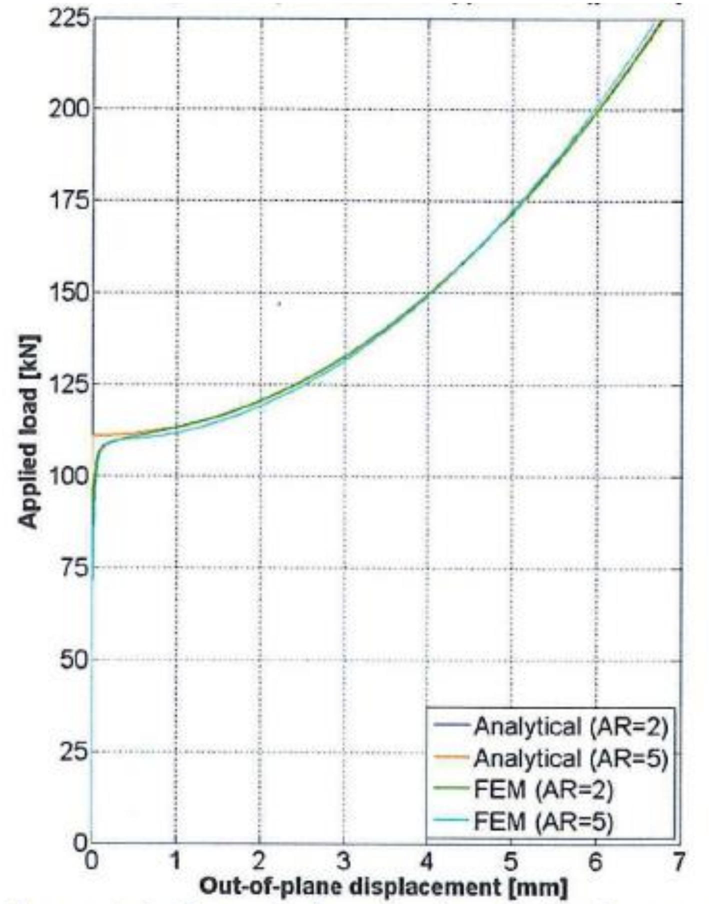
Analysis model versus FE (45-dominated laminate)

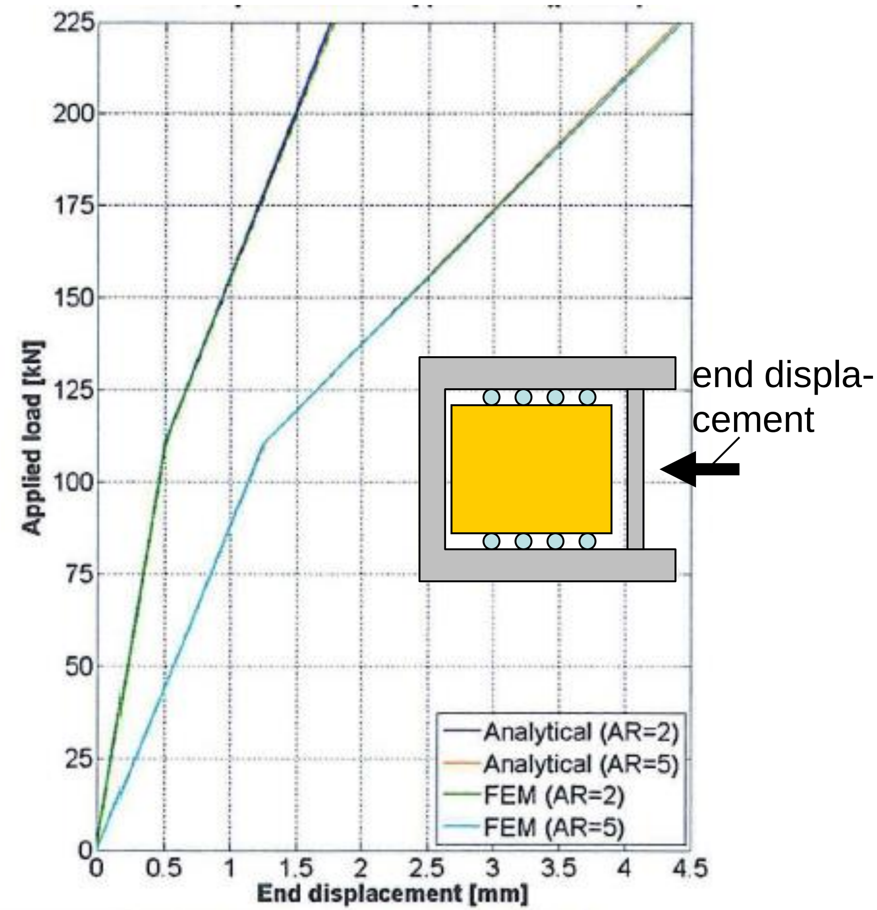




- Results from R. Kroese MS Thesis TUDelft 2013
- Excellent agreement between FE and Analysis

Analysis model versus FE (Quasi-isotropic laminate)





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- Excellent agreement between FE and Analysis

Boundary conditions discussion

- If the stiffeners in a stiffened panel have very high EI and very low GJ, and one end is attached to a frame with also very high EI and low GJ, this analytical model would be quite accurate
- If not, the model may be conservative or unconservative
- Must also use enough terms in the series and account for rectangular as opposed to square panels