1.3

Ideal Rocket Theory (part 1)

Why an “ideal” rocket theory?

- Our objective is to find **simplified equations** for :
- The **jet velocity** v_e
- The **mass flow rate** $\dot{m}$
- The **exit pressure** p_e
- This can be achieved with the **Ideal Flow Theory** , consisting of :
- A simplified **rocket geometry**
- A set of **physical assumptions**

Ideal rocket geometry

Nozzle Exit

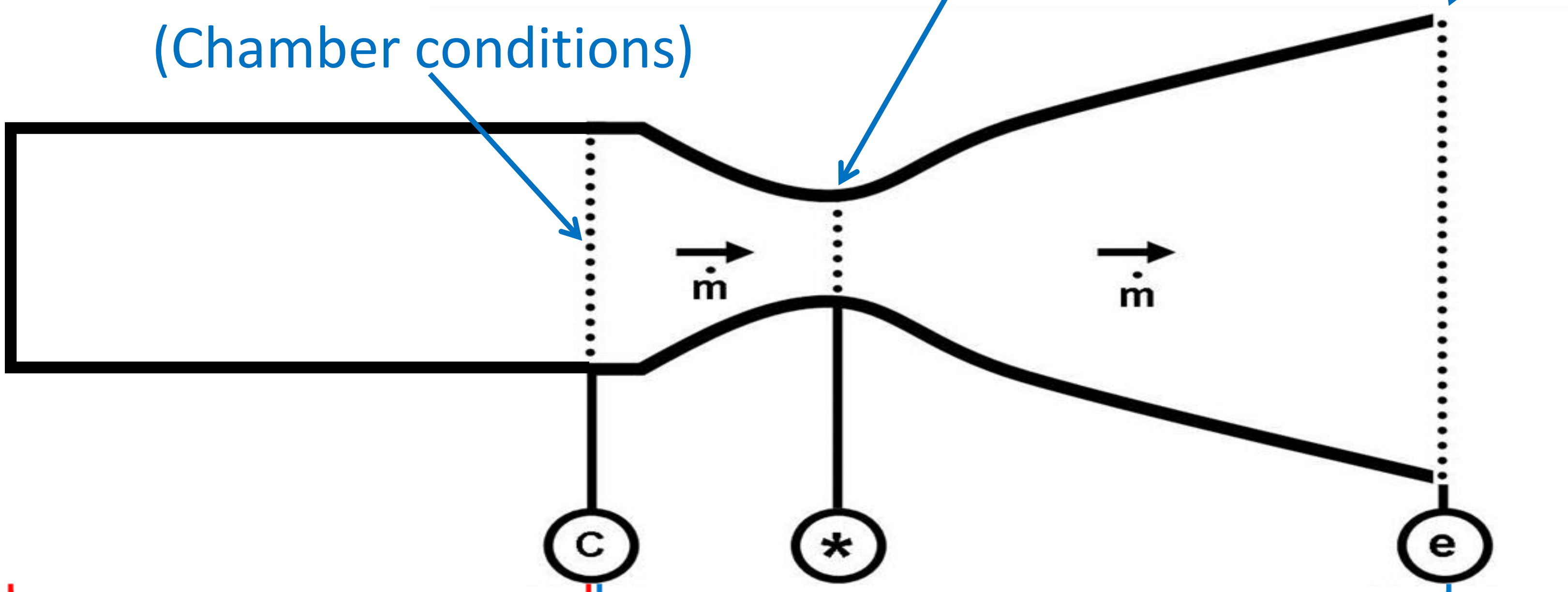
Nozzle Throat

Nozzle Inlet

(Chamber conditions)

Nozzle Expansion Ratio

$$\boxed{\varepsilon = \frac{A_e}{A^*}}$$



In some rockets, no combustion takes place
→ propellant in the chamber is at low temperature

(Combustion) chamber

- High pressure
- (High temperature)
- Low speed

Convergent-Divergent Nozzle

- Propellant is accelerated
- No external energy is provided

Ideal Rocket Theory assumptions

1. The propellant is a **perfect gas**
2. The propellant is a **calorically ideal gas**
3. Propellant has **constant homogeneous** chemical composition
4. Nozzle flow is **steady** (not dependant on time)
5. Nozzle flow is **isentropic** (no energy is provided or lost)
6. Nozzle flow is **1-dimensional** (quantities vary only along axis)
7. Flow velocity is **purely axial**
8. The propellant experiences **no external forces** in the nozzle
9. Propellant in the chamber has **negligible velocity** ($v_c \approx 0$)

Ideal Rocket Theory building blocks

	Conservation Equations
Mass	$\rho \cdot v \cdot A = \dot{m} = \text{constant}$
Momentum	$p + \frac{1}{2}\rho v^2 = \text{constant}$
Energy	$h + \frac{1}{2}v^2 = \text{constant}$

Ideal Gas Equations	
$p = \rho \frac{R_A}{M_W} T$	$\frac{p}{\rho^\gamma} = \text{constant}$
$h = c_p \cdot T$	$c_p = \frac{\gamma}{\gamma - 1} \cdot \frac{R_A}{M_W}$
$M = \frac{v}{a}$	$a^2 = \gamma \cdot \frac{R_A}{M_W} \cdot T = \gamma \cdot \frac{p}{\rho}$

ρ = density [kg/m^3]

v = velocity [m/s]

A = nozzle area [m^2]

p = pressure [Pa]

h = enthalpy [m^2/s^2]

T = temperature [K]

R_A = universal gas constant = 8314 J/(K*kmol)

c_p = constant pressure specific heat [J/K*kg]

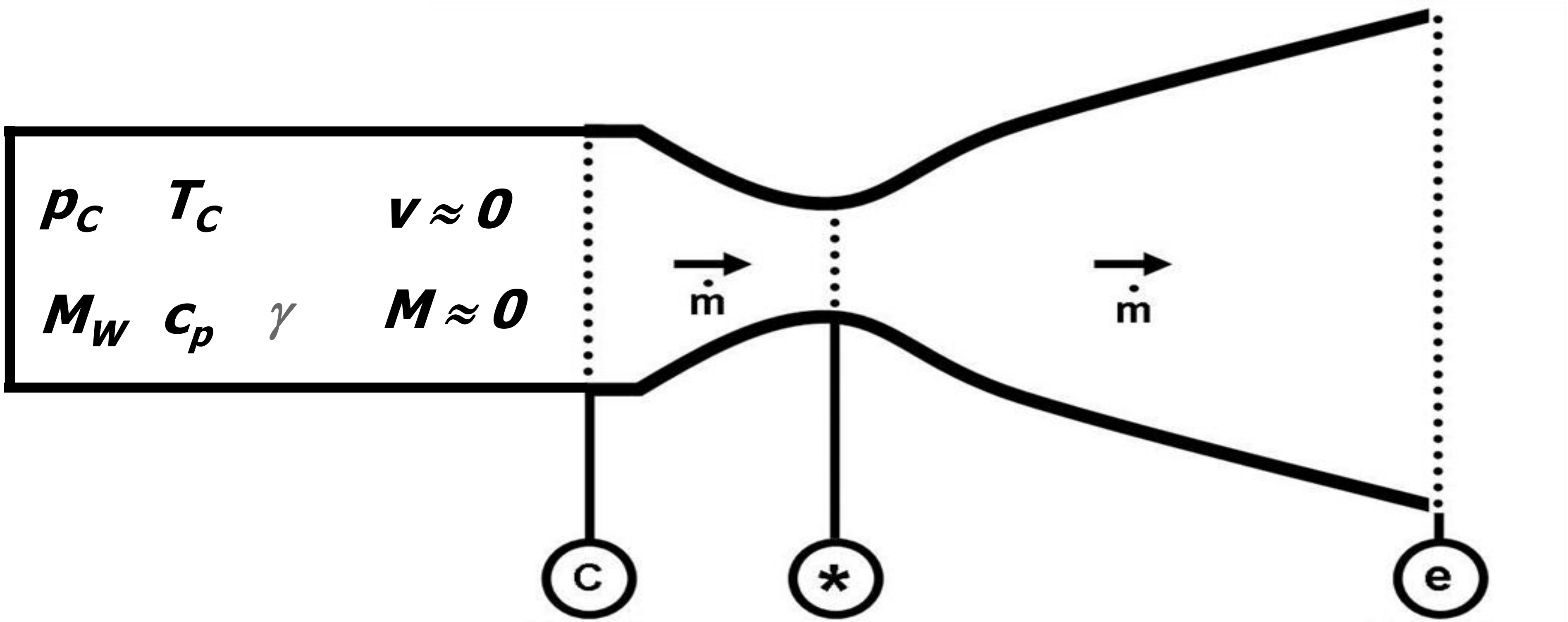
γ = specific heat ratio [-]

M_W = molecular mass [kg/kmol]

a = speed of sound [m/s]

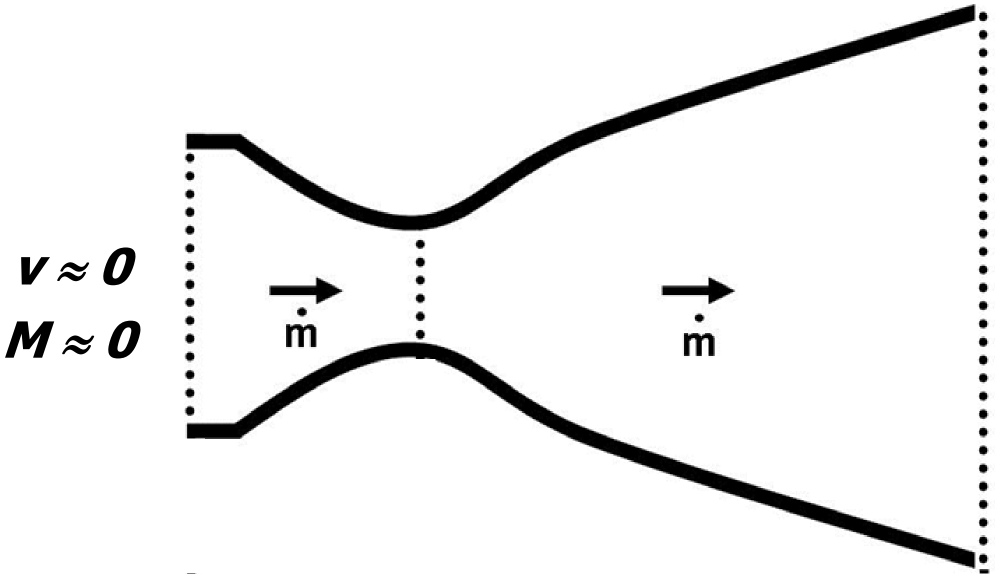
M = Mach number [-]

Two more assumptions…



1. Propellant conditions in the chamber (T_C, p_C) are known
2. Propellant composition and characteristics (γ, c_p, M_W, constant through the nozzle) are known

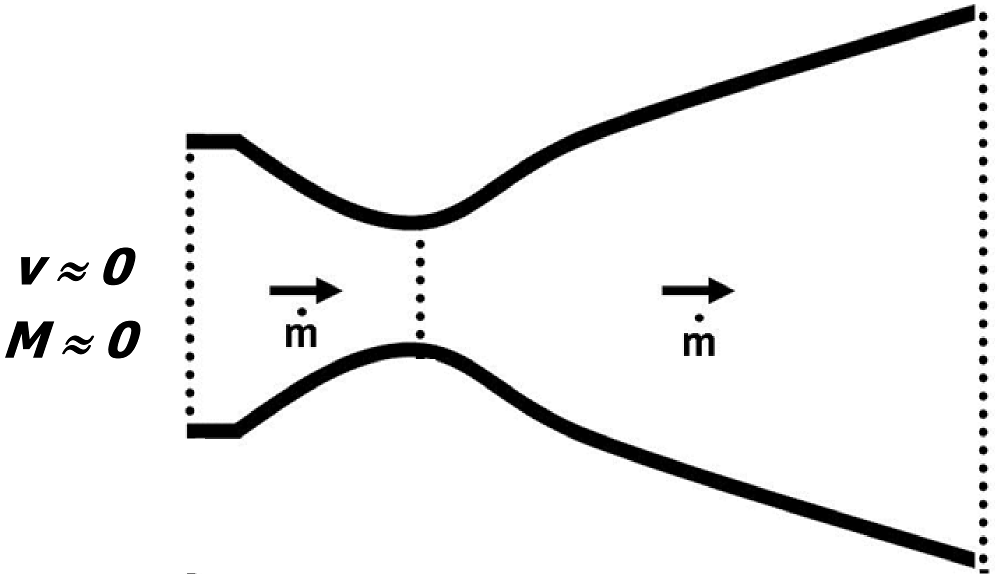
Why a convergent-divergent nozzle?



$$\boxed{\frac{dA}{A} = \left(M^2 - 1\right)\frac{dv}{v}}$$

- **Convergent** ($dA < 0$) → $dv > 0$ <u>only if</u> $M < 1$ (**subsonic flow**)
- **Divergent** ($dA > 0$) → $dv > 0$ <u>only if</u> $M > 1$ (**supersonic flow**)

Why a convergent-divergent nozzle?



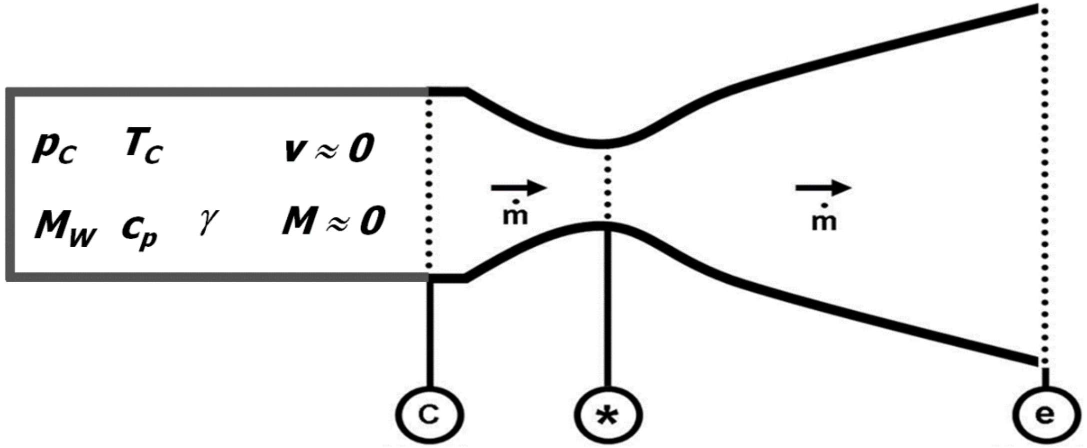
$$\boxed{\frac{dA}{A} = \left(M^2 - 1\right)\frac{dv}{v}}$$

To accelerate the flow **everywhere** in the nozzle ($dv > 0$):

- **Subsonic** convergent ($M < 1$), **supersonic** divergent ($M > 1$)
- **Sonic** throat ($M = 1$)

Jet velocity

$$v_e = \sqrt{\frac{2\gamma}{\gamma - 1} \cdot \frac{R_A}{M_W} \cdot T_C \cdot \left[1 - \left(\frac{p_e}{p_C} \right)^{\frac{\gamma - 1}{\gamma}} \right]}$$



- High jet velocity is obtained with:

✓ **High** *chamber temperature* T_C

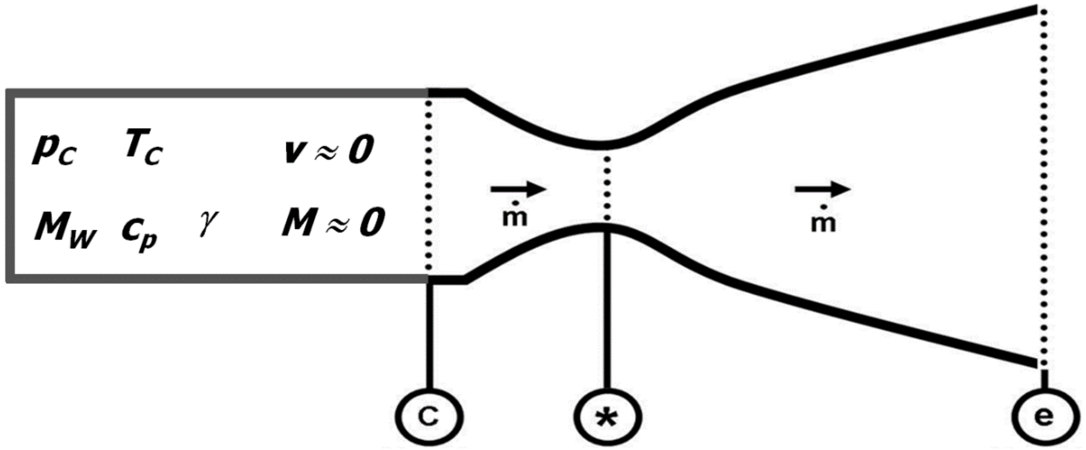
✓ **Low** *molecular mass* M_W

✓ **Low** *pressure ratio* p_e / p_c

Mass flow rate

$$\dot{m} = \frac{p_C \cdot A^*}{\sqrt{\frac{R_A}{M_W} \cdot T_C}} \cdot \Gamma(\gamma)$$

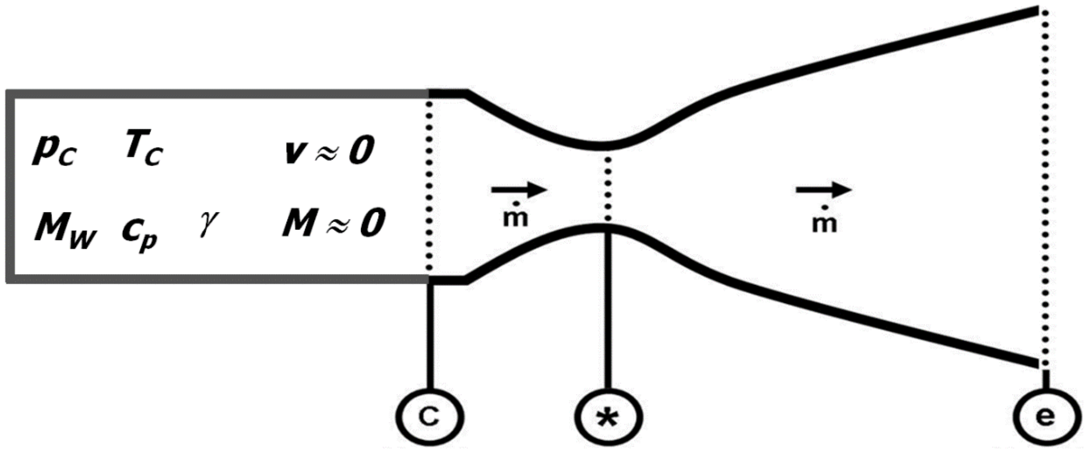
$$\Gamma(\gamma) = \sqrt{\gamma \cdot \left(\frac{1+\gamma}{2}\right)^{\frac{1+\gamma}{1-\gamma}}}$$



- $\Gamma(\gamma)$ = **Vandenkerckhove** function
- High mass flow rate is obtained with:
  - ✓ ***Low*** *chamber temperature* $\boldsymbol{T_C}$
  - ✓ ***High*** *molecular mass* $\boldsymbol{M_W}$
  - ✓ ***High*** *chamber pressure* $\boldsymbol{p_c}$
  - ✓ ***High*** *throat area* $\boldsymbol{A^*}$
- For given T_C, p_C, γ, M_W, A^*, <u>only one</u> mass flow rate makes sonic throat conditions possible (**chocked** flow)

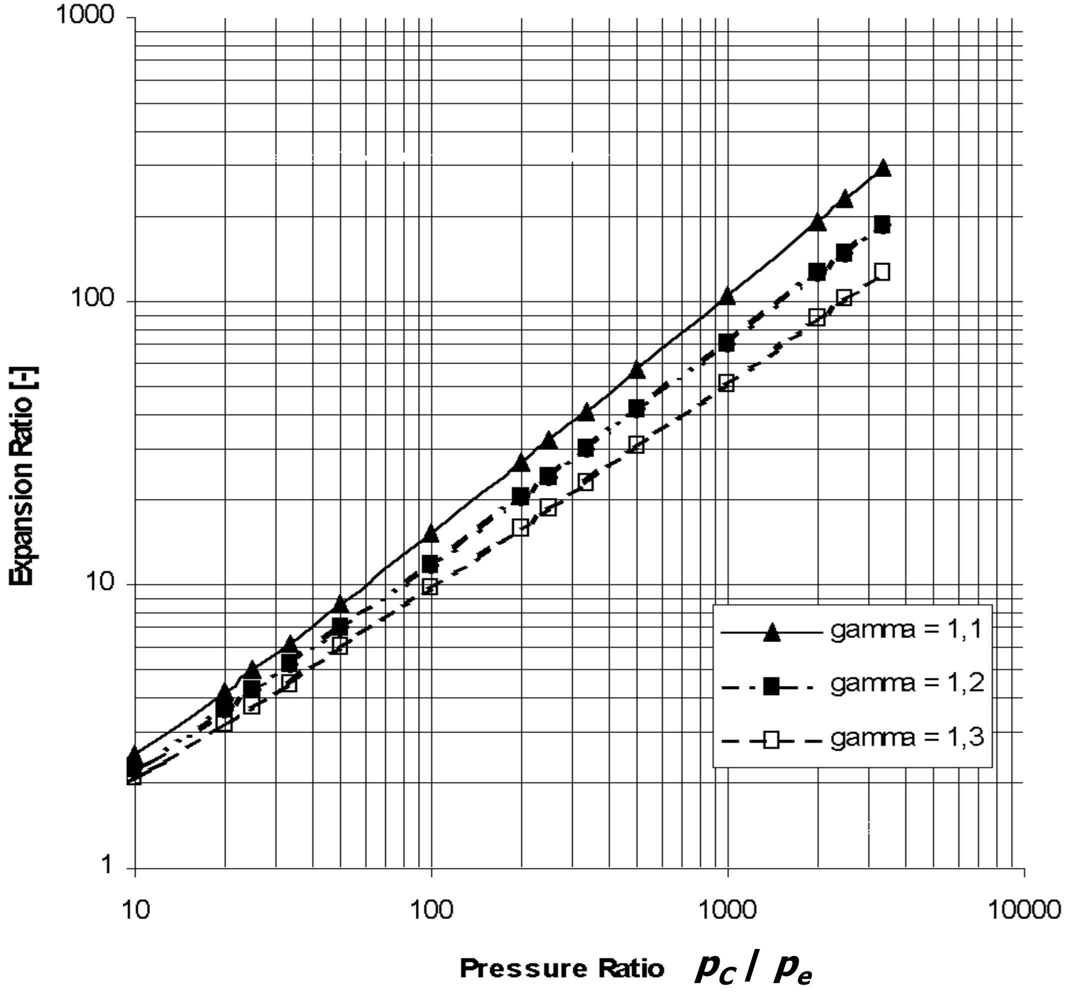
Exit pressure

$$\varepsilon = \frac{A_e}{A^*} = \frac{\Gamma(\gamma)}{\sqrt{\frac{2\gamma}{\gamma-1}\cdot\left(\frac{p_e}{p_C}\right)^{\frac{2}{\gamma}}\cdot\left[1-\left(\frac{p_e}{p_C}\right)^{\frac{\gamma-1}{\gamma}}\right]}}$$



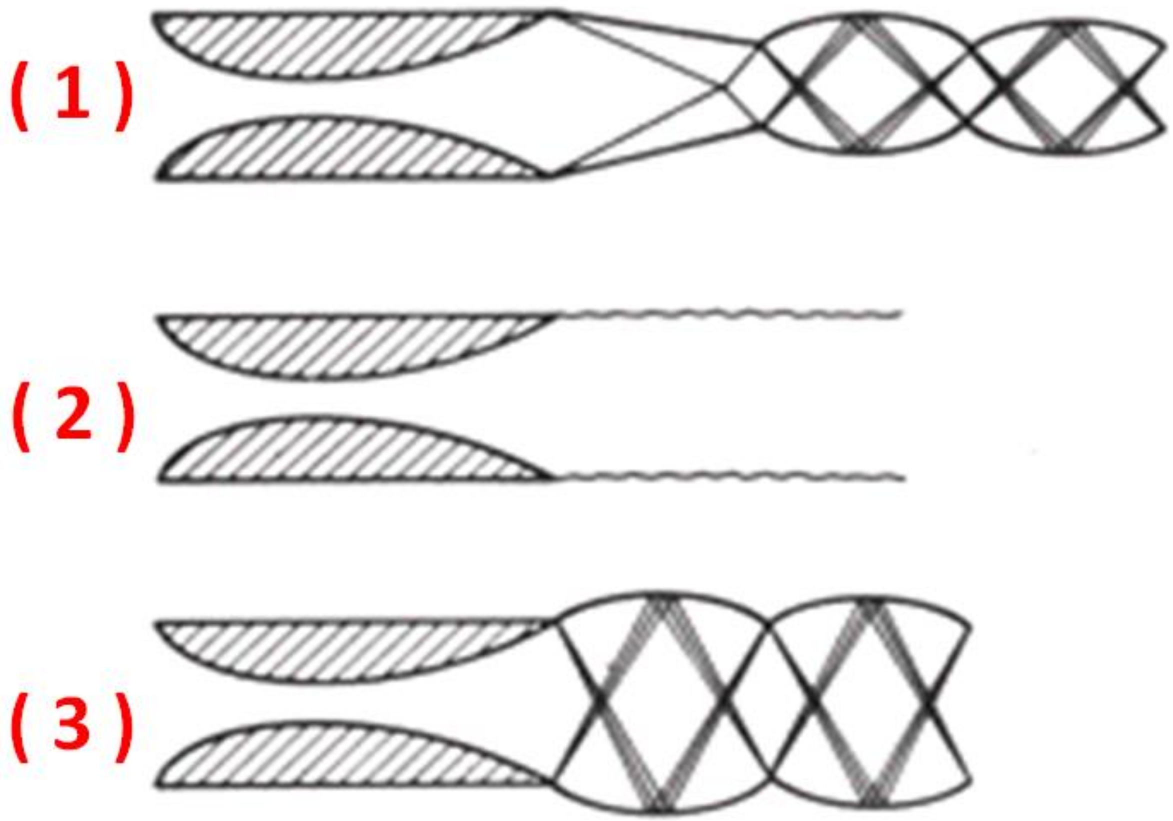
- For given γ, **nozzle geometry** (expansion ratio ε) and **pressure ratio** p_e / p_C are directly related
- **Implicit** equation for p_e / p_C as a function of ε

Exit pressure



- Weak dependence on γ
- **High** ε → high p_C / p_e → **lower p_e** for a given p_C
- For given expansion ratio and chamber pressure, the **exit pressure** p_e is **fixed**

Nozzle expansion conditions



- Three cases are possible:
  1) $p_e < p_a$ $\longrightarrow$ **over-expanded** nozzle
  2) $p_e = p_a$ $\longrightarrow$ **adapted** nozzle
  3) $p_e > p_a$ $\longrightarrow$ **under-expanded** nozzle
- In cases (1) and (3), pressure adjusts to ambient conditions through **shock waves** outside the nozzle
- For a given nozzle geometry, **thrust** is **maximum** when nozzle is **adapted**