Traffic Flow Theory and Simulation

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Lecture 2
Arrival patterns and cumulative curves
Arrival patterns
From microscopic to macroscopic

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Recap traffic flow variables

<table>
<thead>
<tr>
<th>Microscopic (vehicle-based)</th>
<th>Macroscopic (flow-based)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space headway (s [m])</td>
<td>Density (k [veh/km])</td>
</tr>
<tr>
<td>Time headway (h [s])</td>
<td>Flow (q [veh/h])</td>
</tr>
<tr>
<td>Speed (v [m/s])</td>
<td>Average speed (u [km/h])</td>
</tr>
</tbody>
</table>

\[ s = h \times v \]
\[ q = k \times u \]
Excercise – a task for you!

• What is the average speed if you travel
  • 10 km/h from home to university
  • 20 km/h on the way back

• What is the average speed if you make a trip with speed
  • v1 on the outbound trip
  • v2 on the inbound trip

• What is the average speed if you split the trip in n equidistant sections, which you travel in $v_i$
Similar problems arise in traffic

- Local measurements, spatial average speed needed

- Weigh speed measurements
- Inversely proportional to speed
The math...

- Weigh speed measurements \((w)\)
- Inversely proportional to speed: \(w_i = 1/v_i\)

\[
\langle v \rangle_{\text{space}} = \frac{\sum_i w_i v_i}{\sum_i w_i} = \frac{\sum_i \frac{1}{v_i} v_i}{\sum_i \frac{1}{v_i}} = \frac{n}{\sum_i \frac{1}{v_i}}
\]

\[
= \frac{1}{\sum_i \frac{1}{v_i}} n = 1 / \langle \frac{1}{v_i} \rangle_{\text{time}}
\]

Time-average of pace \((1/v)\)
... and the effect

What is higher time mean speed (A) or space mean speed (B)?
Summary

• Speed averaging is not trivial

• Obtain time mean speed from loop detector data by harmonic average (i.e., averaging 1/v)

• Space mean speed is lower, with differences in practice up to factor 2
Overview of remainder of lecture

- Arrival process and relating probability distribution functions
  - Poisson process (independent arrivals)
  - Neg. binomial distribution and binomial distribution
  - Applications facility design (determine length right-or left-turn lane)
- Time headway distributions
  - Distribution functions and headway models
  - Applications
- Speed distribution and free speed distributions
Arrival processes

- Insight into probability distribution of the number of vehicles arriving in a short time interval (e.g. 15 seconds) is important for several applications

**Example:**
- Length of extra lane for left-turning vehicles at an intersection
- Probability that queue exceeds roadway space is limited (e.g. 5%)

**Models for the distribution of vehicles arriving in a short period of time**
- Poisson process
- Binomial process
- Negative binomial process
Poisson process

- Number of vehicles passing $x$ during certain period of length $h$
- $\Lambda(h)$ can be described by a stochastic variable
- Assume independent arrivals
  - Dilute traffic operations w. sufficient passing opportunities
  - No upstream disturbances (e.g. signalized intersection)

Then nr of arrivals $\Lambda(h)$ is **Poisson**:

$$f(k; \lambda) = \frac{\lambda^k \exp(-\lambda)}{k!}$$
Properties of the Poisson distribution

- **Mean** and **variance** are equal (\( = \mu_h \)) for a certain period of length \( h \)
- If “mean = variance” then Poisson is likely
- And thus independence of arrivals is a good assumptions
Poisson process (2)

- Examples of Poisson distributions for different periods / intensities

\[ \mu_h = 0.6 \]

\[ \mu_h = 3 \]
Poisson and real data (arrivals / 15s)

Two-lane rural road in the Netherlands

$q = 280$ veh/h
Poisson and real data (arrivals / 15s)

Two-lane rural road in the Netherlands

$q = 550 \text{ veh/h}$
Poisson and real data (arrivals / 15s)

Two-lane rural road in the Netherlands

\[ q = 800 \text{ veh/h} \]
Poisson process (3): exercise

• Application to left-turn lane design problem
• Intensity for left-turning lane during peak-hour = 360 veh/h
• Duration of the red-phase 50s
• In 95% of the cycles must be undersaturated
• How long must the left-lane be?

Number of arrivals
\[ N = \text{Poisson} \]
Poisson process (3): exercise

- Application to left-turn lane design problem
- Intensity for **left-turning lane** during peak-hour = 360 veh/h
- Duration of the red-phase 50s
- In 95% of the cycles must be undersaturated
- How long must the left-lane be?

- Assume Poisson process: \[ f(k; \lambda) = \frac{\lambda^k \exp(-\lambda)}{k!} \]
- Mean number of arrivals \( 50 \cdot \left( \frac{360}{3600} \right) = 5 \)
- Thus \( \lambda = 5 \) veh per aggregation interval \((= qh \text{ with } q = 360 \text{ and } h = 50/3600)\)
Poisson process (4)

- From graph:
  \[ \Pr(N \geq 8) = 0.9319 \]
  \[ \Pr(N \geq 9) = 0.9682 \]

- If the length of the left-turn lane can accommodate 9 vehicles, the probability of blocking back = 3.18%
Two other possibilities

1. Vehicles are (mainly) following => binomial distribution

2. Downstream of a regulated intersection => negative binomial distribution
Binomial process

• Increase traffic flow yields formation of platoons (interaction)
• Poisson is no longer valid description
• Alternative: Binomial process

\[ f(k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k} \]

• Binomial distribution describes the probability of \( n \) successful, independent trials; the probability of success equals \( p \); \( n_0 = \text{max. number of arrivals within period } h \)
Properties of binomial distribution

• Mean: \( n_0\rho \) and variance: \( n_0(1 - \rho)\rho \)
• Note that \textbf{variance < mean}
• No rationale why approach yields reasonable results
• Choose appropriate model by statistical testing!
Binomial process (3)

- Left-turn example revisited
- Note: variance < mean!
  - Assume a variance of 2.5 \(\text{veh}^2 / \text{cycle} (=50 \text{sec})\)
- Determine how long the left-turning lane must be under these assumptions
- **Hint:** use the recursive formula in reader (Home work)
Downstream of controlled intersection

• More likely to have bunched vehicles (Why?)
• Negative binomial distribution
How to determine model to use?

Some guidelines:
• Are arrivals independent => Poisson distribution
  • Low traffic volumes
  • Consider traffic conditions upstream (but also downstream)
• Consider mean and variance of arrival process
• Platooning due to regular increase in intensity? => Binomial
• Downstream of signalized intersection? => Negative binomial

Definitive answer?
• Use statistical tests (Chi-square tests) to figure out which distribution is best
Time headways

- A time headway of a vehicle is defined by the period between the passing moment of the preceding vehicle and the vehicle considered.

- Distinction between
  - Net headway (gap): rear bumper – front bumper
  - Gross headway: rear bumper – rear bumper

- Headway distribution models generally pertain to a single lane.
Example headway distribution two-lane road

- Site: Doenkade
Exponential distribution

Lane of a two-lane road

Intensity = 639 veh/h
Average headway = 0.0835 s

Headway distribution from data

Exponential p.d.f. with $\lambda = 0.0835$

P.d.f. $f(h)$ vs. headways (s)
Maximum likelihood

- Maximum-likelihood method aims to maximize the ‘probability’ of the parameters given the sample (observation)
- The likelihood of a single observation $h_i$ equals (lambda parameter)

$$P(H = h_i) = f(h_i; \lambda)$$

- The likelihood of an entire sample \( \{h_i\} \) equals

$$\mathcal{L} = \prod_i P(H = h_i) = \prod_i f(h_i; \lambda)$$
Optimisation of parameters

- The probability of an entire sample \( \{h_i\} \) equals

\[
\mathcal{L} = \prod_i P(H = h_i) = \prod_i f(h_i; \lambda)
\]

- ML entails maximizing this likelihood, i.e.

\[
\lambda^* = \arg\max_{\lambda} (\mathcal{L}(\lambda))
\]
Composite headway models

**Composite** headway models distinguish

- Vehicles that are **driving freely** (and thus arrive according to some Poisson process, or whose headways are exponentially distributed)

- Vehicles that are **following at some minimum headway** (so-called empty zone, or constrained headway)

  *which distribution is suitable?*
Basic idea

- Combine two distribution functions
  - Constraint: fraction $\phi$
  - Free: fraction $(1-\phi)$

$$P(H = h) = \phi (P_{\text{constraint}}(h)) + (1 - \phi) (P_{\text{free}}(h))$$

- Application: capacity estimation
  - How?
Estimation: likelyhood or graphically

Survival function (log. scale)

Probability density

- deviation from straight line at $h^* = 7$ s
- total headway p.d.f.
- empty zone p.d.f.
- $\phi = 0.69$
- $q = 639$ veh/h
- $C = 1767$ veh/h
- free headway p.d.f.
Summary

• Arrival distribution:
  Free flow: headways exponential,
  nr of per aggregation time: Poisson
  Following: binomial

• Calculation of required dedicated lane length

• Composite headway models => capacity estimation
Binomial process (2)

• Examples of the binomial process
Cowan’s M3 model

- Drivers have a certain **minimum headway** \( x \) which can be described by a **deterministic variable**
- Minimum headway \( x \) describes the headway a driver needs for safe and comfortable following the vehicle in front
- If a driver is **not following**, we assume that he / she is driving at a headway \( u \) which can be described by a random variate \( U \sim p_{\text{free}}(u) \)
- The headway \( h \) of a driver is **the sum** of the free headway \( u \) and the minimum headway \( x \), and thus \( H = x + U \)

- **The free headway is assumed distributed according to the exponential distribution** (motivation?)
Cowan’s M3 model

- Probability density function of $H$

\[ p(h) = \phi \delta(x - h) + (1 - \phi) p_{\text{free}}(h) = p_1(h) + p_2(h) \]

- Unless there are major upstream disturbances, the free headways are **exponentially distributed**, i.e.

\[ p_2(h) = (1 - \phi) p_{\text{free}}(h) = (1 - \phi) H(h - x) \lambda e^{-\lambda(h-x)} \quad \text{for} \quad h > x \]

- The model thus assumes that all constrained drivers maintain headway $x$, while the headway of the unconstrained drivers is distributed according to a shifted exponential distribution
Cowan’s M3 model

- Example of Cowan’s M3 model
- (note that $h_0 = x$)
Branston’s headway distribution model

• Rather that assuming one fixed $x$ for all drivers, we can assume that different drivers maintain different minimum headways.
• Inter-driver differences are reflected by assuming that $x$ is also a stochastic variable $X$ with a specific distribution function $p_{\text{follow}}(x)$.
• This yields the composite headway distribution model of Branston:

$$p(h) = \phi p_{\text{follow}}(h) + (1 - \phi) p_{\text{free}}(h)$$

$$= \phi p_{\text{follow}}(h) + (1 - \phi) \lambda e^{-\lambda h} \int_{0}^{h} p_{\text{follow}}(s)e^{-\lambda s} ds$$
Buckley’s composite model

- Drivers have a certain **minimum headway** \( x \) which can be described by a random variable \( X \sim p_{\text{follow}}(x) \).
- Minimum headway describes the headway a driver needs for safe and comfortable following the vehicle in front.
- \( X \) describes differences between drivers and within a single driver.

- If a driver is **not following**, we assume that he / she is driving at a headway \( u \) which can be described by a random variate \( U \sim p_{\text{free}}(u) \).

- The headway \( h \) of a driver is then the minimum between the free headway \( u \) and the minimum headway \( x \), and thus \( H = \min\{X, U\} \).
Buckley’s composite model\(^3\)

- Consider a threshold value \( h^* \) such that all observed headways \( h \) are larger than \( h^* \) are free headways, i.e. \( p_{\text{follow}}(h) = 0, \ h > h^* \)

![Graph of probability density showing \( p(h) \), total density, and \( \phi p_{\text{follow}}(h) \), density of following vehicles.](image)
Buckley’s composite model\textsuperscript{4}

• Unless there are major upstream disturbances, the free headways are \textbf{exponentially distributed}, i.e.

\[ p(h) = (1 - \phi) p_{\text{free}}(h) = A\lambda e^{-\lambda h} \quad \text{for} \quad h > h^* \]

• For headways \( h \) smaller than \( h^* \), we need to correct the total headway. This is done by \textit{removing from the exponential distribution the fraction of vehicles that have preferred following times larger than} \( h \)

• The fraction of drivers with headway \( h \) that are not following equals

• Then for \( h < h^* \) we have for the \textbf{free headway part}

\[ p_2(h) = (1 - \phi) p_{\text{free}}(h) = (1 - \theta(h)) A\lambda e^{-\lambda h} = A\lambda e^{-\lambda h} (1 - \phi) \int_0^h p_{\text{follow}}(\eta) d\eta \]
Buckley’s composite model\textsuperscript{5}

- For the total headway distribution, we then get

\[
p(h) = \phi p_{\text{follow}}(h) + (1 - \phi) A \lambda e^{-\lambda h} \int_{0}^{h} p_{\text{follow}}(\eta) d\eta
\]

where \( A \) denotes the \textbf{normalization constant} and can be determined from

\[
\int_{0}^{\infty} p(\eta) d\eta = 1 \implies A = \int_{0}^{\infty} \lambda e^{-\lambda \eta} p_{\text{follow}}(\eta) d\eta
\]

- Special estimation procedures exist to determine \( \lambda, \phi \) and p.d.f. \( p_{\text{follow}}(h) \)
  - Parametric: specify p.d.f. \( p_{\text{follow}}(h) \) with unknown parameters
  - Distribution free, non-parametric, i.e. without explicit specification \( p_{\text{follow}}(h) \)
Determination of $h^*$

- For $h > h^*$, distribution is exponential, i.e.
  \[ p(h) = A\lambda e^{-\lambda h} \quad \text{for } h > h^* \]

- Consider survival function $S(h)$
  \[ S(h) = \Pr(H > h) = \int_h^\infty A\lambda e^{-\lambda \eta} d\eta = Ae^{-\lambda h} \]

- Obviously, for $h > h^*$, we have
  \[ \ln[S(h)] = \ln[Ae^{-\lambda h}] = \ln(A) - \lambda h \]

- Suppose we have headway sample \( \{h_i\} \)
- Consider the empirical distribution function
  \[ \hat{S}_n(h) = 1 - \frac{1}{n} \sum H(h - h_i) \]

- Now draw $\ln[\hat{S}_n(h)]$ and determine $h^*$, $A$, and $\lambda$
Applications of headway distributions

- Analysis of crossing a street / gap-acceptance
- Calculate waiting time or delay
- Capacity estimation (without capacity observations) using composite headway models of Cowan / Branston / Buckley
  - Estimate all parameters of composite headway distribution
  - Assume all vehicles are following ($\phi = 1$) to see that ($\rho(h) = \rho_{\text{follow}}(h)$), i.e. headway is minimal: we have $\text{mean}(H) = \text{mean}(X)$
  - Recall that $q = 1/\text{mean}(H)$
  - Estimate mean headway (mean empty zone) and capacity

\[ C = \frac{1}{\text{mean}(X)} = \frac{1}{\int_0^\infty x p_{\text{follow}}(x) dx} \]
Binomial process (3)

- Left-turn example revisited
- Assume a variance of $2.5 \text{ veh}^2 / \text{cycle}$ (=50 sec)
- Determine how long the left-turning lane must be under these assumptions

- Mean number of arrivals 5: $n_0 \rho = 5$
- ‘Assume’ variance of $2.5 \text{ veh}^2$: $n_0(1 - \rho)\rho = 2.5$
- Then we can determine:
  \[ \rho = 0.5 \text{ and } n_0 = 10 \]

- Results are very sensitive to variance!!
Binomial process (4)

- From graph:
  \[ \Pr(N \geq 7) = 0.9453 \]
  \[ \Pr(N \geq 8) = 0.9893 \]

- If the length of the left-turn lane can accommodate 7 vehicles, the probability of blocking back = 5.47%
Negative Binomial distribution

• Also consider **negative Binomial distribution** (see course notes)

\[
\Pr(N(h) = n) = \binom{n_0 + n - 1}{n} p^{n_0} (1 - p)^n
\]

• Appears to hold when upstream disturbances are present

• Choose appropriate model by trial and error and statistical testing
Neg. bin. distribution & observations

Two-lane rural road in the Netherlands

$q = 280 \text{ veh/h}$
Neg. bin. distribution & observations

Two-lane rural road in the Netherlands

$q = 550 \text{ veh/h}$
Neg. bin. distribution & observations

Two-lane rural road in the Netherlands

q = 800 veh/h
Example real-life headway distribution

- Pedestrian experiments
- Headways of pedestrians passing a certain cross-section (wide bottleneck scenario)
Exponential distribution (2)

- Exponential distribution function / survival function

\[
\text{mean}(H) = \bar{H} = \frac{1}{q}
\]

\[
\text{var}(H) = \frac{1}{q^2}
\]
Cumulative curves
Calculation of delays and queues

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Cumulative vehicle plots

- Cumulative flow function $N_x(t)$: number of vehicles that have passed cross-section $x$ at time instant $t$
- $N_x(t)$: step function that increases with 1 each time instant vehicle passes
- Horizontal axis: trip times
- Vertical axis: vehicle count (storage)
Examples of cumulative curves?
Construction of cumulative curves
Information in cumulative curves

\[ \frac{dN}{dt} = \text{flow} \]
Cumulative vehicle plots

- Flow = number of vehicles passing \( x \) (observer) during \( T \)
- What is the flow in this case?

Observer 1

- \( N_1(7.5) = 1000 \)
- \( N_1(8.0) = 1500 \)
- \( T = 0.5 \) h
Information in cumulative curves

Which line corresponds with detector 1?
Information in cumulative curves

![Diagram showing cumulative curves for travel time over time, with two curves labeled 1 and 2, and N as a common point on the vertical axis.](image)
Intermezzo - capacity

- Capacity is the maximum flow on a cross section
- What determines the capacity
- Nr of lanees
- Minimum headway influenced by
- Speed limit

1.5 s/veh = 2400 veh/h
Bottleneck in section
What if bottleneck is present

Flow limited to capacity
Travel time increases
More vehicles in the section
Total delay = sum delay over vehicles
Total delay = sum # extra vehicles in section
Controlled intersection

signalised intersection

- downstream
- upstream
Delay (2)

moved cumcurves

Capacity

Capacity
Real-life curves

Number of vehicles between observers can be used to determine density!
Application

- Applications: identification of stationary periods (constant flow)
- In this case, cumulative curves are (nearly) straight lines
Oblique curves

- Amplify the features of the curves by applying an oblique scaling rate $q_0$
- Use transformation: $N_2 = N - q_0 t$
Oblique curves

- Notice that density can still be determined directly from the graph; accumulation of vehicles becomes more pronounced.
- This holds equally for stationary periods.
- For flows note that $q_0$ needs to be added to the flow determined from the graph by considering the slope of the slanted cumulative curve.
Use of oblique curves

• Can delay be read directly from oblique cumulative curves
  A=yes
  B=no
• Can travel times be read directly from oblique cumulative curves?
  A=yes
  B=no
Delay determination simpler

moved cumcurves

moved slanted cumcurves
Deeper analysis

demand vs. capacity

Bottleneck
Simplest queuing model

• “Vertical queuing model”
• Given the following demand profile, and capacity, give the (translated) cumulative curves (and determine the delay)
Answer: cumulative curves
Answer: flow representation
Identifying capacity drop
Identifying capacity drop

• Using oblique (slanted) cumulative curves show that capacity...
Learning goals

• You now can:
  • Construct (slanted) fundamental diagrams
  • Use these to calculate:
    delays, travel times, density, flow
• In practice shortcoming:
  Data is corrupt, and errors accumulate

• Test yourself: tomorrow in excersise