LATERALLY LOADED PILE

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1 Introduction

In this note an approximate analytic solution is presented for the problem of a laterally loaded pile in a homogeneous sandy soil, taking into account plastic deformations of the soil. The purpose of the analysis is to evaluate the stiffness of a single pile to a lateral load applied at a certain level above the soil surface.

A fairly realistic assumption for the response of the soil to a lateral displacement is to consider this to be elasto-plastic, see Figure 1. In this figure the response is assumed to be

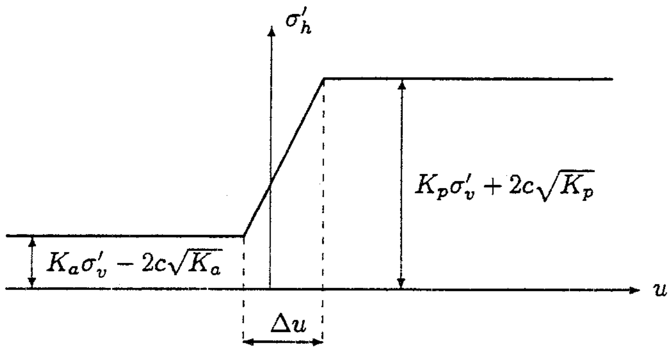

Figure 1: Elasto-plastic soil response.

elastic if the displacement is small. It reaches its maximum value (the *passive* lateral soil pressure),

$$\sigma'_{h-\max} = K_p\sigma'_v + 2c\sqrt{K_p}, \tag{1}$$

when the displacement reaches a certain positive value, and it reaches its minimum value (the *active* lateral soil pressure),

$$\sigma'_{h-\min} = K_a\sigma'_v - 2c\sqrt{K_a}, \tag{2}$$

when the displacement reaches a certain negative value. If the value obtained from eq. (2) is found to be negative the horizontal effective stress is zero, because tensile stresses cannot be transferred in a granular material. In the equations given above c is the cohesion of the soil, and K_a and K_p are the active and passive pressure coefficients,

$$K_a = \frac{1-\sin\phi}{1+\sin\phi}, \tag{3}$$

$$K_p = \frac{1+\sin\phi}{1-\sin\phi}, \tag{4}$$

where ϕ is the friction angle of the soil. The length of the range of elastic displacements (Δu in figure 1) is called the *stroke*. This is a similar quantity as the quake, used in the analysis of axially loaded piles. It represents the displacement difference between generating active and passive soil pressures, respectively.

The response shown in figure 1 can be used to develop a numerical model. In the present note a simplified analysis will be used, originally developed by Blum (1931) for sheet pile walls. In this analysis it is assumed that the elastic range is extremely small, see figure 2. This is called a perfectly plastic response. As soon as there is a displacement the response is

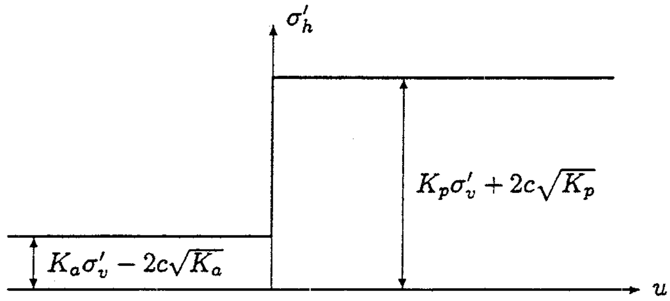

Figure 2: Perfectly plastic soil response.

either at its maximum or its minimum. This means that the stiffness of the soil is assumed to be very large. It can be expected that the displacements will be somewhat underestimated.

If the horizontal force on the pile acts towards the right the soil response on the right side of the pile will be the passive earth pressure, and the soil response on the left side of the pile will be active soil pressure. Thus the total response is

$$f = -(K_p - K_a)D\sigma'_v - 2cD(\sqrt{K_p} + \sqrt{K_a}\,), \tag{5}$$

or, assuming that the vertical effective stress increases linearly with depth,

$$f = -(K_p - K_a)D\gamma' z - 2cD(\sqrt{K_p} + \sqrt{K_a}\,), \tag{6}$$

where γ' is the submerged unit weight of the soil, usually $\gamma' \approx 10$ kN/m^2.

It is assumed that the distribution of the soil reaction is as shown in figure 3. The basic idea is that the pile is displaced towards the right by the applied force, except at the lower end, where a displacement towards the left occurs. The soil reaction on this deeper part of the pile, is replaced by a concentrated force (Blum's *Ersatzkraft*), further assuming that the pile is clamped at that depth. In the problem considered here the length of the pile above the soil surface is l, and the pile is loaded at its top by a horizontal force of magnitude P.

Compared to the more realistic soil response shown in Figure 1 the Blum model is stiffer. It underestimates the displacements. More refined computations can be made by a numerical model based on an elasto-plastic representation of the soil. This is recommended for a final design.

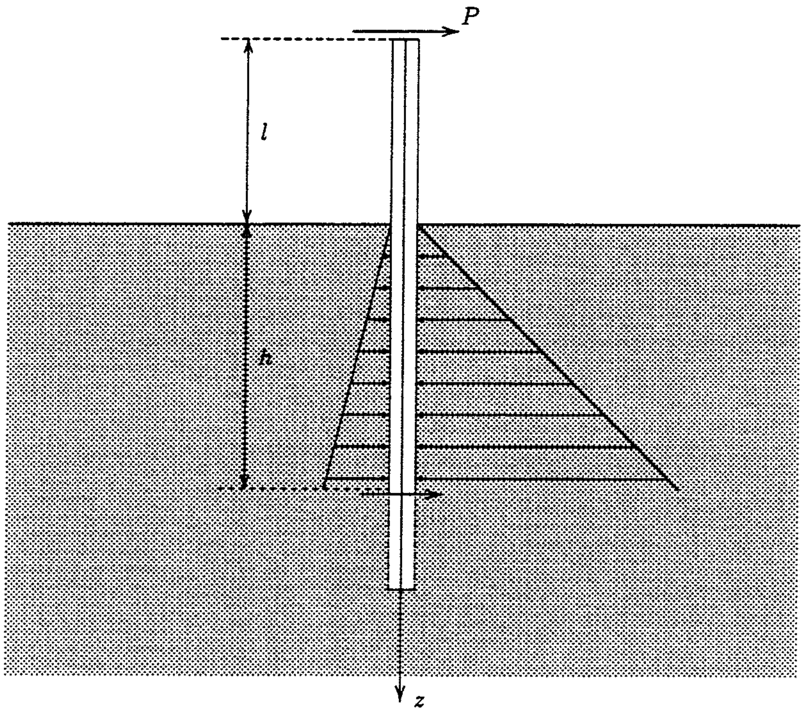

Figure 3: Blum's schematization.

1.1 Solution of the basic problem

The differential equation is, assuming that the soil is cohesionless sand ($c = 0$),

$$EI\frac{d^4u}{dz^4} = f = -(K_p - K_a)D\gamma' z. \tag{7}$$

It is usually assumed that the three-dimensional effects on the lateral stress can be taken into account by taking the width D somewhat larger than the actual width of the pile. An alternative method, which will be used here, is to maintain D as the actual diameter of the pile, and to take the value of the soil parameter $K_p - K_a$ somewhat larger (say by a factor 2).

The boundary conditions are

$$z = 0: \quad Q = -EI\frac{d^3u}{dz^3} = -P \tag{8}$$

$$z = 0: \quad M = -EI\frac{d^2u}{dz^2} = -Pl, \tag{9}$$

$$z = h: \quad u = 0, \tag{10}$$

$$z = h: \quad \frac{du}{dz} = 0, \tag{11}$$

$$z = h: \quad \frac{d^2u}{dz^2} = 0. \tag{12}$$

The value of the parameter h, the depth below the sea bottom at which the pile is considered to be clamped, is unknown. The actual length of the pile should be considerably larger than the value of h, in order that the pile is fully clamped at the depth h. The additional unknown parameter h in the problem is balanced by the fact that there are five boundary conditions for the bending problem, rather than the usual four. A possible line of reasoning leading to the last three conditions is that the pile is very well clamped at great depths, so that the displacement u and its first two derivatives are zero there. The weak point in this argument, at least from the mathematical side, is that the assumption that the first two derivatives of the displacement are zero, but not the third order derivative (which cannot be zero, because of the shear force acting there), and this seems to be somewhat arbitrary. It may be noted, however, that the assumed soil reaction cannot be in equilibrium with the applied load without the concentrated force, because equilibrium of moments is impossible without that force. The analysis using the assumptions for the pressure distribution as shown in figure 3 is an excellent example of the art of engineering, especially since it has been shown that the results are in good agreement with those of more refined models (Verruijt, 1995).

The general solution of the differential equation (7) is

$$u = -\frac{\sigma_0 z^5}{120\,EI} + C_1 z^3 + C_2 z^2 + C_3 z + C_4, \tag{13}$$

where σ_0 is a reference stress,

$$\sigma_0 = (K_p - K_a)\gamma' D. \tag{14}$$

The integration constants can be determined from the boundary conditions. The first boundary condition gives

$$C_1 = \frac{P}{6EI}, \tag{15}$$

and the second boundary gives

$$C_2 = \frac{Pl}{2EI}. \tag{16}$$

It now follows from the fifth boundary condition that

$$P = \frac{\sigma_0 h^2}{6\,(1 + l/h)}. \tag{17}$$

This equation gives a relation between P and h. If the load P is given the value of h can be determined from this equation. It can of course also be written as

$$\sigma_0 h^2 = 6\,P(1 + l/h). \tag{18}$$

It now follows from the first and second boundary conditions that

$$C_3 = \frac{-Ph(h + 3l)}{4EI}, \tag{19}$$

$$C_4 = \frac{Ph^2(h + 3l)}{4EI}. \quad \frac{Ph^2(4h+9l)}{30\,EI} \quad \text{See, eq. (21)} \tag{20}$$

This completes the solution.

The most important results are the lateral displacement and the rotation at the top of the pile. These are found to be

$$z = 0: \quad u = u_0 = \frac{Ph^2(4h + 9l)}{30\,EI}, \tag{21}$$

$$z = \overset{0?}{\cancel{h}} : \quad \phi = \phi_0 = -\frac{du}{dz} = \frac{Ph(h+3l)}{4\,EI}. \tag{22}$$

The displacement of the top of the pile (for $z = -l$) consists of three terms,

$$z = -l : \quad u = u_t = u_1 + u_2 + u_3 = u_0 + \phi_0 l + \frac{Pl^3}{3\,EI}, \tag{23}$$

where the last term represents the contribution of the bending of the pile above the sea bottom.

2 Design considerations

For a preliminary design let it be considered that the bending moment at the level of the sea bottom is characteristic. The local bending stress is

$$\sigma_b = \frac{MD}{2I} \tag{24}$$

Let the force P be expressed as

$$P = \gamma_w w^2 D, \tag{25}$$

The moment of inertia is $I = \pi D^3 d/8 = AD^2/8$, where A is the area of the cross section, $A = \pi Dd$. This now gives

$$A = \frac{4\gamma_w w^2 l}{\sigma_b}. \tag{26}$$

Assume that $\gamma_w = 10$ kN/m^3, $w = 7$ m, $l = 20$ m and $\sigma_b = 160 \times 10^3$ kN/m^2. Then the area should be about $A = 0.25$ m^2. A possible combination is that the diameter is $D = 2$ m and that the wall thickness is $d = 0.04$ m. In that case the bending stiffness is $EI = 25 \times 10^6$ kNm^2.

The value of h can be determined from (17),

$$\gamma_w w^2 D = (K_p - K_a)\gamma' D \frac{h^2}{6(1 + l/h)}.$$

or

$$h^2 = \frac{6w^2}{K_p - K_a}(1 + \frac{l}{h}).$$

Assuming that $K_p - K_a = 6$ (taking into account the threedimensional effect), this gives

$$h^2 = w^2(1 + \frac{l}{h}).$$

This leads to a value of about $h = 12$ m.

The displacements of the top of the pile are now as follows. The first term, the displacement at the sea bottom is

$$u_1 = u_0 = \frac{Ph^2(4h + 9l)}{30\,EI} = 0.044 \text{ m},$$

The second term, the displacement due to the rotation at the sea bottom, is

$$u_2 = \phi_0 l = \frac{Ph(4h + 3l)l}{4\,EI} = 0.172 \text{ m},$$

compare eq (22)

The third term, due to the bending of the free standing part of the pile above the sea bottom is

$$u_3 = \frac{Pl^3}{3\,EI} = 0.080 \text{ m}.$$

Thus the total displacement is about 0.3 m, which is probably inadmissible. The conclusion must be that the static strength of the pile is not the determining factor, but rather the stiffness.

3 Parameter study

In order to study the influence of the water depth on the foundation a computer program has been written that calculates the length of the pile in the soil as a function of the allowable displacement. The results are shown in table 1. The quantity L is the length of the pile below water level, including the depth of the pile in the soil (h), and 30 % extra length to ensure the fixation of the pile. The quantity V is the volume of the pile below water level. It may be noted that in all these calculations the actual wind turbine has been disregarded.

l	h	D	d	L	V
5	8.771	2.250	0.038	16.402	3.651
10	9.920	2.608	0.043	22.895	7.093
15	10.815	2.951	0.049	29.059	11.770
20	11.565	3.272	0.055	35.034	17.696
25	12.217	3.578	0.060	40.883	24.941
30	12.800	3.869	0.064	46.640	33.537
35	13.329	4.148	0.069	52.328	43.532
40	13.816	4.415	0.074	57.960	54.937

Table 1: Pile dimensions.

In these calculations the following assumptions have been made.

- The lateral force is generated at the water level, and is represented by equation 25, with $w = 7$ m. This means that for a depth of 20 m and a pile of 3 m diameter the force is about 1400 kN.
- The ratio of diameter to wall thickness has been assumed to be $D/d = 60$.
- The maximum displacement has been assumed to be 1/200 of the water depth.

4 Conclusions

- The calculations confirm that for water depths of 20 m or more (*water levels during storm conditions*), the dimensions of the pile become very large.

- The analysis also confirms that the stiffness of the pile-soil system is not very large. Stiffness is the determining factor for the design, not the strength of the pile material.

- The calculations indicate that the apparent depth of the point of fixation of the pile is located at a depth of about 12 meter below the soil surface. For a pile having a diameter of 3 meter this means that $h \approx 4D$.

- It should be noted that the location of the point of application of the force above the soil surface (indicated by l) is considerably larger than the average low water level used in the maps and the reports of the Near Shore Windpark study. The value of l is the height of the wave force acting during a storm, when the water level is considerably higher (say 5 meter) than average low water. Thus a value $l = 20$ m corresponds to a low water depth of about 15 meter.

- A possible lowering of the soil surface by scour has been neglected. This would have a negative influence on the stiffness and the required pile length, but may be prevented by a gravel pack.

- The influence of the wind force on the turbine, at a considerable height above the water level, has also been neglected. This means that the actual deformations will be considerably greater than the ones calculated here. This may be balanced by the perhaps rather severe criterion (1/200) used in the present analysis.

- For an actual design the lateral stiffness of the pile-soil system should be determined by a numerical analysis, using one of the many available software packages.