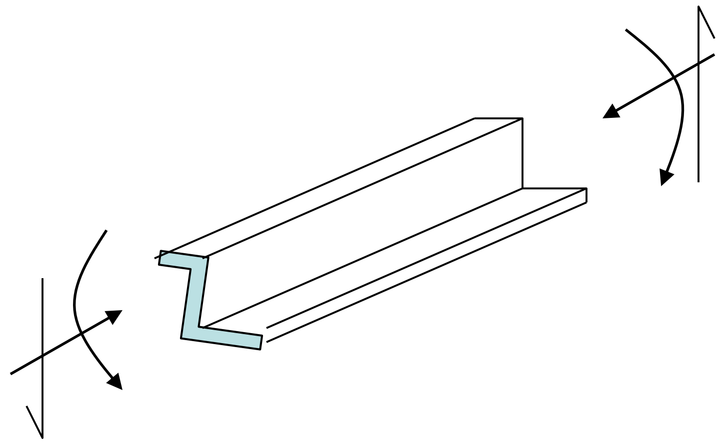
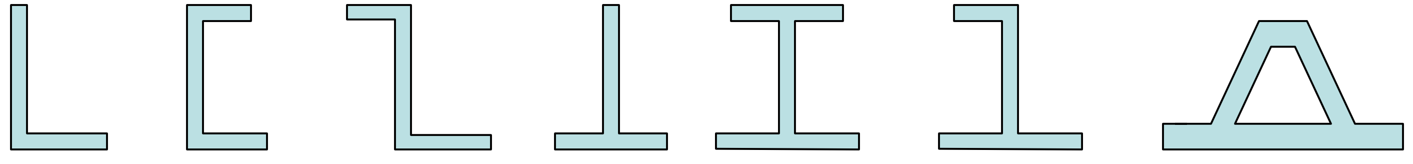
# Beams (Stringers, Stiffeners, Panel Breakers)

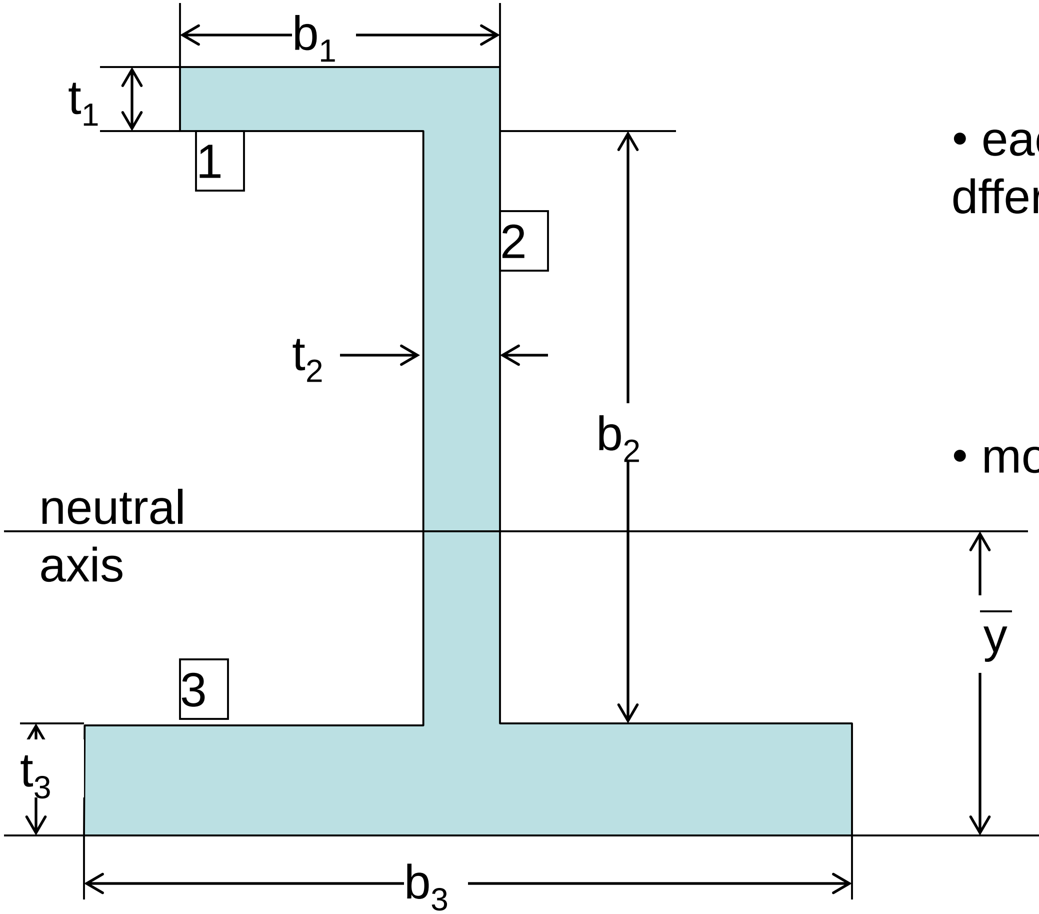
- axial (longitudinal) loads
- bending loads
- stiffening elements



- Different cross-sectional shapes



# Beams…cross-section properties




• each section or member can have dfferent layup =>

  – different stiffness

  – different strength

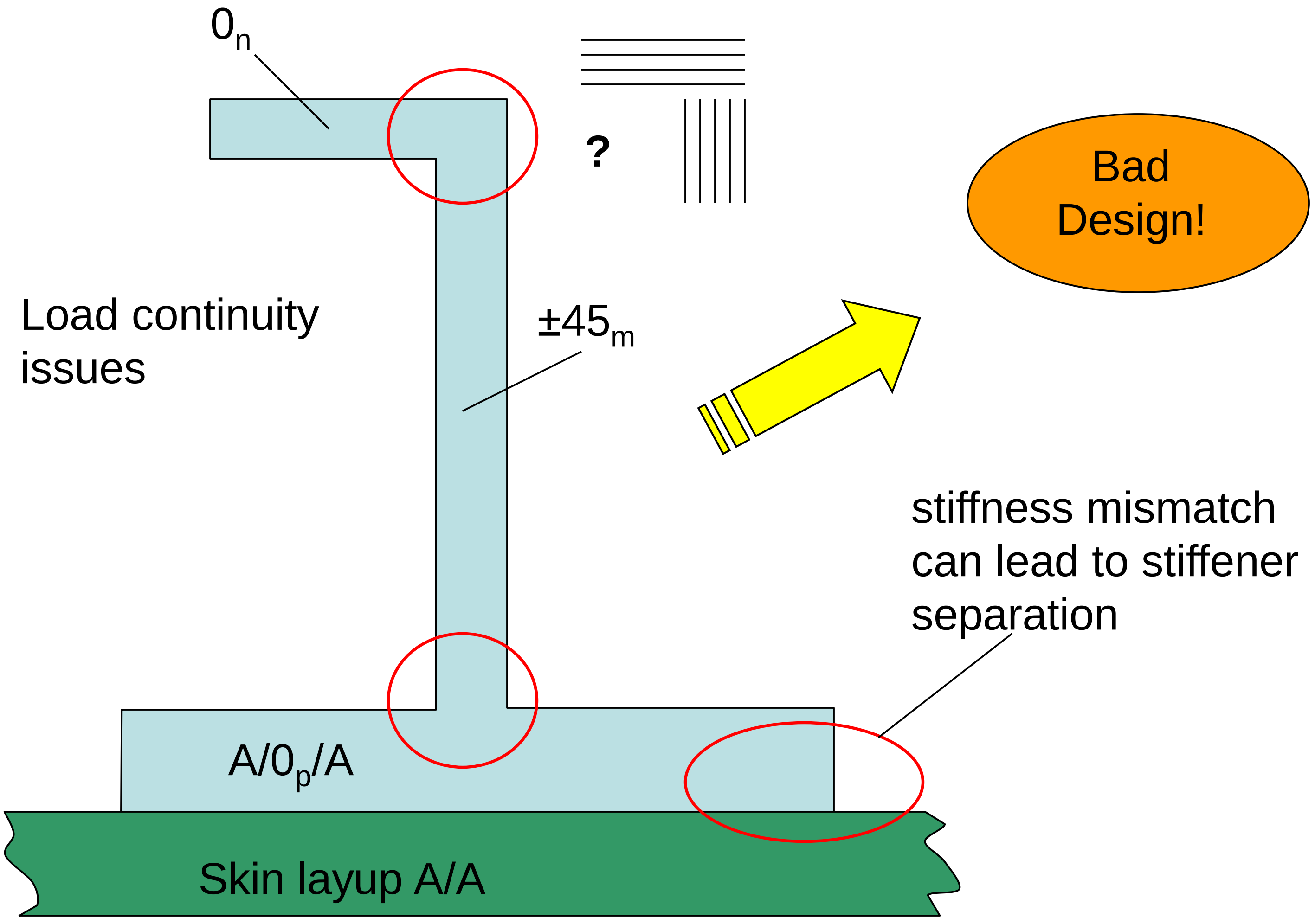
• more efficient structure by tailoring

# Beams…Layup Guidelines (qualitative)

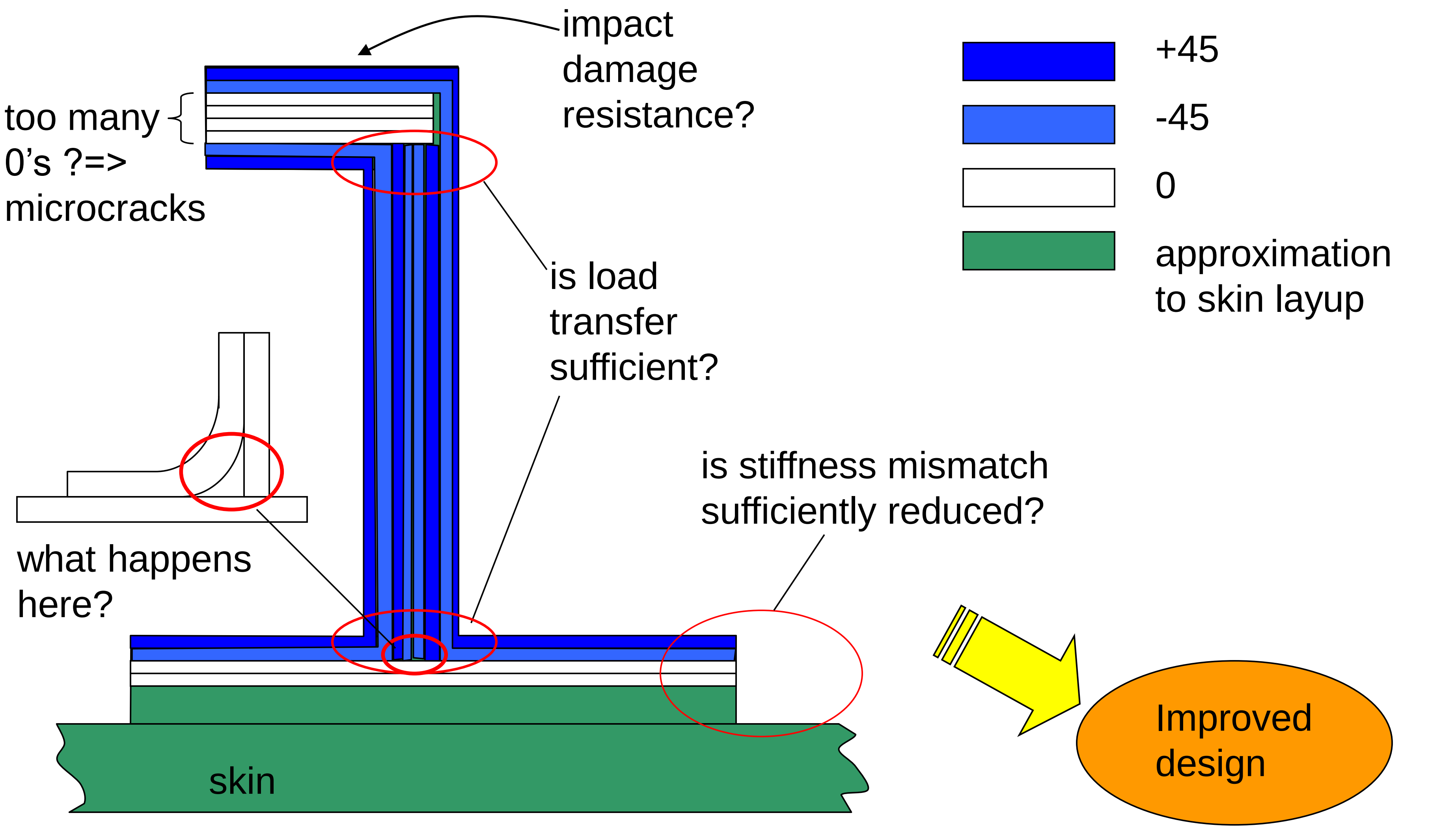
| Location | Layup[1] | Reason |
|---|---|---|
| Flange away from skin | 0 degree plies | need stiff material away from neutral axis |
| Web | ±45 degree plies | buckling resistance under shear; high shear strength |
| Flange next to skin | 0 degree plies **and** as close to the skin layup as possible | Stiff material away from neutral axis; reduced stiffness mismatch with skin |

(1) 0 degrees aligned with axis of beam

# Beams…Layup Guidelines (qualitative)



# First order correction of beam layup

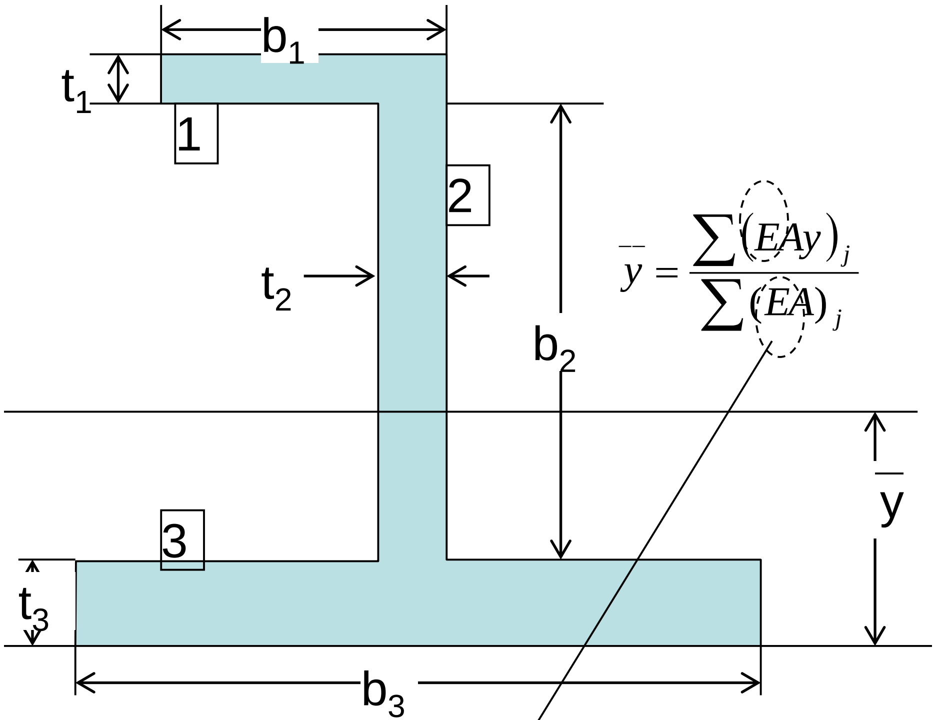


# Second order correction of beam layup

- to be continued…

# Beam cross section properties

- Equivalent axial stiffness $(EA)_{eq}$
- From axial strain compatibility:




$$\bar{y} = \frac{\sum (EAy)_j}{\sum (EA)_j}$$

$$\varepsilon_x^{(1)} = \varepsilon_x^{(2)} = \varepsilon_x^{(3)} = \varepsilon_a$$

$$\varepsilon_x^{(i)} = \frac{F_i}{(EA)_i} \qquad \text{i=1-3}$$

$$(EA)_i = E_i b_i t_i$$

$$E_i = \frac{1}{(a_{11})_i t_i}$$

$F_i$ is applied force on ith member

$E_i$ is **membrane** stiffness of ith member

note difference from isotropic case where E is not present!

# Beam cross-section properties (axial loading)

- Force sum:

$$F_{TOT} = F_1 + F_2 + F_3$$

- Three eqns in the three unknowns $F_1$-$F_3$

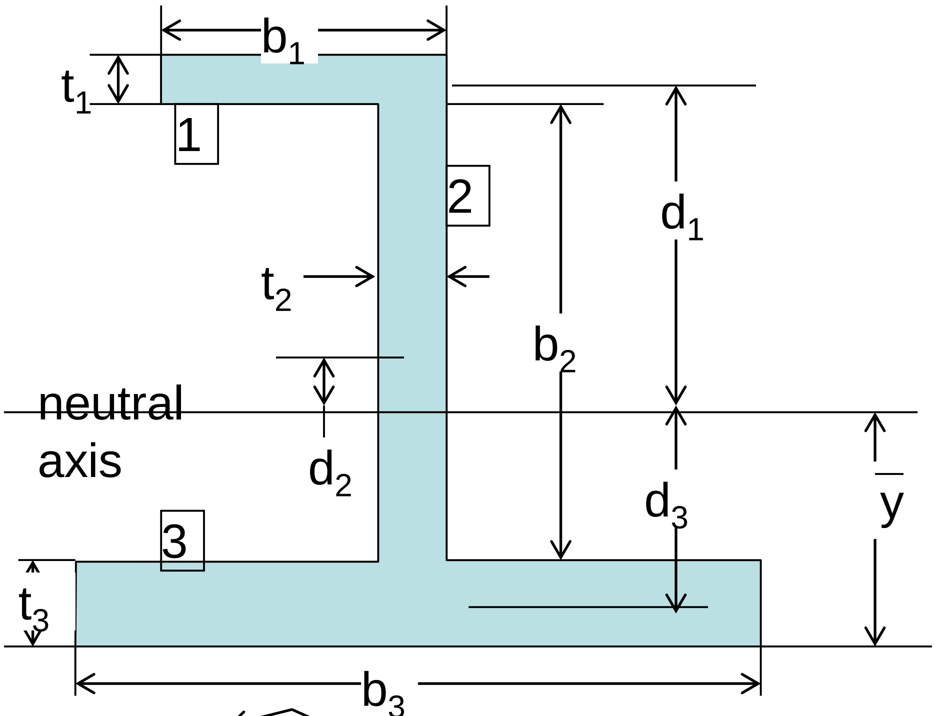
$$F_i = \frac{(EA)_i}{\sum_{j=1}^{3}(EA)_j} F_{TOT} = \frac{E_i b_i t_i}{\sum_{j=1}^{3} E_j b_j t_j} F_{TOT}$$

- Equivalent axial stiffness:

$$\left.\begin{aligned} \varepsilon_a &= \frac{F_{TOT}}{(EA)_{eq}} \Rightarrow (EA)_{eq} = \frac{F_{TOT}}{\varepsilon_a} \\ \varepsilon_a &= \frac{F_1}{(EA)_1} = \frac{\cancel{(EA)_1}}{\cancel{(EA)_1}\sum (EA)_j} F_{TOT} \end{aligned}\right\} \Rightarrow (EA)_{eq} = \sum_j (EA)_j$$

# Beam cross-section properties (bending load)

- Equivalent bending stiffness $(EI)_{eq}$




$$\bar{y} = \frac{\sum (E_b A y)_j}{\sum (E_b A)_j}$$

$E_b$ not present in isotropic case!

Bending of all members is characterized by same radius of curvature (that of beam neutral axis):

$$R_{c1} = R_{c2} = R_{c3} = R_{ca}$$

$$R_{ci} = \frac{(EI)_i}{M_i}$$

$$(EI)_i = E_{bi}\left[\frac{(width)_i (height)_i^3}{12} + A_i d_i^2\right]$$

$$E_{bi} = \frac{12}{t_i^3 (d_{11})_i}$$

$E_{bi}$ is **bending** stiffness of ith member

BUT not for beams deeper than 1cm!

# Beam Cross-section properties (bending load)

- Moment sum:

$$M_{TOT} = M_1 + M_2 + M_3$$

- Three eqns in the three unknowns $M_1$-$M_3$

$$M_i = \frac{(EI)_i}{\sum_{j=1}^{3} (EI)_j} M_{TOT}$$

analogous to the relation for the force on each member

- Equivalent bending stiffness for cross-section:

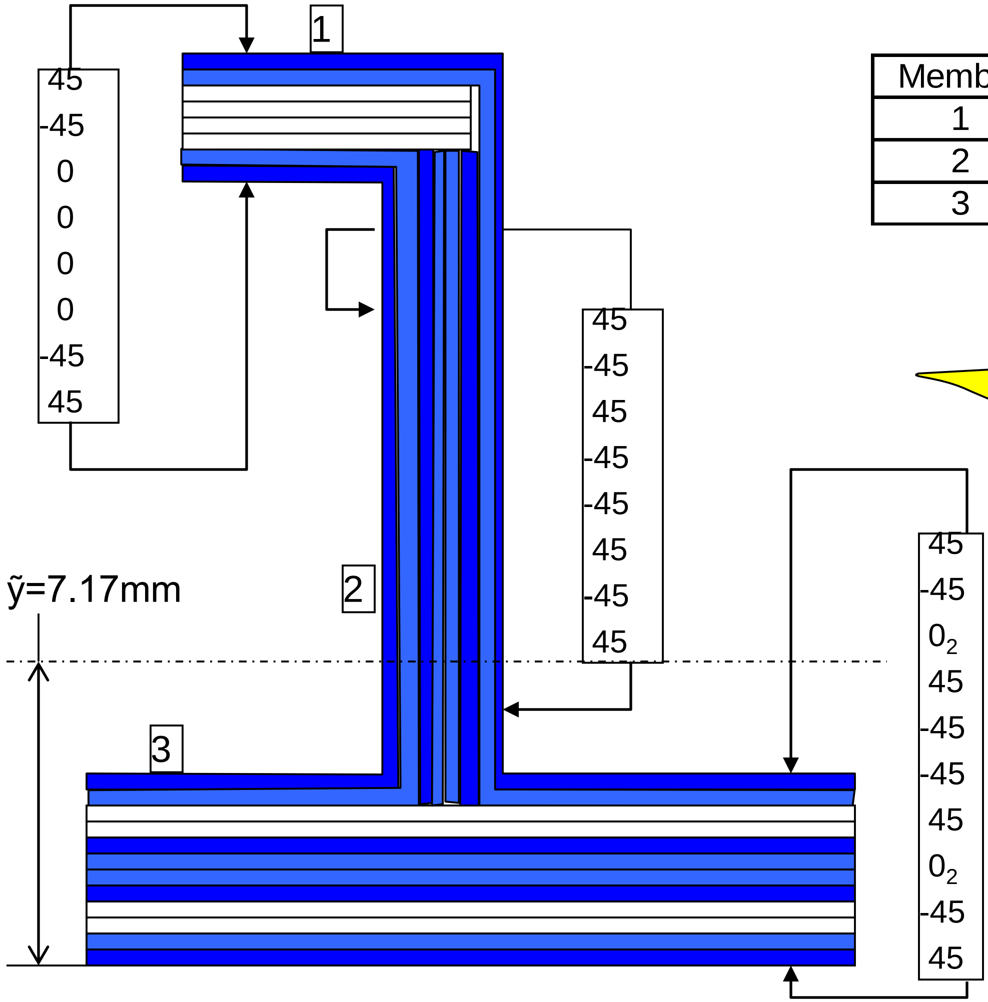
$$\frac{M_1}{(EI)_1} = \frac{M_{TOT}}{(EI)_{eq}} \Rightarrow (EI)_{eq} = \frac{M_{TOT}}{M_1}(EI)_1$$

$$M_1 = \frac{(EI)_1}{\sum_j (EI)_j} M_{TOT}$$

$$(EI)_{eq} = \sum_j (EI)_j$$

# Beam cross-sectional properties: Example

- same as before but with bottom flange better defined




| Member | b (mm) | t (mm) | Em (GPa) | Eb (GPa) |
|---|---|---|---|---|
| 1 | 12.7 | 1.2192 | 75.6 | 32.4 |
| 2 | 31.75 | 1.2192 | 18.2 | 17.9 |
| 3 | 38.1 | 1.8288 | 56.5 | 47.9 |

| | Alum | Comp | Δ(%) |
|---|---|---|---|
| EA (kN) | 8525 | 5803 | 46.9 |
| EI (Nm^2) | 1401 | 631 | 121.8 |

same thickness aluminum is more efficient but also 72% heavier

calculated using Eb

# Beam cross-sectional properties: Example (cont'd)

- increase (ply) thickness of composite by 46.9% and re-shuffle one flange layup:

| WAS | Layup | Member | b (mm) | t (mm) | Em (GPa) | Eb (GPa) | WAS |
|---|---|---|---|---|---|---|---|
| [45/-45/0$_2$]s | [45/0$_2$/-45]s | 1 | **12.7** | 1.791005 | 75.6 | **62.7** | **32.4** |
| same | [45/-45/45/-45]s | 2 | **31.75** | 1.791005 | 18.2 | 17.9 | 17.9 |
| same | [45/-45/0$_2$/45/-45]s | 3 | **38.1** | 2.686507 | 56.5 | 47.9 | 47.9 |

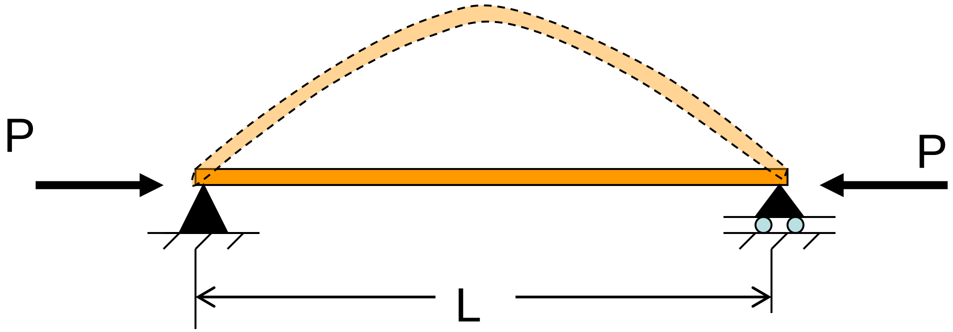
| | Alum | Comp | Δ(%) |
|---|---|---|---|
| EA (kN) | 8525 | 8525 | 0.0 |
| EI (Nm^2) | 1401 | 1441 | -2.8 |

$$\frac{W_{Gr/Ep}}{W_{Al}} = 0.58\frac{1.469}{1} = 0.852$$

density ratio for carbon/epoxy

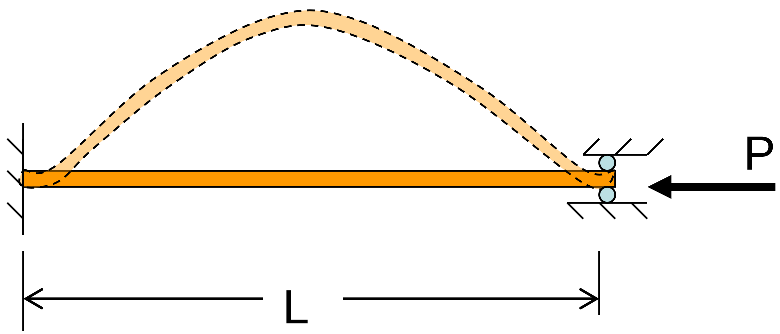
composite matches Al properties and is 15% lighter!

# Beams - Column buckling




both ends pinned

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$



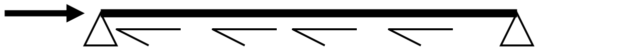
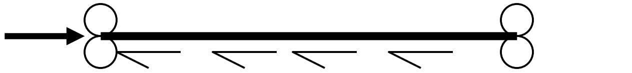
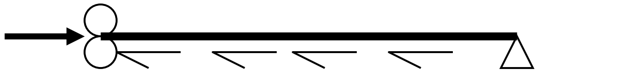
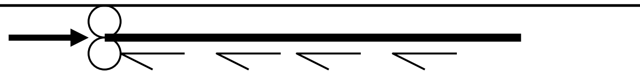

both ends fixed

$$P_{cr} = \frac{4\pi^2 EI}{L^2}$$

EI calculated as before but with $E_{membrane}$ for buckling calculation

# Column buckling – Effect of BCs and loading

$$P_{cr} = \frac{c\pi^2 EI}{L^2}$$

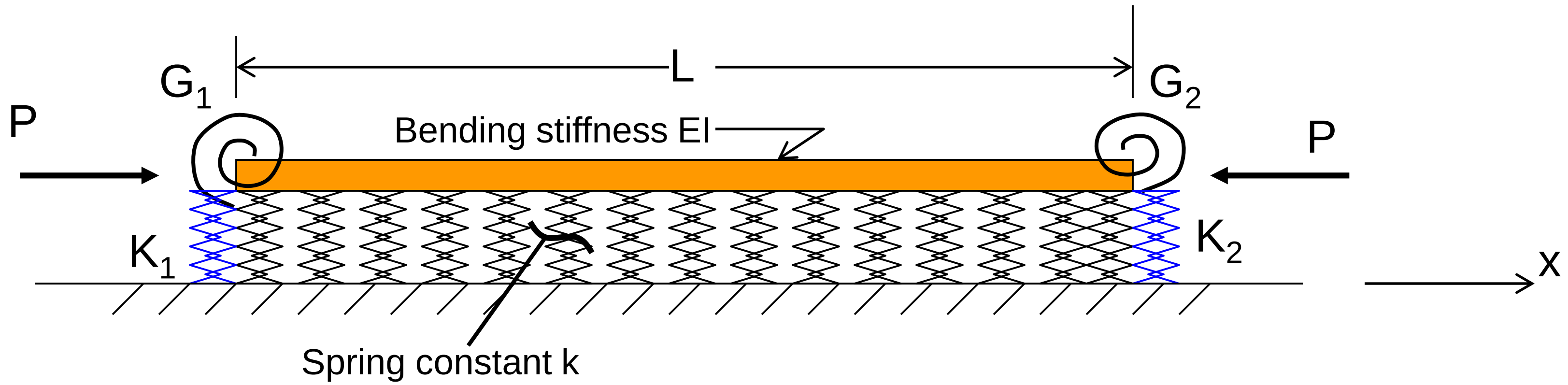
| Configuration | BC at left,right end | c |
|---|---|---|
|  | pinned, pinned | 1.88 |
|  | fixed, fixed | 7.56 |
|  | fixed, pinned | 2.05 |
|  | fixed, pinned | 5.32 |
|  | fixed, free | 0.25 |
|  | fixed, free | 0.80 |

free: free rotation and free translation

pinned: free rotation, fixed translation

fixed: fixed rotation, fixed translation

# Beam on elastic foundation under compression




- Foundation with spring constant k (appropriate units)
- Ends with different translational and rotational spring constants (BC's)

Governing Equation

$$EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} + kw = 0$$

# Beam on elastic foundation under compression

- can solve ODE which gives solutions of the form

$$w = Ae^{px}$$

$$p = \pm\sqrt{\frac{-\frac{P}{EI} \pm \sqrt{\left(\frac{P}{EI}\right)^2 - \frac{4k}{EI}}}{2}}$$

Combinations of sines, cosines, and exponentials depending on the magnitudes of P and k

satisfying the BC's leads to a 4x4 eigenvalue problem in the buckling load P

# Beam on elastic foundation under compression (pinned ends)

- or, which is faster, can use an energy approach

- The total energy in the beam is given by

$$\Pi_c = \underbrace{\frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx}_{\text{bending potential energy}} + \underbrace{\frac{1}{2}\int_0^L (-P)\left(\frac{dw}{dx}\right)^2 dx}_{\text{external work}} + \underbrace{\frac{1}{2}\int_0^L kw^2 dx}_{\text{spring potential energy}}$$

- Assume deflection w in the form

$$w = \sum A_m \sin\frac{m\pi x}{L}$$

Satisfies the BC's

$A_m$ are unknown coefficients

also happens to be the exact solution

# Beam on elastic foundation under compression (pinned ends)

- Substitute in the energy expression and note the following:

$$\left[\sum A_m \sin\frac{m\pi x}{L}\right]^2 = \sum\sum A_m A_n \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right) \quad \alpha\neq\beta$$

$$= \frac{1}{2}\left(1-\cos 2\alpha\right) \quad \alpha=\beta$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right) \quad \alpha\neq\beta$$

$$= \frac{1}{2}\left(1+\cos 2\alpha\right) \quad \alpha=\beta$$

# Beam on elastic foundation under compression (pinned ends)

- Energy expression becomes:

$$\Pi_c = \sum \left[ \frac{(EI)m^4\pi^4}{4L^3} - \frac{Pm^2\pi^2}{4L} + \frac{kL}{4} \right] A_m{}^2$$

- Minimizing with respect to $A_m$:

$$\frac{\partial \Pi_c}{\partial A_m} = 0$$

- leads to:

$$2\left[ \frac{(EI)m^4\pi^4}{4L^3} - \frac{Pm^2\pi^2}{4L} + \frac{kL}{4} \right] A_m = 0$$

# Beam on elastic foundation under compression (pinned ends)

- which after rearranging reads

$$\underbrace{\left[\frac{\pi^2 EI}{L^2}\left(m^2+\frac{kL^4}{\pi^4 (EI) m^2}\right)-P\right]}_{\text{diagonal matrix}}\underbrace{A_m}_{\text{column vector}}=0$$

- either $A_m$=0 (trivial solution, no bending)
- or the determinant of the diagonal matrix = 0
- set

$$K_{mm}=\frac{\pi^2 EI}{L^2}\left(m^2+\frac{kL^4}{\pi^4 (EI) m^2}\right)$$

# Solution (buckling load of beam on elastic foundation)

$$\begin{bmatrix} K_{11}-P & 0 & 0 & 0 & 0 \\ 0 & K_{22}-P & 0 & 0 & 0 \\ 0 & 0 & K_{33}-P & 0 & 0 \\ 0 & 0 & 0 & \dots & \\ 0 & 0 & \dots & \dots & \dots \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \dots \\ \dots \\ \dots \end{Bmatrix} = 0$$

$$\Rightarrow \quad (K_{11}-P)(K_{22}-P)(K_{33}-P)\dots = 0$$

$$\Rightarrow \quad \boxed{P_{cr} = \min(K_{ii})}$$

# Special case: k=0

- pinned beam under compression

$$P_{cr} = K_{mm} = \frac{\pi^2 EI}{L^2}\left( m^2 + \cancel{\frac{kL^4}{\pi^4 (EI) m^2}} \right)$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} m^2$$

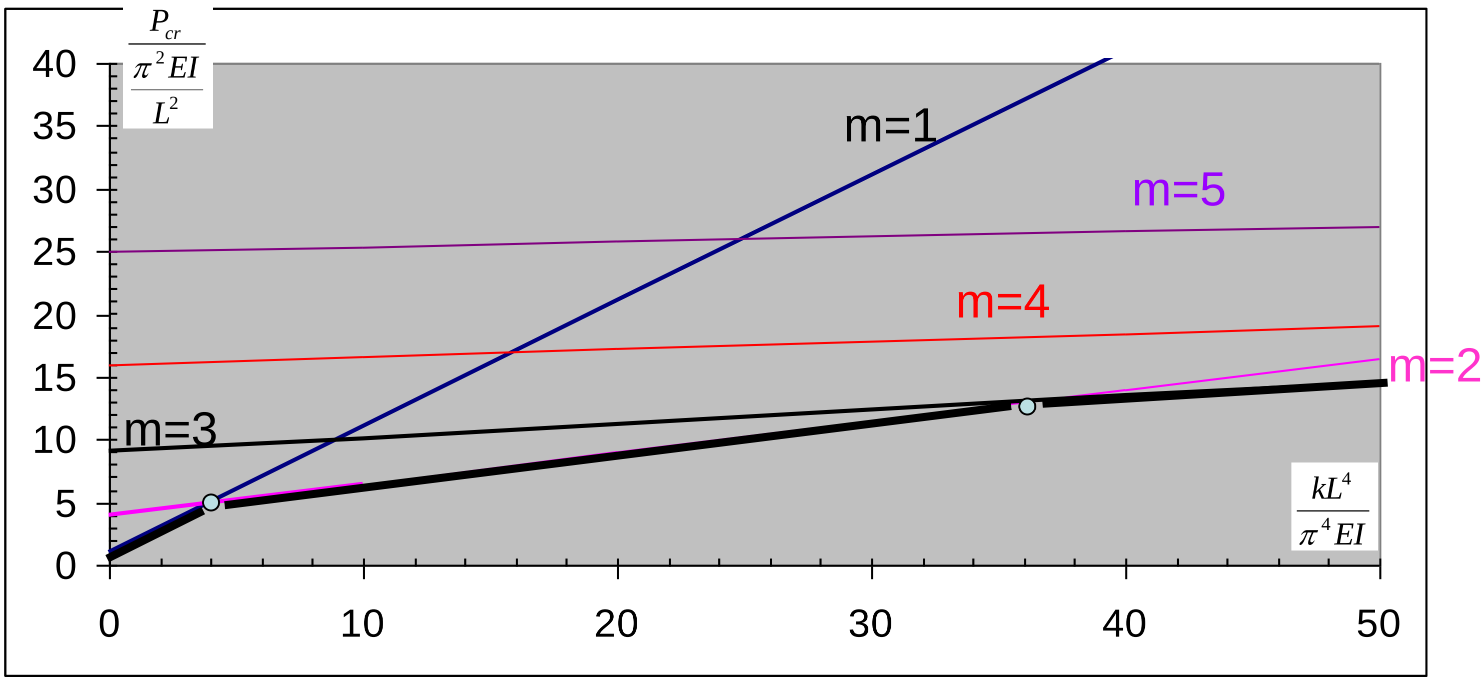
- minimized for m=1 => exact solution for buckling load

# Beam on elastic foundation under compression (pinned ends)

• the dependence of the buckling load on the spring constant k can be seen more easily in a graph.

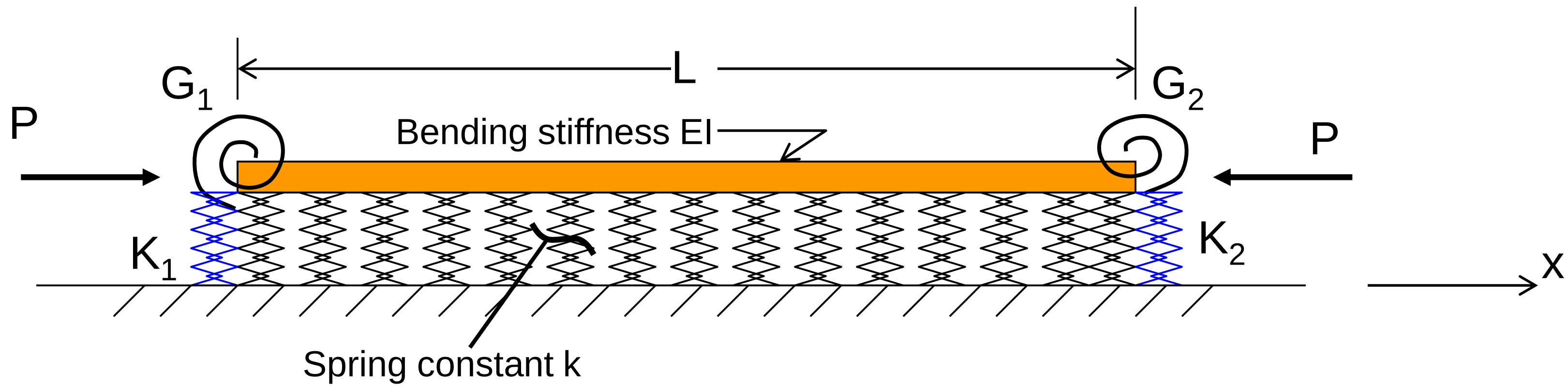
Rearranging:

$$\frac{P_{cr}}{\dfrac{\pi^2 EI}{L^2}} = m^2 + \frac{kL^4}{\pi^4 EI}\frac{1}{m^2}$$



# Beam on elastic foundation: other BC's

- ref: Aristizabal-Ochoa, J.D. "Classical stability of beam columns with semi-rigid connections on elastic foundation" American Soc. Civil Engineers, 16th Engr Mechanics Conf., 2003, paper 67




- general BC's at the two ends (moment and shear force balance):

$$\left.\begin{aligned} -EI\frac{d^2w}{dx^2}+G_1\frac{dw}{dx}=0 \\ EI\frac{d^3w}{dx^3}+P\frac{dw}{dx}+K_1w=0 \end{aligned}\right\} \quad \text{at } x=0$$

$$\left.\begin{aligned} -EI\frac{d^2w}{dx^2}+G_2\frac{dw}{dx}=0 \\ EI\frac{d^3w}{dx^3}+P\frac{dw}{dx}+K_2w=0 \end{aligned}\right\} \quad \text{at } x=L$$

# Beam on elastic foundation: Other BC's

- Define:

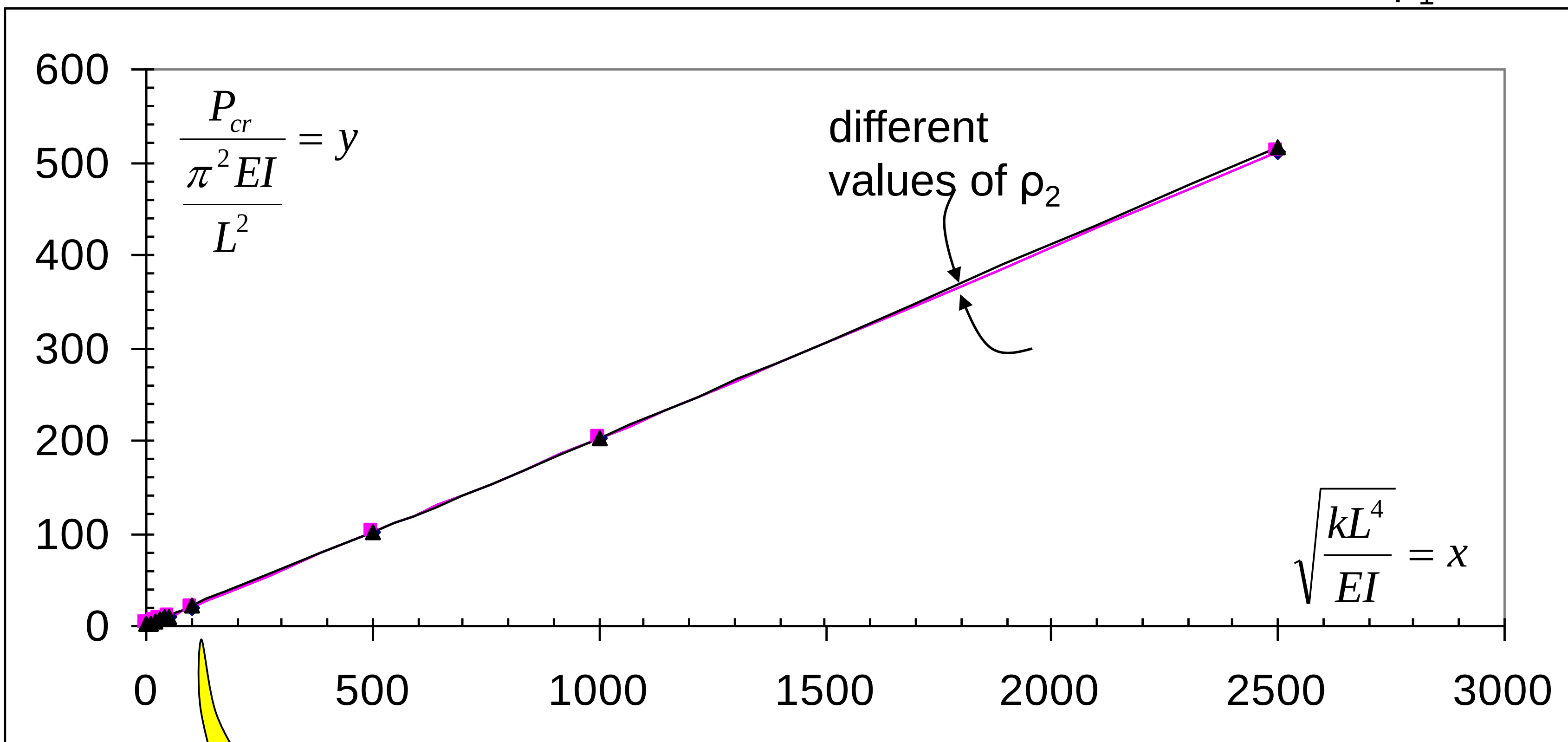
$$R_i = \frac{G_i L}{EI}$$

$$\rho_i = \frac{1}{1+\dfrac{3}{R_i}} \quad \begin{cases} \rho=1 \Rightarrow \text{fixed} \\ \rho=0 \Rightarrow \text{pinned} \end{cases}$$

- w=0 at both ends; dw/dx specified at one end ($0 \le \rho_2 \le 1$) and pinned at the other ($\rho_1=0$)

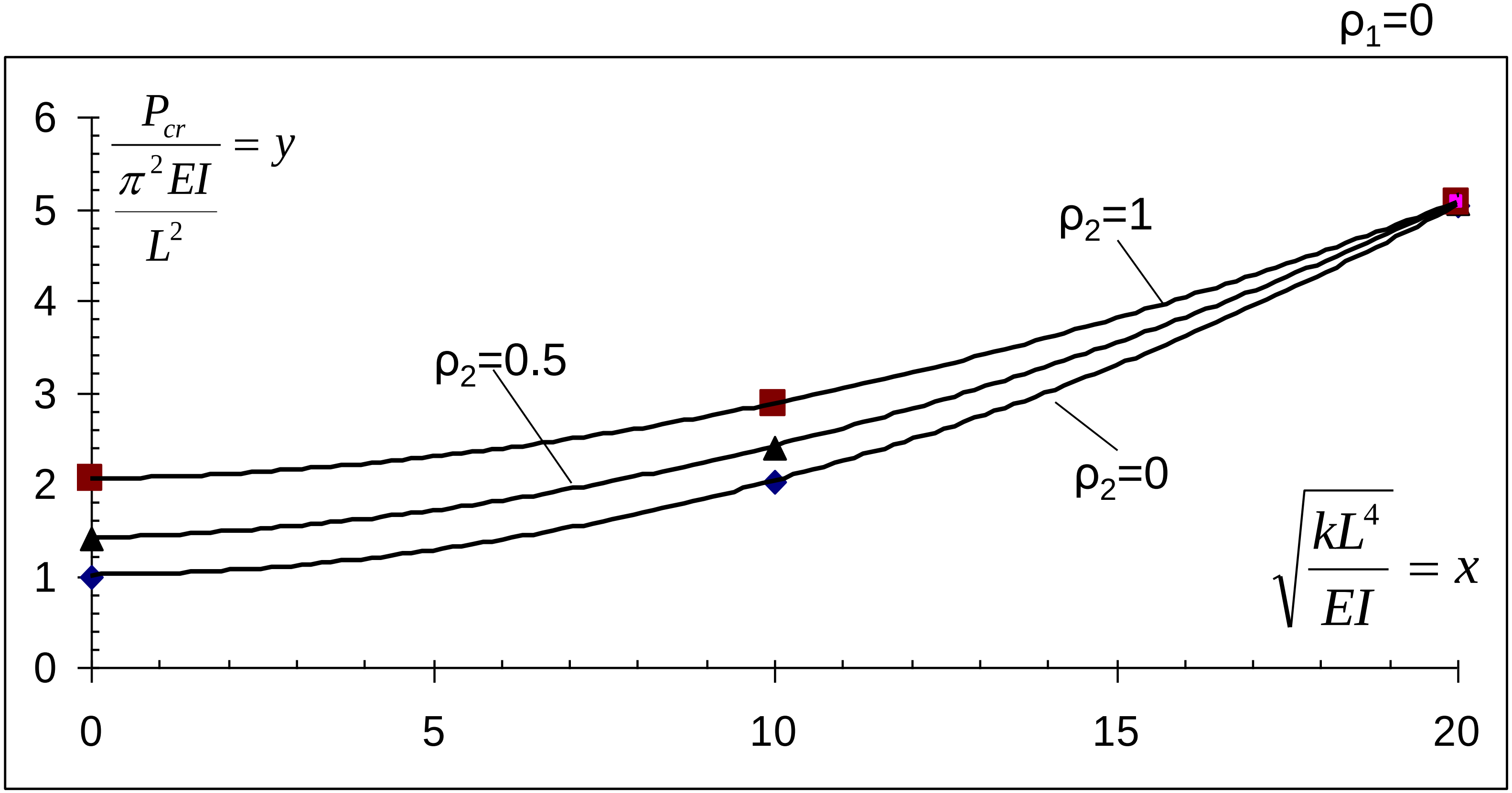
# Beam on elastic foundation: other BC's

$\rho_1$=0




see next Figure

# Beam on elastic foundation: other BC's



# Beam on elastic foundation: Other BC's

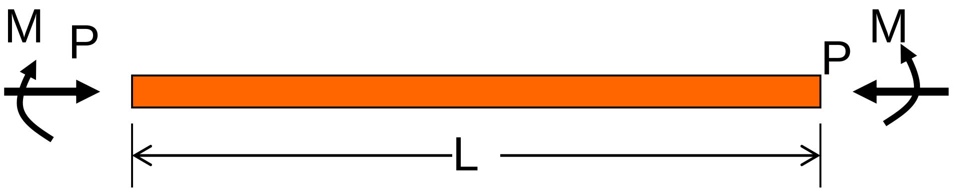
$\rho_1=0$

| $\rho_2$ | x | y | $r^2$ (goodness of fit) |
|---|---|---|---|
| 0 | 0≤x≤20 | $0.0099x^2+0.0041x+1$ | 1.0000 |
| 0 | 20<x≤100 | $0.0004x^2+0.1517x+1.6658$ | 0.9987 |
| 0.5 | 0≤x≤20 | $0.0084x^2+0.0169x+1.4069$ | 1.0000 |
| 0.5 | 20<x≤100 | $0.0002x^2+0.1714x+1.4518$ | 0.9988 |
| 1 | 0≤x≤20 | $0.0069x^2+0.01134x+2.046$ | 1.0000 |
| 1 | 20<x≤100 | $7x10^{-5}x^2+0.1924x+1.2722$ | 0.9998 |

# Beam on elastic foundation: Other BC's

| $\rho_1 = \rho_2$ | x | y | $r^2$ (goodness of fit) |
|---|---|---|---|
| 0.2 | $0 \le x \le 20$ | $0.0099x^2 + 0.0039x + 1.28$ | 1.0000 |
| 0.2 | $20 < x \le 100$ | $0.0004x^2 + 0.1517x + 1.9512$ | 0.9987 |
| 0.5 | $0 \le x \le 20$ | $0.0099x^2 + 0.0019x + 1.916$ | 1.0000 |
| 0.5 | $20 < x \le 100$ | $0.0003x^2 + 0.1539x + 2.5361$ | 0.999 |
| 1 | $0 \le x \le 20$ | $0.0051x^2 + 0.0265x + 4$ | 1.0000 |
| 1 | $20 < x \le 100$ | $-0.0003x^2 + 0.2385x + 2.3368$ | 0.9943 |

# Beams under combined compression and bending




Governing equation (same as for beam on elastic foundation with k=0):

$$EI\frac{d^4w}{dx^4}+P\frac{d^2w}{dx^2}=0$$

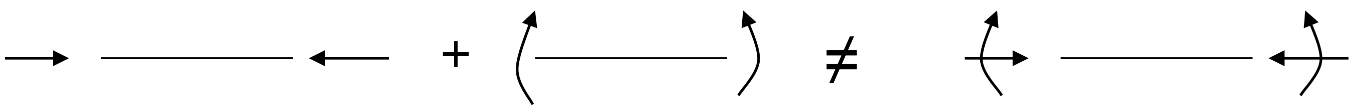
Solution

$$w=C_o+C_1\sin\left(\sqrt{\frac{P}{EI}}x\right)+C_2\cos\left(\sqrt{\frac{P}{EI}}x\right)+C_3x$$

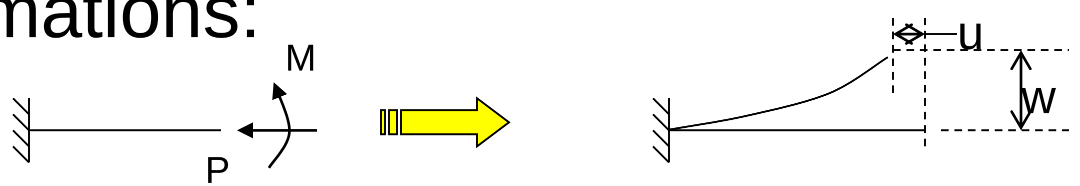
$C_o$, $C_1$, $C_2$, and $C_3$ determined from BC's

# Beams under combined compression and bending

- “Traditional” superposition is NOT applicable



- nor can one separate bending from axial deformations:




cannot calculate u from P and w from M!

- **why is the problem non-linear?**

# Beams under combined compression and bending

- special type of “superposition” is applicable:
  - since the source of non-linearity is in the axial load, if the axial load is kept equal to P in each constituent problem, superposition is valid

Example: