From: Léon Harland, Jan Vugts (WOT)
To: Tim Cockerill (US), Murray Ferguson (KEW), Martin Kühn and Wim Bierbooms (IvW), Anita Sandström (KT)
cc:
Our reference: --
Date: 18 October 1996

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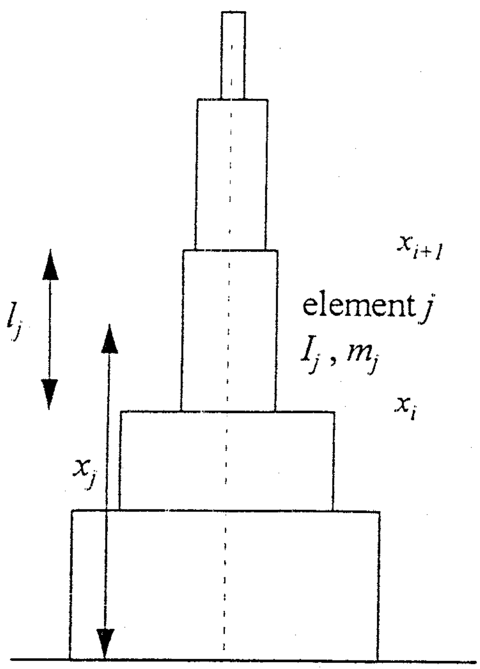
Project: OPTI-OWECS
Subject: Analytic expression for the first natural period of a stepped tower.
Confidential mark:
Work Package: 2.1.2 Cost model
Action point:
Your reference:

Lady and Gentlemen,

In this short note an expression is given for the natural period of a stepped tower. This expression will be used in the cost model of Tim. The expression is determined from two influences which are assumed to be uncorrelated. The first is the bending of the tower. Second is the foundation influence which is assumed to be of a small to moderate influence compared to bending. The derivation is given below. Those who belief us right away: the expression is given in equation (10) on page 5. The equation is equally valid for a stepped monotower and for a lattice tower structure; only the determination of I_j and m_j is different.

Regards,
Jan Vugts and Léon Harland



Bending influence

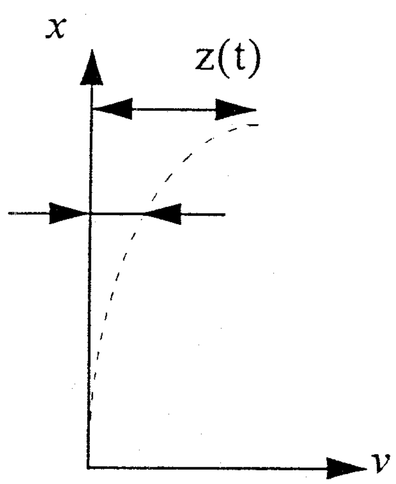
Basis of the derivation is a stepped tower with n segments, e.g. n=5 in the figure. It is assumed that the length l_j (m), the moment of inertia I_j (m^4) and the distributed mass m_j (kg/m) are known for each segment.

The basic equation (1) on the next page is obtained from Rayleigh's method using the principle of equating maximum potential and maximum kinetic energy in the vibration mode. For further detail see 'Dynamics of structures' by Clough and Penzien.

The numerator in (1) represents the generalised stiffness and the denominator the generalised mass.

$$\omega^2 = \frac{\int_0^L EI(x)\left[\Psi''(x)\right]^2 dx}{\int_0^L m(x)\left[\Psi(x)\right]^2 dx} \tag{1}$$

$\Psi(x)$ = mode shape



The displacement at any point and time:

$v(x,t) = \Psi(x)z(t)$

$z(t) = Z_0 sin(\omega t)$ = top displacement used as a normalisating factor for the mode shape.

The mode shape is to be assumed, as an approximation to the true mode shape which is, of course, unknown. However, the method is not very sensitive to the assumption made. The assumed mode shape must in any event satisfy the boundary conditions, at least geometrically. $\psi(x)$ is non-dimensional and found from:

$$\Psi(x) = \frac{v(x,t)}{z(t)}$$

The exact mode shape for the vibration in bending of a uniform beam is taken as an approximation (2):

$$\Psi(x) = \left(1 - \cos\left(\frac{\pi x}{2L}\right)\right) \tag{2}$$

Over each segment of the stepped beam *m(x)* and *I(x)* are constant, whereas $\psi(x)$ and $\psi''(x)$ will be approximated at mid length and taken as constant. ($\psi''(x)$ is trivial to determine). Substitution into equation (1) gives:

$$\omega^2 = \frac{\pi^4}{16L^4} \cdot E \cdot \frac{\sum_{j=1}^{n} I_j l_j \cos^2\left(\frac{\pi x_j}{2L}\right)}{\sum_{j=1}^{n} m_j l_j \left(1 - \cos\left(\frac{\pi x_j}{2L}\right)\right)^2} \tag{3}$$

Next define an equivalent moment of inertia and mass per unit length of the stepped beam.

$$I_{eq} = \frac{\sum_{j=1}^{n} I_j l_j \cos^2\left(\frac{\pi x_j}{2L}\right)}{L}$$

$$m_{eq} = \frac{\sum_{j=1}^{n} m_j l_j \left(1 - \cos\left(\frac{\pi x_j}{2L}\right)\right)^2}{L}$$

There are the numerator and denominator, respectively, of equation (3), each divided by the full length of the tower to ensure that I_{eq} and m_{eq} continue to have the correct units of a moment of inertia and a mass per unit length.

To include a top mass M_{top} is a straightforward addition to the equivalent mass in the denominator and the equation (3) hence becomes:

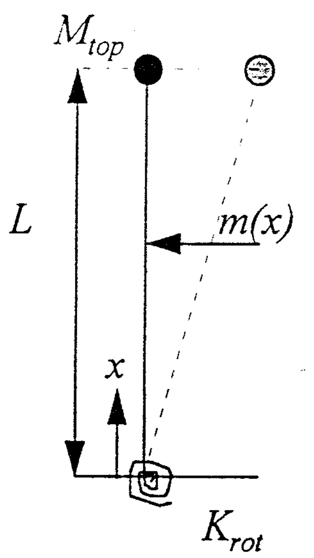
$$\omega^2 = \frac{\pi^4}{16L^4} \cdot \frac{EI_{eq}L}{\left(M_{top} + m_{eq}L\right)} = \frac{\pi^4}{16} \cdot \frac{EI_{eq}}{\left(M_{top} + m_{eq}L\right)L^3} \qquad (4)$$

Obviously equation (4) has been checked via some calculations. For a uniform beam a division in 5 elements gave a frequency which is 5.4% higher than the exact result, while 10 elements results in a 4.5% higher frequency. For an engineering approximation this is in our view good enough.

Foundation influence

We want to bring the final equation including foundation influence in the same form as the equation presented earlier in our contribution on natural periods.

Rigid body rotation of a tapered beam with a topp mass



$$\omega_{rot}^2 = \frac{K_{rot}}{I}$$

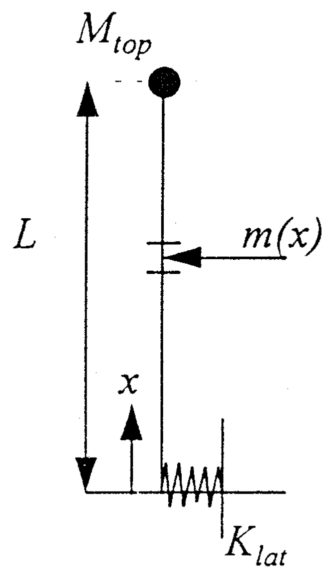
$$I = \int_0^x m(x) \cdot x^2 \cdot dx + M_{top}L^2$$

with: $I_{rot} = m_{rot} \cdot L^2 = \int_0^x m(x) \cdot x^2 \cdot dx$

gives: $I = \left(M_{top} + m_{rot}\right) \cdot L^2$

$$\omega_{rot}^2 = \frac{K_{rot}}{\left(M_{top} + m_{rot}\right) \cdot L^2} \qquad (5)$$

Translation of a tapered beam with top mass



$$\omega_{lat}^2 = \frac{K_{lat}}{mass}$$

$$mass = \int_0^x m(x)dx + M_{top}$$

$$= M_{top} + m_{lat}$$

$$\omega_{lat}^2 = \frac{K_{lat}}{\left(M_{top} + m_{lat}\right)} \tag{6}$$

Composite equation (ignoring shear effect)

For the total system subjected to the three springs in series we then have:

$$\omega_t^2 = \frac{\text{total generalised stiffness of complete system}}{\text{total generalised mass of complete system}}$$

As we are more interested in natural periods (note: $\omega=2\pi/T$), this can be easily transformed (with bending as the governing influence):

$$T_t^2 = 4\pi^2 \cdot \frac{\text{total generalised mass of complete system}}{\text{total generalised stiffness of complete system}}$$

$$\cong 4\pi^2 \cdot \frac{\text{equivalent mass for bending}}{\text{sum of stiffnesses for springs in series}}$$

$$= 4\pi^2 \cdot \left(M_{top} + m_{eq} L\right) \cdot \left[\frac{L^3}{\frac{\pi^4}{16} EI_{eq}} + \frac{L^2}{K_{rot}} + \frac{1}{K_{lat}}\right] \tag{7}$$

$$T_t^2 = \frac{4\pi^2\left(M_{top} + m_{eq} L\right) L^3}{3EI_{eq}} \cdot \left[\frac{48}{\pi^4} + \frac{3EI_{eq}}{K_{rot} L} + \frac{3EI_{eq}}{K_{lat} L^3}\right] \tag{8}$$

Next the rotational and translational influences of the foundation can be combined:

$$T_t^2 = \frac{4\pi^2 \left(M_{top} + m_{eq} L\right) L^3}{3EI_{eq}} \cdot \left[\frac{48}{\pi^4} + \frac{3EI_{eq}}{K_{eq} L}\right] \tag{9}$$

where: $$K_{eq} = \frac{K_{rot} K_{lat} L^2}{K_{rot} + K_{lat} L^2}$$

So the **final** expression:

$$T_t^2 = \frac{4\pi^2 \left(M_{top} + m_{eq} L\right) L^3}{3EI_{eq}} \cdot \left[\frac{48}{\pi^4} + C_{found}\right] \tag{10}$$

with:

$$I_{eq} = \frac{\sum_{j=1}^{n} I_j l_j \cos^2\left(\frac{\pi x_j}{2L}\right)}{L}$$

$$m_{eq} = \frac{\sum_{j=1}^{n} m_j l_j \left(1 - \cos\left(\frac{\pi x_j}{2L}\right)\right)^2}{L}$$

$$C_{found} = \frac{3EI_{eq}}{K_{eq} L}$$

$$K_{eq} = \frac{K_{rot} K_{lat} L^2}{K_{rot} + K_{lat} L^2}$$

Final remark: C_{found} is a factor reflecting the influence of the flexibility of the foundation. The value of C_{found} may vary between 0 for very stiff foundation behaviour and, say, 0.5 for reasonably flexible foundation behaviour.

LAH - JHV 18-10-96