Offshore Hydromechanics Module 1

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4. Potential Flows part 2

Introduction

Topics of Module 1

• Problems of interest	Chapter 1
• Hydrostatics	Chapter 2
• Floating stability	Chapter 2
• **Constant potential flows**	**Chapter 3**
• Constant real flows	Chapter 4
• Waves	Chapter 5

Learning Objectives

Chapter 3

- Understand the basic principles behind potential flow
- *To schematically model flows applying basic potential flow elements and the superposition principle*
- *To perform basic flow computations applying potential flow theory*

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Fluid Mechanics Laws

Summarizing:

- Conservation of mass (continuity):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0$$

Incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \nabla \cdot \vec{V} = 0$$

- Conservation of momentum (inviscid flow):

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \vec{\nabla} p$$

- Rotation of a fluid element:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Vorticity:

$$\vec{\zeta} = 2\vec{\omega} = \vec{\nabla} \times \vec{V}$$

Fluid Mechanics Laws

Velocity Potential

- Assumptions
  - Homogeneous
  - Continuous
  - Incompressible
  - Non-viscous (inviscid)
  - **Irrotational flow**

- The velocity potential is a function of time and position:

$$\Phi(x, y, z, t)$$

- The spatial derivatives of the velocity potential equal the velocity components at a time and position:

$$\frac{\partial \Phi}{\partial x} = u \qquad \frac{\partial \Phi}{\partial y} = v \qquad \frac{\partial \Phi}{\partial z} = w$$

Fluid Mechanics Laws

Velocity Potential

- Continuity equation for potential flow:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$\nabla^2 \Phi = 0$ Laplace equation

- Irrotationality

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$ in the (x,y) plane

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$$ in the (y,z) plane

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$ in the (x,z) plane

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Fluid Mechanics Laws

Bernoulli Equation

- Potential flow: $\vec{V} = \nabla\Phi$

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - gz + \frac{p}{\rho} = constant$$

Bernoulli equation

- More general: also valid on a streamline

Potential Flow

Stream funtion (2D)

- Ψ is the (2D) stream function, with:

$$\frac{d\Psi}{dy} = u \qquad \frac{d\Psi}{dx} = -v$$

- Difference of Ψ between neighboring stream lines: rate of flow between streamlines

- Orthogonality: $$\frac{d\Psi}{dy} = \frac{d\Phi}{dx} = u \qquad \frac{d\Psi}{dx} = -\frac{d\Phi}{dy} = -v$$

- Impervious boundaries equals streamline:

$$\frac{d\Phi}{dn} = 0 \qquad \Psi = constant$$

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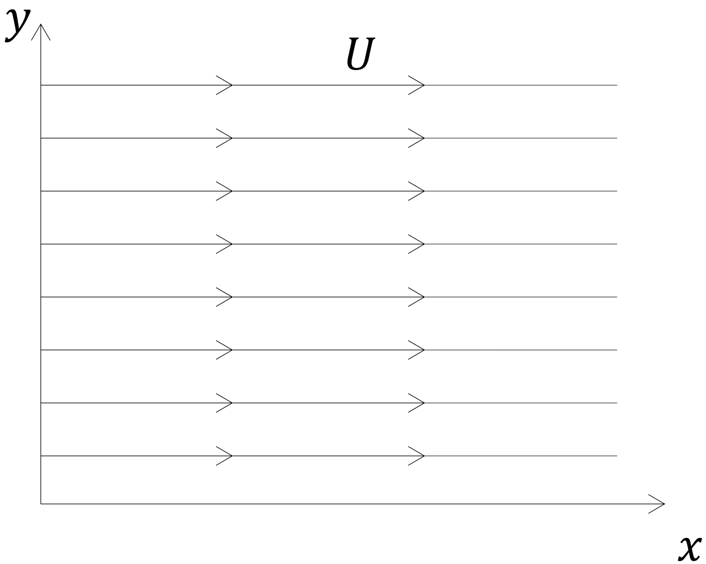
Potential flow elements

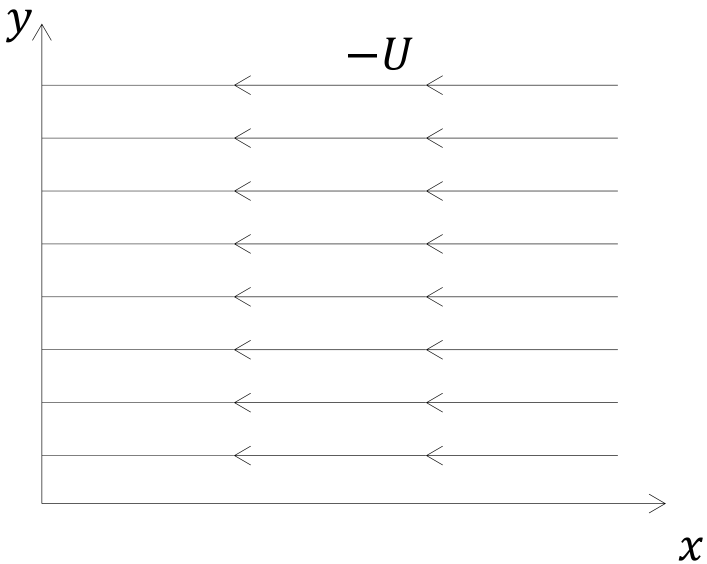
Introduction

- Using the previous we can define 'flow elements'
- We can add these elements up to construct realistic flow patterns
- Modeling of submerged bodies by matching streamlines to body outline
- Using the velocity potential, stream function and Bernoulli equation to find velocities, pressures and eventually fluid forces on bodies
- Discussed:
    - Uniform flow element
    - Source/sink element
    - Doublet element
    - Vortex element

Potential flow elements

Uniform flow





$$\Phi = U \cdot x$$

$$\Psi = U \cdot y$$

$$u = \frac{d\Phi}{dx} = \frac{d\Psi}{dy} = U$$

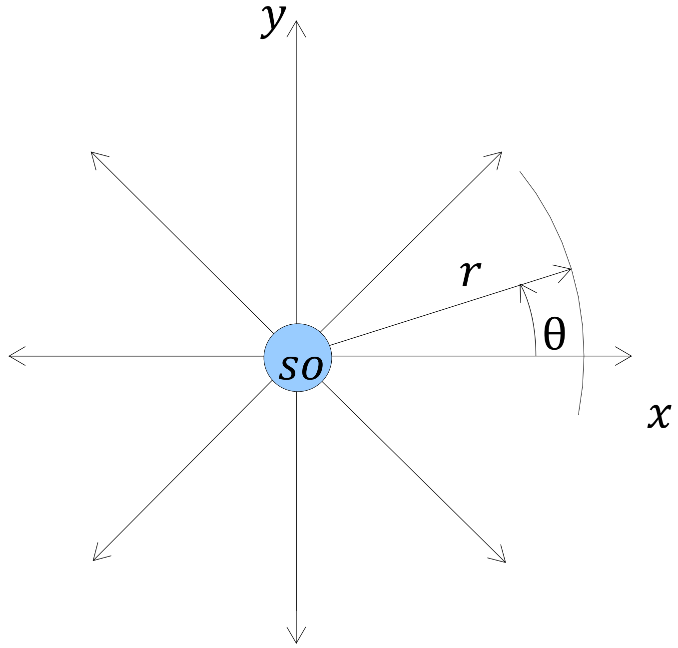
$$\Phi = -U \cdot x$$

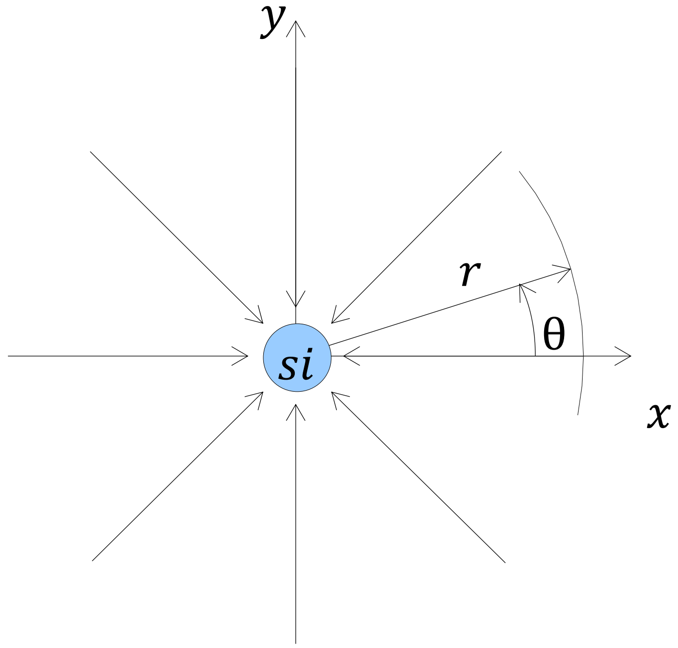
$$\Psi = -U \cdot y$$

$$u = \frac{d\Phi}{dx} = \frac{d\Psi}{dy} = -U$$

Potential flow elements

Source and sink flow





$$\Phi = +\frac{Q}{2\pi} \cdot \ln r = +\frac{Q}{2\pi} \cdot \ln\sqrt{x^2 + y^2}$$

$$\Psi = +\frac{Q}{2\pi} \cdot \theta = +\frac{Q}{2\pi} \cdot \arctan\frac{y}{x}$$

$$\Phi = -\frac{Q}{2\pi} \cdot \ln r = -\frac{Q}{2\pi} \cdot \ln\sqrt{x^2 + y^2}$$

$$\Psi = -\frac{Q}{2\pi} \cdot \theta = -\frac{Q}{2\pi} \cdot \arctan\frac{y}{x}$$

Potential flow elements

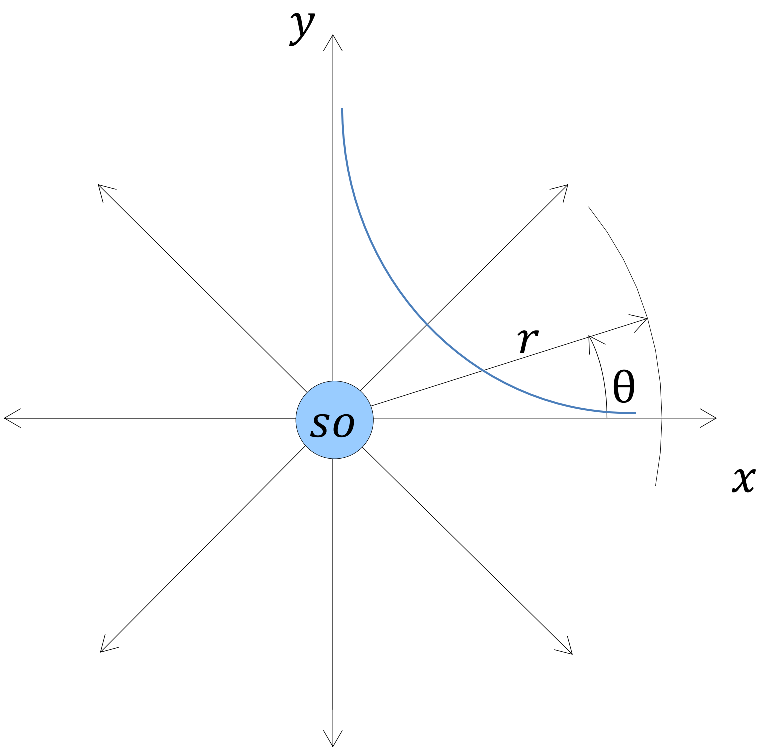
Source and sink flow

$$\Phi = +\frac{Q}{2\pi} \cdot \ln r$$

$$\Psi = +\frac{Q}{2\pi} \cdot \theta$$

$$v_r = \frac{\partial \Phi}{\partial r} = \frac{1}{r} \cdot \frac{\partial \Psi}{\partial \theta} = \frac{Q}{2\pi r}$$

$$v_\theta = \frac{1}{r} \cdot \frac{\partial \Phi}{\partial \theta} = -\frac{\partial \Psi}{\partial r} = 0$$



Potential flow elements

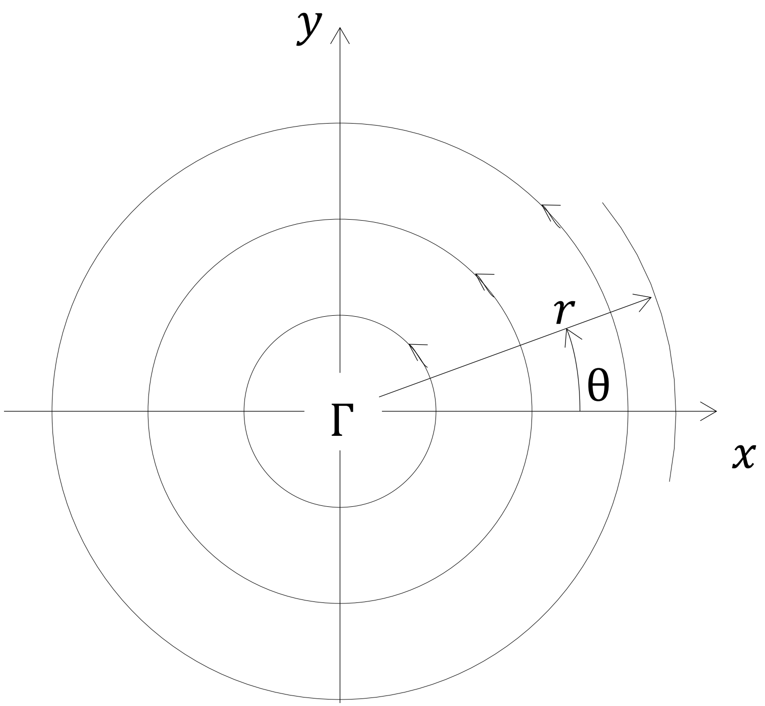
Circulation or vortex elements

$$\Phi = +\frac{\Gamma}{2\pi} \cdot \theta$$

$$\Psi = +\frac{\Gamma}{2\pi} \cdot \ln r$$

$$v_r = \frac{\partial \Phi}{\partial r} = \frac{1}{r} \cdot \frac{\partial \Psi}{\partial \theta} = 0$$

$$v_\theta = \frac{1}{r} \cdot \frac{\partial \Phi}{\partial \theta} = -\frac{\partial \Psi}{\partial r} = \frac{\Gamma}{2\pi r}$$



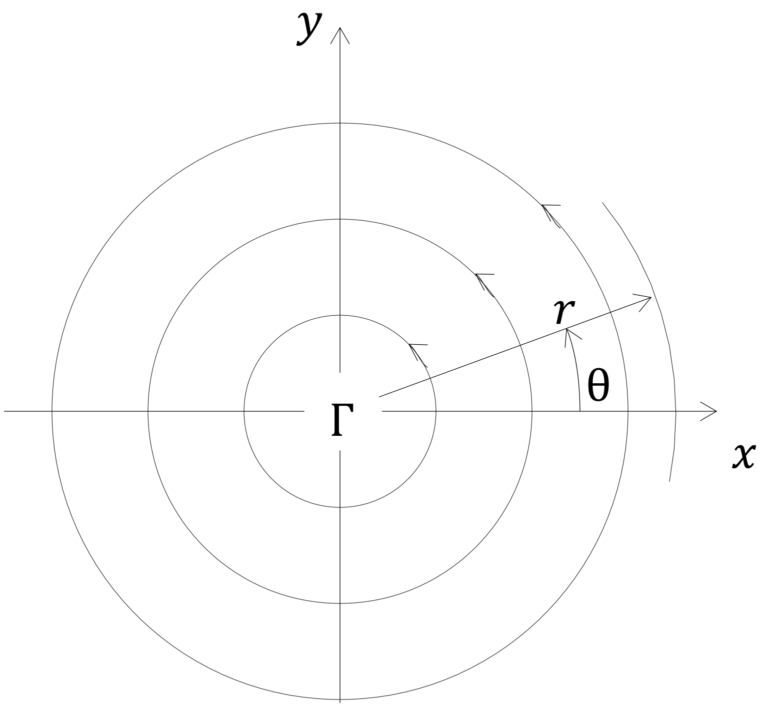
Potential flow elements

Circulation or vortex elements

$$\Phi = +\frac{\Gamma}{2\pi}\cdot\theta \qquad \Psi = +\frac{\Gamma}{2\pi}\cdot \ln r$$

$$v_r = \frac{\partial\Phi}{\partial r} = \frac{1}{r}\cdot\frac{\partial\Psi}{\partial\theta} = 0$$

$$v_\theta = \frac{1}{r}\cdot\frac{\partial\Phi}{\partial\theta} = -\frac{\partial\Psi}{\partial r} = \frac{\Gamma}{2\pi r}$$



Circulation strenght constant:

$$\Gamma = \oint v_\theta \cdot ds = 2\pi r\cdot v_\theta = constant$$

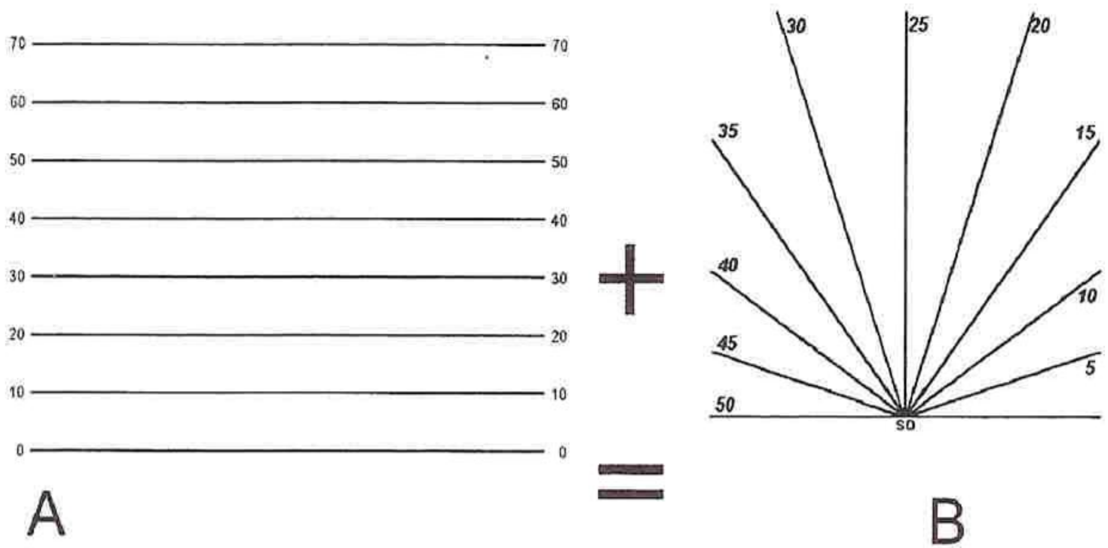
Therefore: no rotation, origin singular point: velocity infinite

Superposition

Methodology (source in positive uniform flow)

- The resulting velocity fields, potential fields or stream function fields may be simply superposed to find the combined flow patterns

(Using stream function values)



Uniform flow field

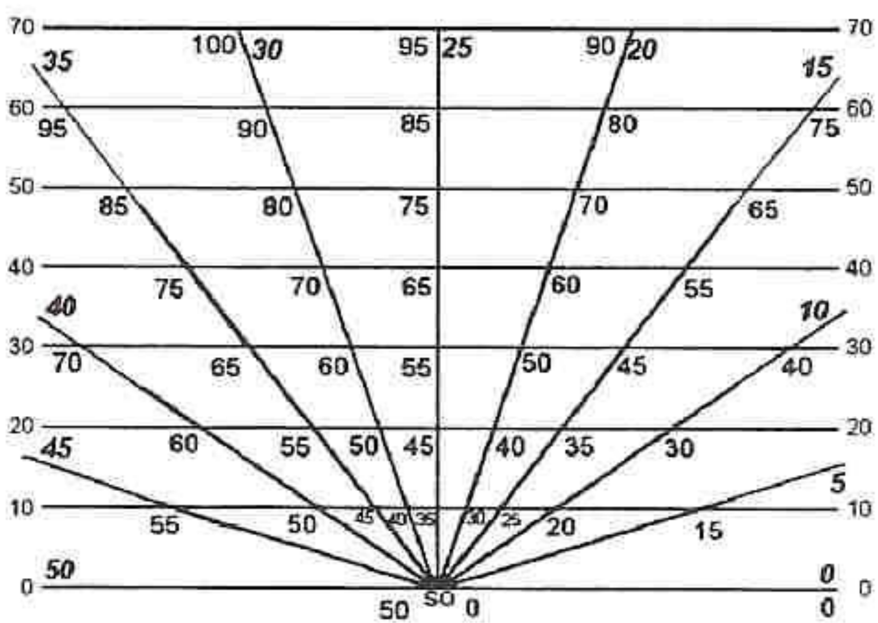
Source flow field

Superposition

Methodology (source in positive uniform flow)

- The resulting velocity fields, potential fields or stream function fields may be simply superposed to find the combined flow patterns

(Using stream function values)



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Superposition

Methodology (source in positive uniform flow)

- The resulting velocity fields, potential fields or stream function fields may be simply superposed to find the combined flow patterns

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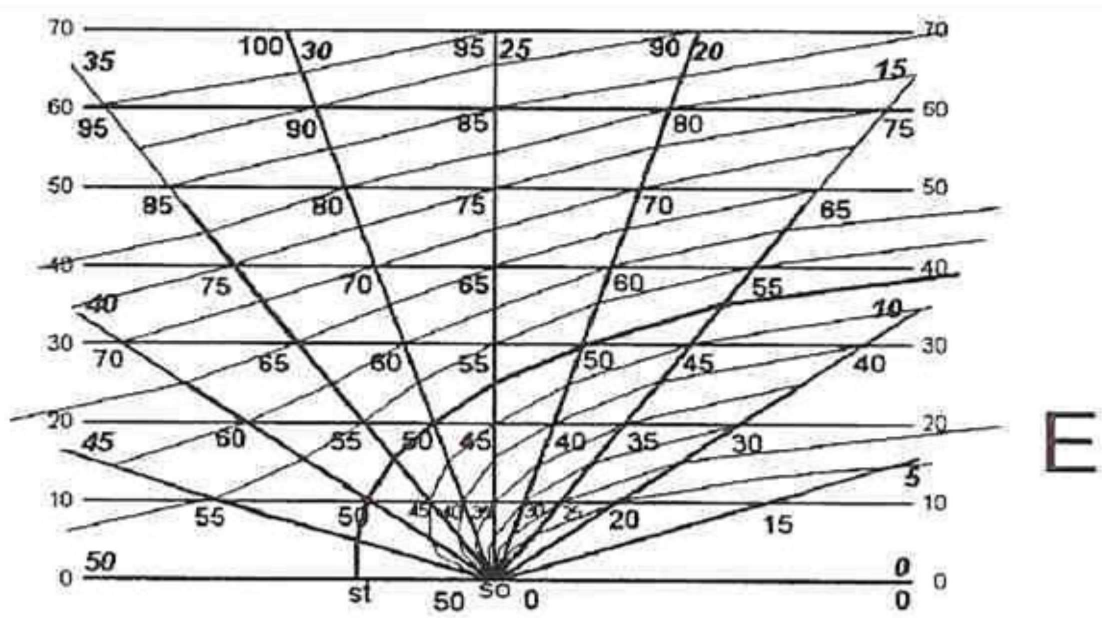


Superposition

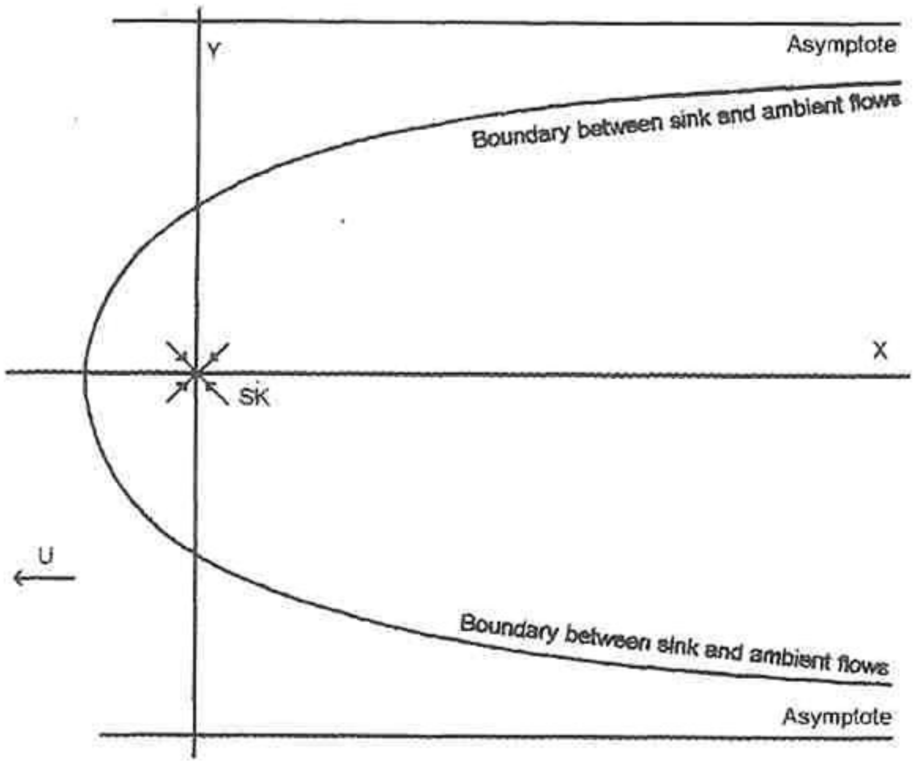
Sink in negative uniform flow

- Besides graphically this works also with formulas:

$$\Psi = -\frac{Q}{2\pi} \cdot \arctan\frac{y}{x} - U_\infty \cdot y$$

$$\Phi = -\frac{Q}{2\pi} \cdot \ln\sqrt{x^2 + y^2} - U_\infty \cdot x$$

For instance:
Find location stagnation point (Blackboard...)



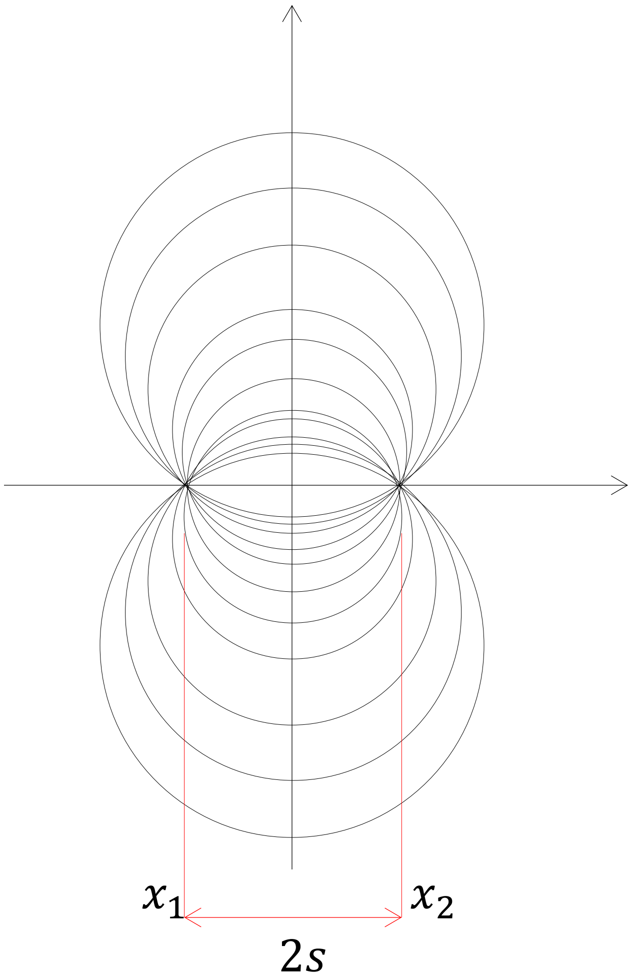
Superposition

Separated source and sink

$$\Psi_{source} = +\frac{Q}{2\pi}\cdot\theta_1 = +\frac{Q}{2\pi}\cdot\arctan\frac{y}{x_1}$$

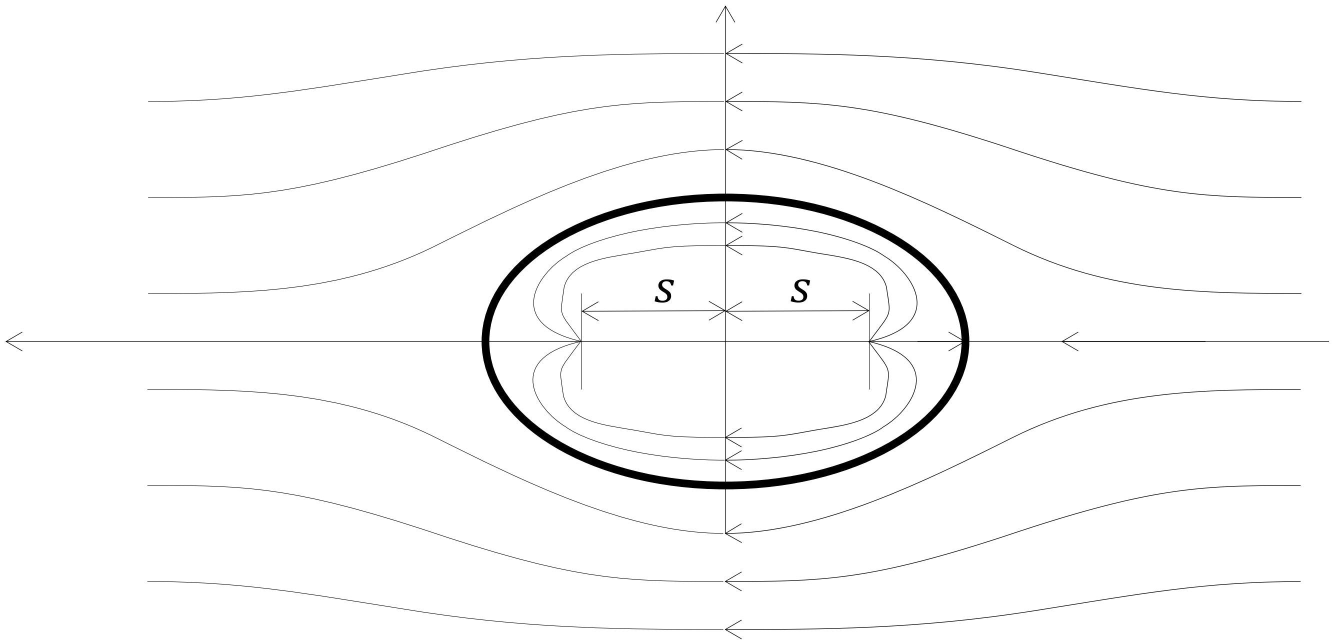
$$\Psi_{sink} = -\frac{Q}{2\pi}\cdot\theta_2 = -\frac{Q}{2\pi}\cdot\arctan\frac{y}{x_2}$$

$$\Psi = \frac{Q}{2\pi}\cdot\arctan\frac{2ys}{x^2+y^2-s^2}$$



Superposition

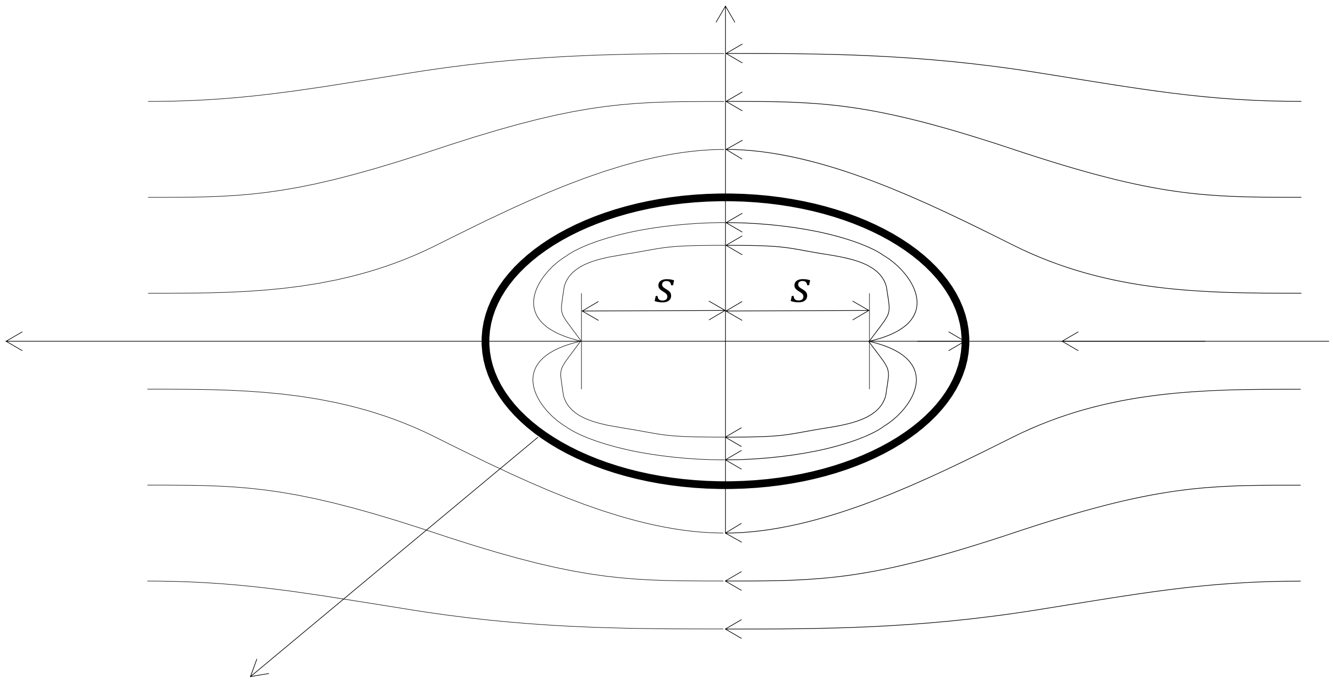
Separated source and sink in uniform flow



$$\Psi = \frac{Q}{2\pi} \cdot \arctan\frac{2ys}{x^2 + y^2 - s^2} + U_\infty y$$

Superposition

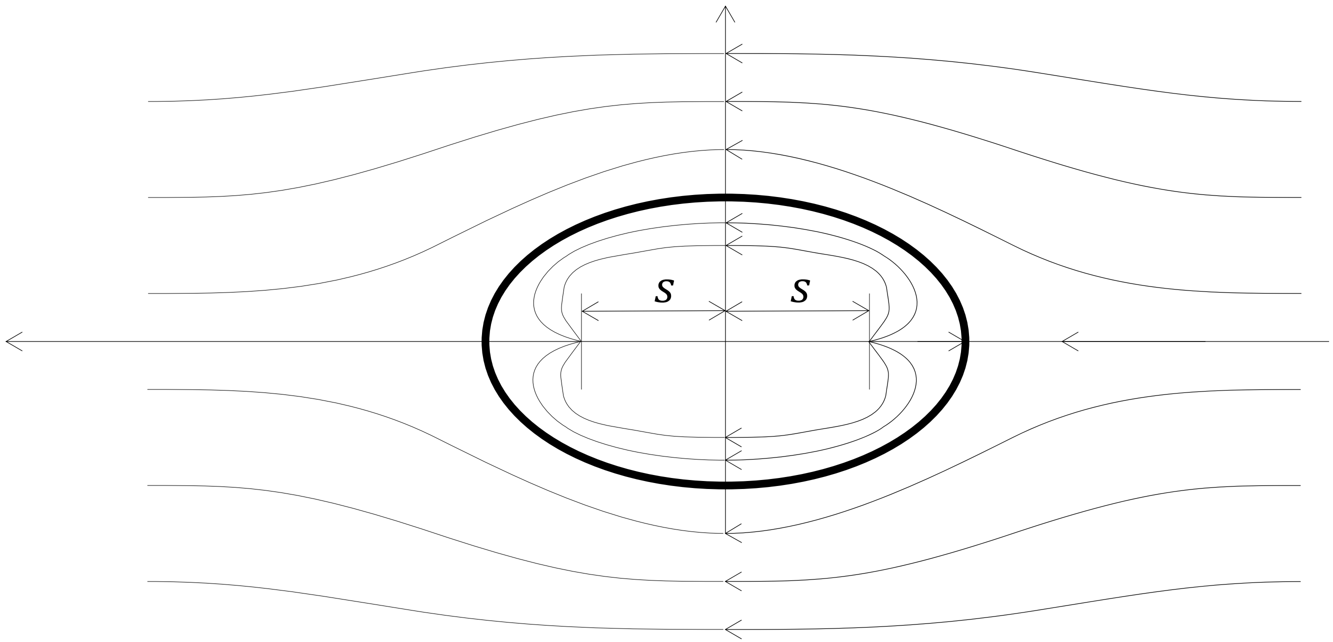
Separated source and sink in uniform flow



Streamline resembles fixed boundary (Rankine oval)
The flow outside this streamline resembles flow around solid boundary with this shape
Shape can be changed by using more source-sinks along x-axis with different strenghts

Superposition

Separated source and sink in uniform flow



This approach can be extended to form ship forms in 2D or 3D:

Rankine ship forms

Useful for simple flow computations

Potential flow elements

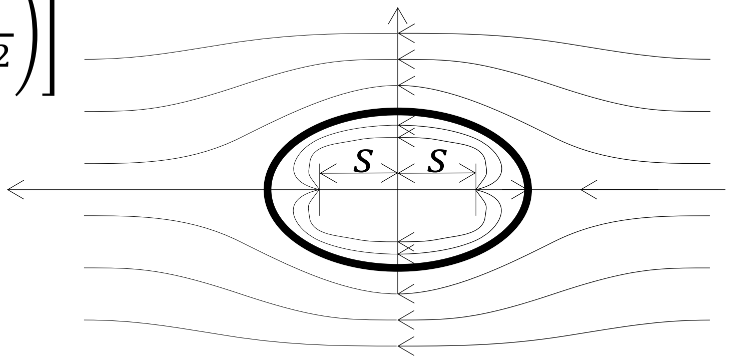
Doublet or dipole

When distance 2s becomes zero a new basic flow element is produced:

Doublet or dipole producing flow in positive x-direction

$$\Psi = \lim_{s \to 0} \left[\frac{Q}{2\pi} \cdot \arctan\left(\frac{2ys}{x^2 + y^2 - s^2} \right) \right]$$

$$\Psi = \lim_{s \to 0} \left[\frac{Q}{\pi} s \cdot \left(\frac{y}{x^2 + y^2 - s^2} \right) \right]$$



Note: in book errors wrt to doublet and its orientation!

Potential flow elements

Doublet or dipole

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Doublet or dipole producing flow in positive x-direction

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$$\Psi = \lim_{s \to 0} \left[\frac{Q}{\pi} s \cdot \left(\frac{y}{x^2 + y^2 - s^2} \right) \right]$$

Set constant: $\mu = \frac{Q}{\pi} s$

Potential flow elements

Doublet or dipole

When distance 2s becomes zero a new basic flow element is produced:

Doublet or dipole producing flow in positive x-direction

$$\Psi = \lim_{s \to 0}\left[\frac{Q}{2\pi} \cdot \arctan\left(\frac{2ys}{x^2 + y^2 - s^2}\right)\right]$$

$$\Psi = \lim_{s \to 0}\left[\frac{Q}{\pi} s \cdot \left(\frac{y}{x^2 + y^2 - s^2}\right)\right]$$

Disappears when: $s \to 0$

Set constant: $\mu = \frac{Q}{\pi} s$

Potential flow elements

Doublet or dipole

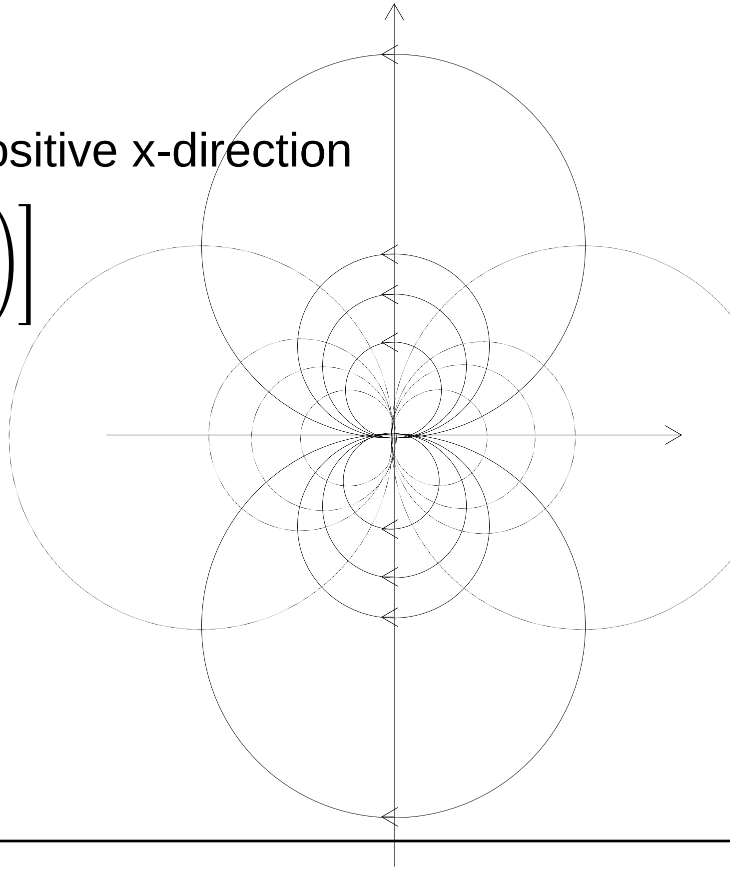
When distance 2s becomes zero a new basic flow element is produced:

Doublet or dipole producing flow in positive x-direction

$$\Psi = \lim_{s\to 0}\left[\frac{Q}{2\pi}\cdot \arctan\left(\frac{2ys}{x^2+y^2-s^2}\right)\right]$$

$$\Psi = \mu \cdot \frac{y}{x^2+y^2} = \mu \cdot \frac{\sin\theta}{r}$$

$$\Phi = -\mu \cdot \frac{x}{x^2+y^2} = -\mu \cdot \frac{\cos\theta}{r}$$



Superposition

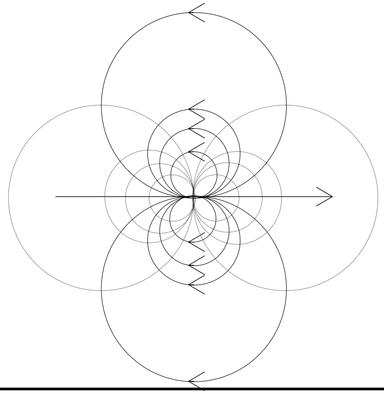
Doublet in a uniform flow

$$\Phi = -\mu \cdot \frac{x}{x^2 + y^2} - U_\infty x \qquad \Psi = \mu \cdot \frac{y}{x^2 + y^2} - U_\infty y$$

$$\Phi = -\mu \cdot \frac{\cos\theta}{r} - U_\infty r\cos\theta \qquad \Psi = \mu \cdot \frac{\sin\theta}{r} - U_\infty r\sin\theta$$

Wrong in book!

Doublet pointing in positive x-direction, uniform flow in negative x-direction:



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Superposition

Doublet in a uniform flow

$$\Phi = -\mu \cdot \frac{x}{x^2 + y^2} - U_\infty x \qquad \Psi = \mu \cdot \frac{y}{x^2 + y^2} - U_\infty y$$

$$\Phi = -\mu \cdot \frac{\cos\theta}{r} - U_\infty r \cos\theta \qquad \Psi = \mu \cdot \frac{\sin\theta}{r} - U_\infty r \sin\theta$$

Set $\Psi = 0$ then:

$$\Psi = y\left[\frac{\mu}{x^2 + y^2} - U_\infty\right] = 0$$

True when:

$$y = 0$$

$$\frac{\mu}{x^2 + y^2} - U_\infty = 0 \rightarrow x^2 + y^2 = \frac{\mu}{U_\infty}$$

Superposition

Doublet in a uniform flow: flow around a circle

- The radius of the circle:

$$R = \sqrt{\frac{\mu}{U_\infty}}$$

- Doublet strength needed for radius R:

$$\mu = U_\infty R^2$$

$$\Phi = -\mu \cdot \frac{\cos\theta}{r} - U_\infty r \cos\theta$$

$$\Psi = \mu \cdot \frac{\sin\theta}{r} - U_\infty r \sin\theta$$

- This yields the following:

$$\Phi = -\frac{U_\infty R^2 \cos\theta}{r} - U_\infty r \cos\theta = -R U_\infty \left[\frac{R}{r} + \frac{r}{R}\right] \cos\theta$$

$$\Psi = \frac{U_\infty R^2 \sin\theta}{r} - U_\infty r \sin\theta = R U_\infty \left[\frac{R}{r} - \frac{r}{R}\right] \sin\theta$$

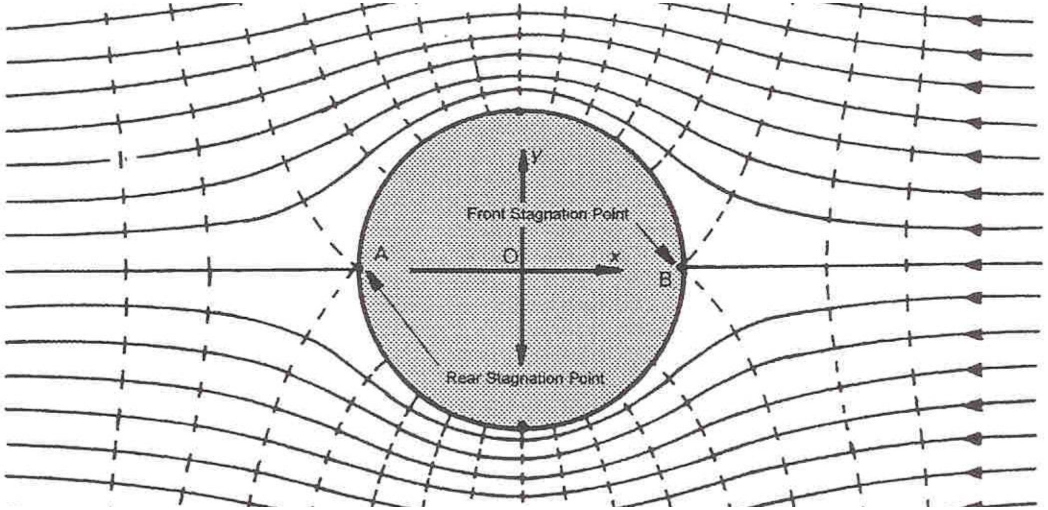
Superposition

Doublet in a uniform flow: flow around a circle

$$\Phi = -RU_\infty \left[\frac{R}{r} + \frac{r}{R}\right] \cos\theta$$

$$\Phi = -U_\infty R^2 \cdot \frac{x}{x^2 + y^2} - U_\infty x$$

$$u = \frac{d\Phi}{dx} = U_\infty R^2 \frac{x^2 - y^2}{(x^2 + y^2)^2} - U_\infty$$



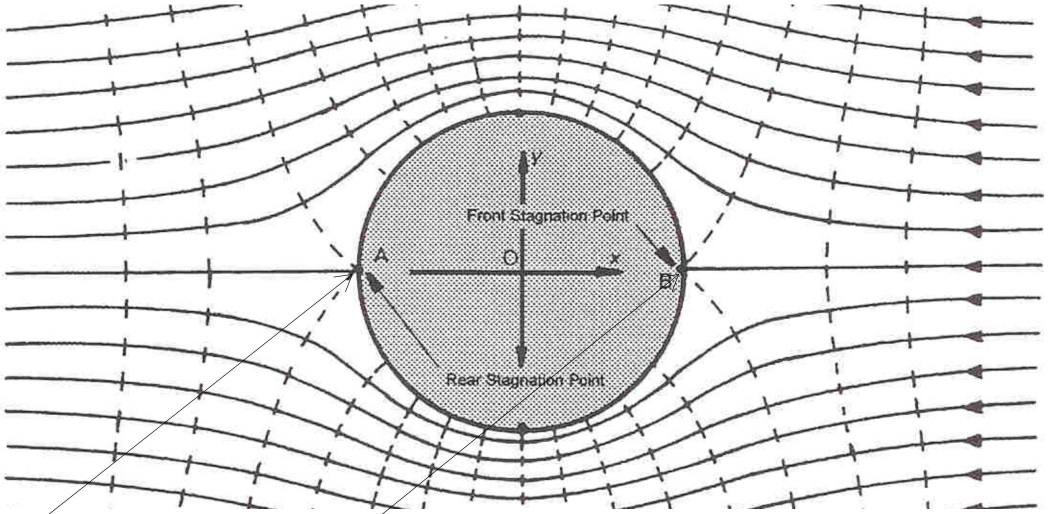
Superposition

Doublet in a uniform flow: flow around a circle

$$\Phi = -RU_\infty\left[\frac{R}{r} + \frac{r}{R}\right]\cos\theta$$

$$\Phi = -U_\infty R^2 \cdot \frac{x}{x^2 + y^2} - U_\infty x$$

$$u = \frac{d\Phi}{dx} = U_\infty R^2 \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} - U_\infty$$



$$x = \pm R, \qquad y = 0$$

$$u = \frac{d\Phi}{dx} = U_\infty R^2 \frac{R^2 - 0^2}{\left(R^2 + 0^2\right)^2} - U_\infty = U_\infty R^2 \frac{R^2}{R^4} - U_\infty = 0$$

Stagnation points!

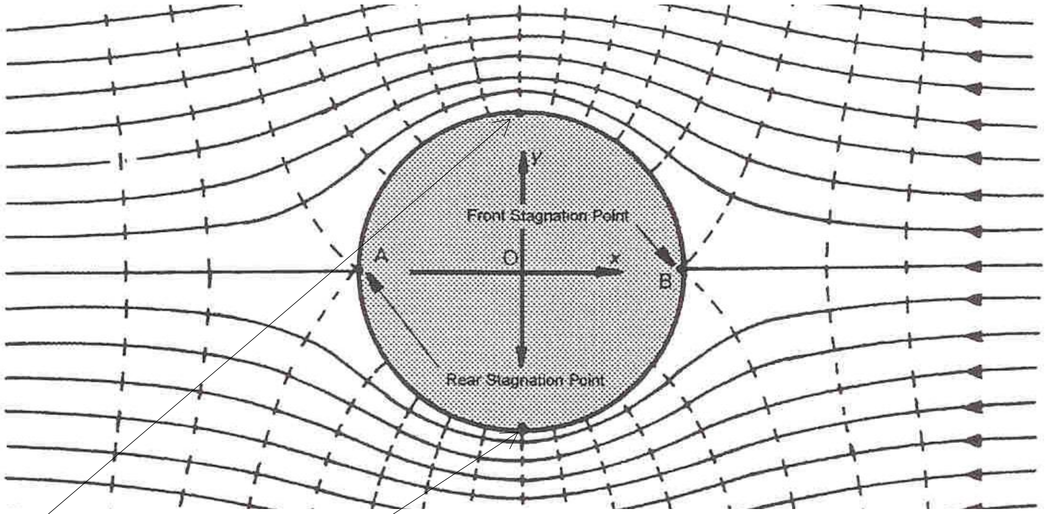
Superposition

Doublet in a uniform flow: flow around a circle

$$\Phi = -RU_\infty\left[\frac{R}{r}+\frac{r}{R}\right]\cos\theta$$

$$\Phi = -U_\infty R^2 \cdot \frac{x}{x^2+y^2} - U_\infty x$$

$$u = \frac{d\Phi}{dx} = U_\infty R^2 \frac{x^2-y^2}{(x^2+y^2)^2} - U_\infty$$



$$x = 0, \qquad y = \pm R$$

$$u = \frac{d\Phi}{dx} = U_\infty R^2 \frac{0^2-R^2}{(0^2+R^2)^2} - U_\infty = -U_\infty R^2 \frac{R^2}{R^4} - U_\infty = -2U_\infty$$

Superposition

Evaluate velocities on cylinder wall

$$\Psi = \mu \cdot \frac{\sin\theta}{r} - U_\infty r \sin\theta$$

- Generally, velocity on cylinder wall:

$$v_\theta(r = R) = -\left[\frac{\partial \Psi}{\partial r}\right]_{r=R} = -\frac{\partial}{\partial r}\left[\frac{U_\infty R^2 \sin\theta}{r} - U_\infty r \sin\theta\right] = \ldots = 2U_\infty \sin\theta$$

Superposition

Evaluate pressures on cylinder wall

- Use the Bernoulli equation:

$$\frac{1}{2}\rho U_\infty^2 + 0 = p + \frac{1}{2}\rho v_\theta^2$$

$$v_\theta = -2U_\infty \sin\theta$$

Pressure at stagnation points:

$$v = 0$$

Pressure at cylinder boundary:

$$v_r = 0$$

Assuming constant elevation

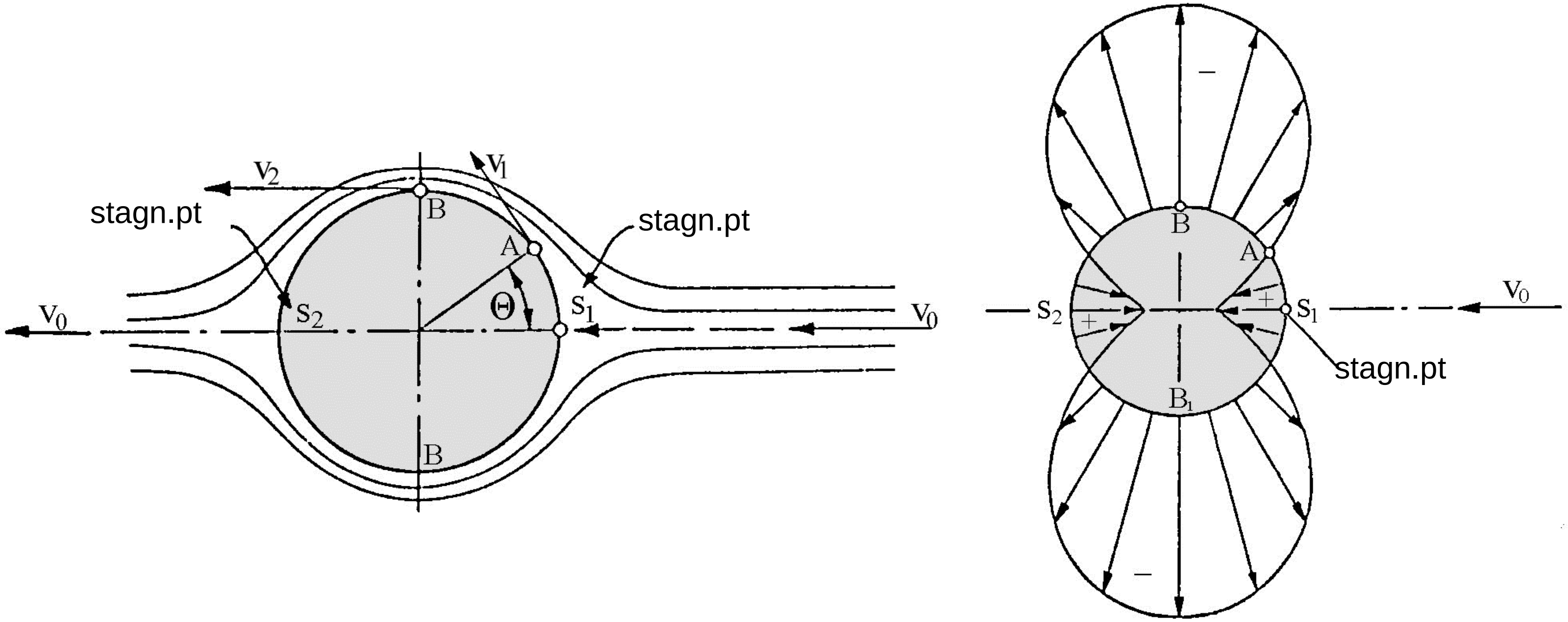
- Result:

$$p = \frac{1}{2}\rho U_\infty^2 [1 - 4\sin^2\theta]$$

Superposition

Evaluate pressures on cylinder wall

- Velocity profile: $$v_\theta = -2U_\infty \sin\theta$$
- Pressure profile: $$p = \frac{1}{2}\rho U_\infty^2[1 - 4\sin^2\theta]$$



Superposition

Evaluate pressures on cylinder wall

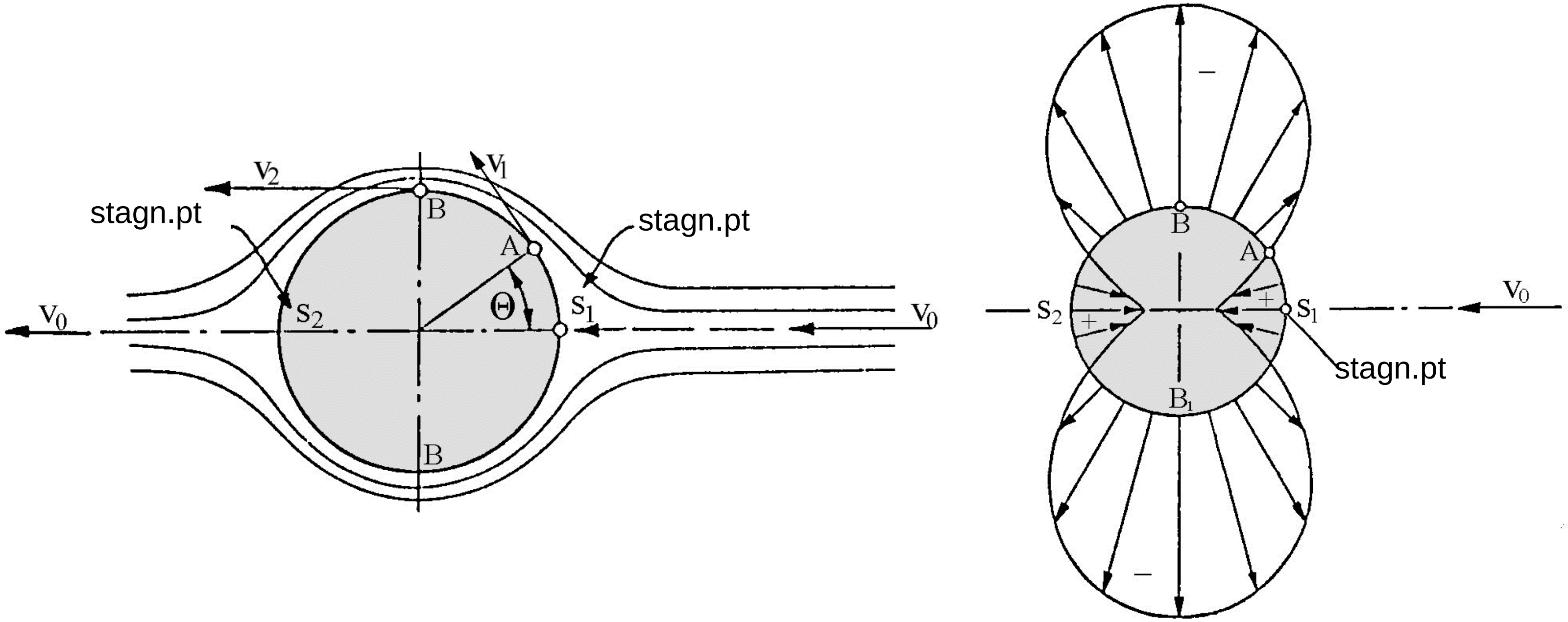
- Velocity profile:

$$v_\theta = -2U_\infty \sin\theta$$

- Pressure profile:

$$p = \frac{1}{2}\rho U_\infty^2[1 - 4\sin^2\theta]$$

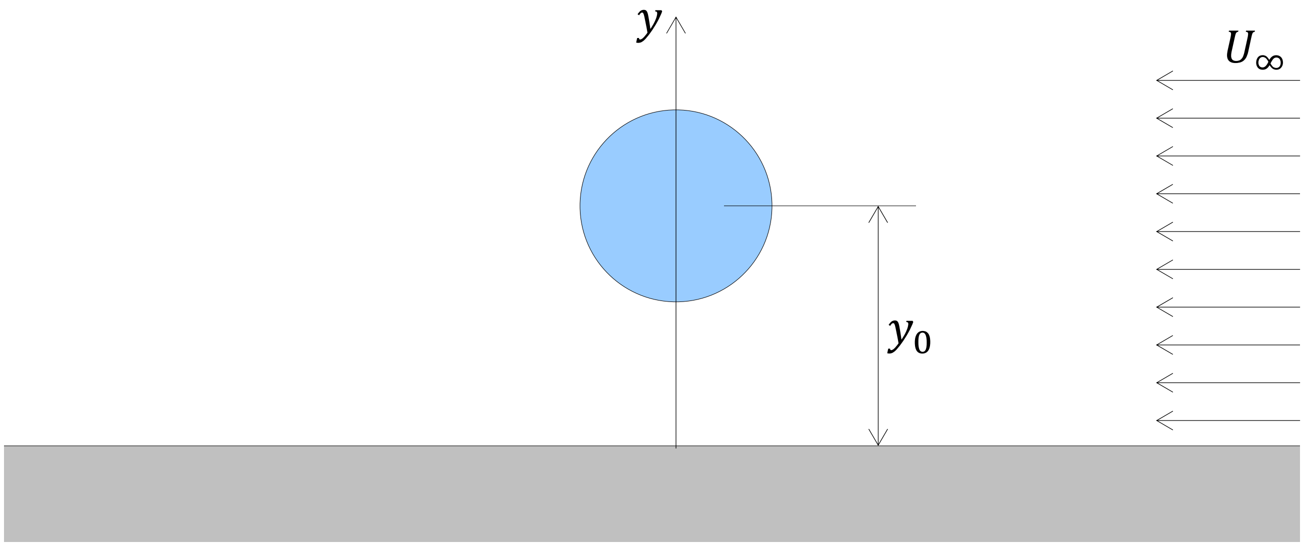
- **No net resulting force!!!**
- **D'Alembert's Paradox**



Superposition

Pipeline near seabed

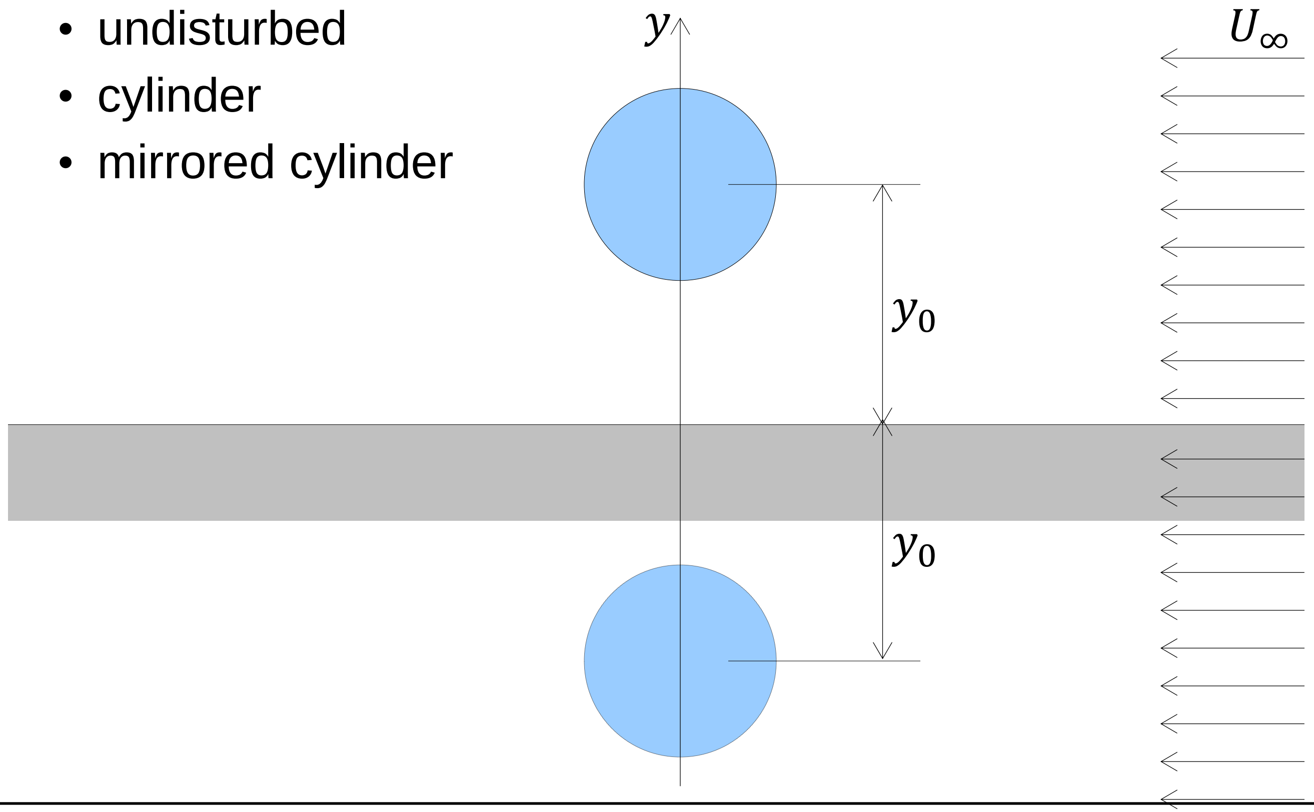
- How to calculate the flow around a pipeline near the seabed?



Superposition

Pipeline near seabed

- Mirror flow in seabed!
- Superpose flows:
  - undisturbed
  - cylinder
  - mirrored cylinder

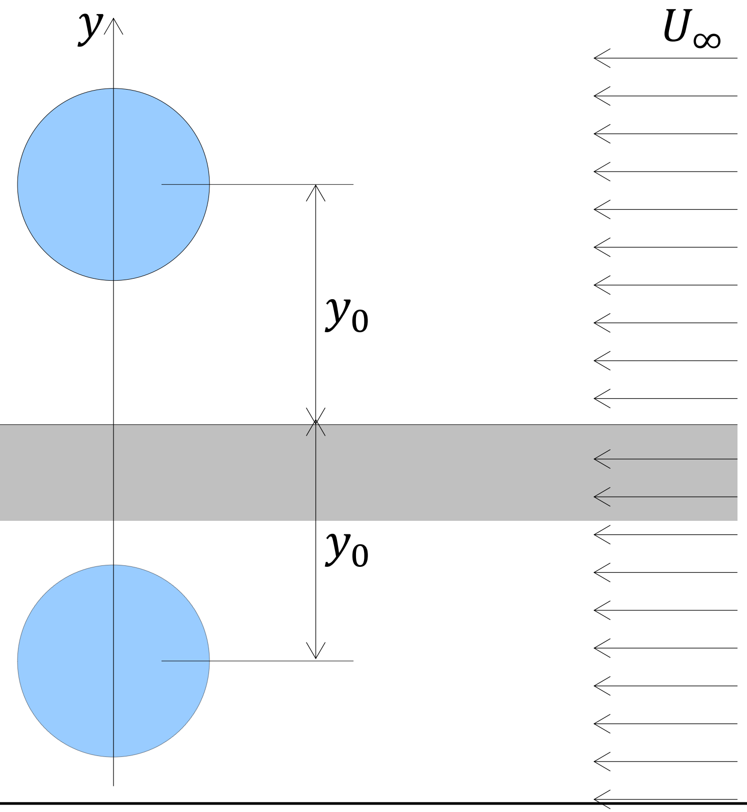


Superposition

Pipeline near seabed

- Mirror flow in seabed!
- Superpose flows:
  - undisturbed
  - cylinder
  - mirrored cylinder

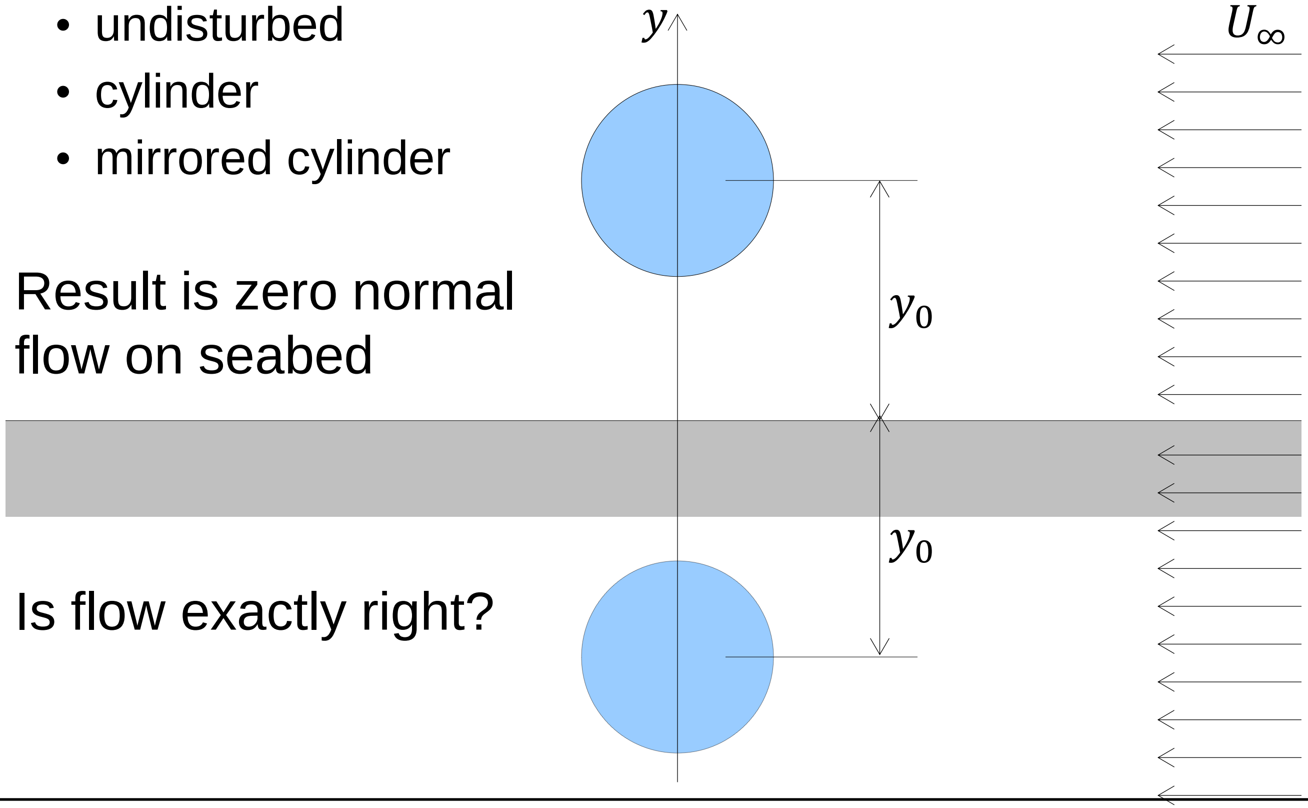
- Result is zero normal flow on seabed



Superposition

Pipeline near seabed

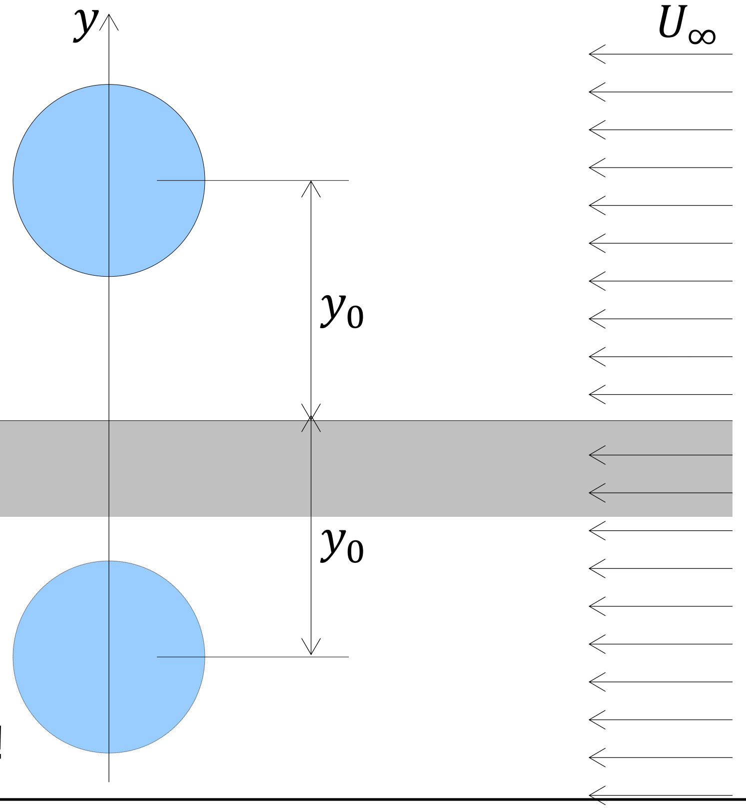
- Mirror flow in seabed!
- Superpose flows:
  - undisturbed
  - cylinder
  - mirrored cylinder
- Result is zero normal flow on seabed
- Is flow exactly right?



Superposition

Pipeline near seabed

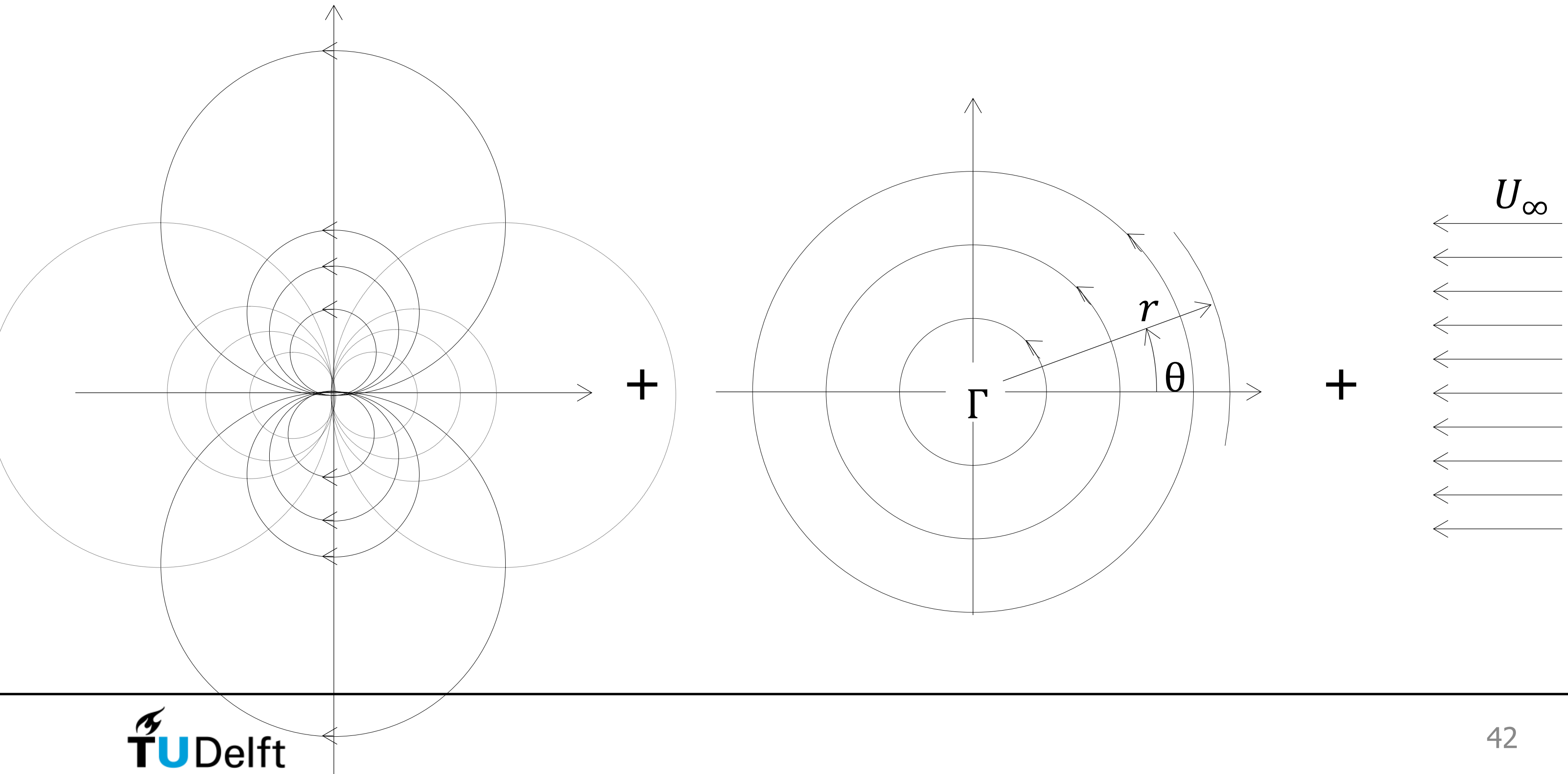
- Mirror flow in seabed!
- Superpose flows:
  - undisturbed
  - cylinder
  - mirrored cylinder
- Result is zero normal flow on seabed
- Is flow exactly right?
  NO: interaction
  cylinders not modeled!



Superposition

Circulation

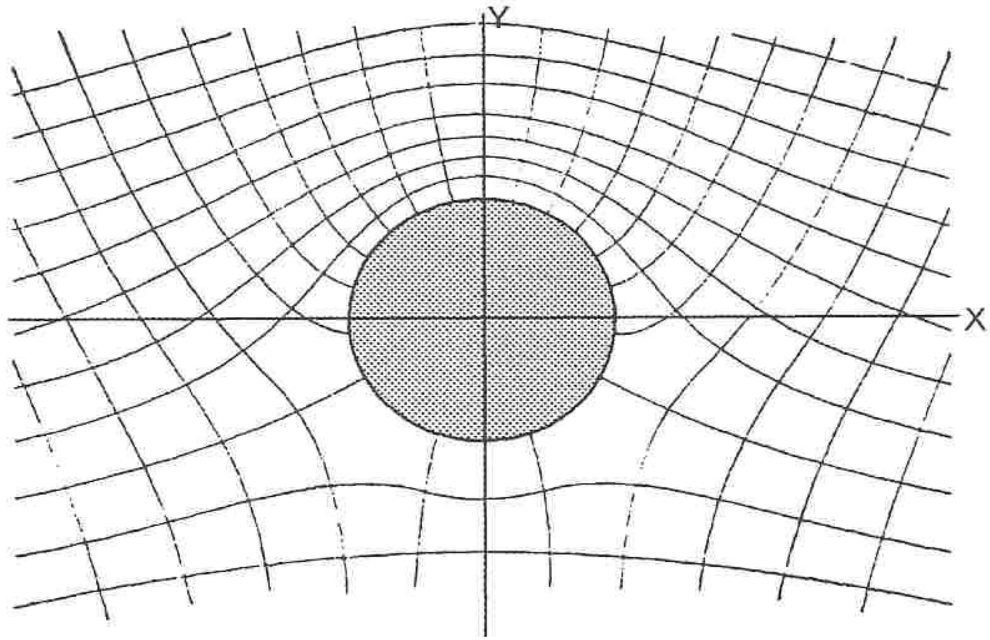
- Add circulation (or vortex flow element) to doublet in uniform flow
- Resulting velocity field:



Superposition

Circulation

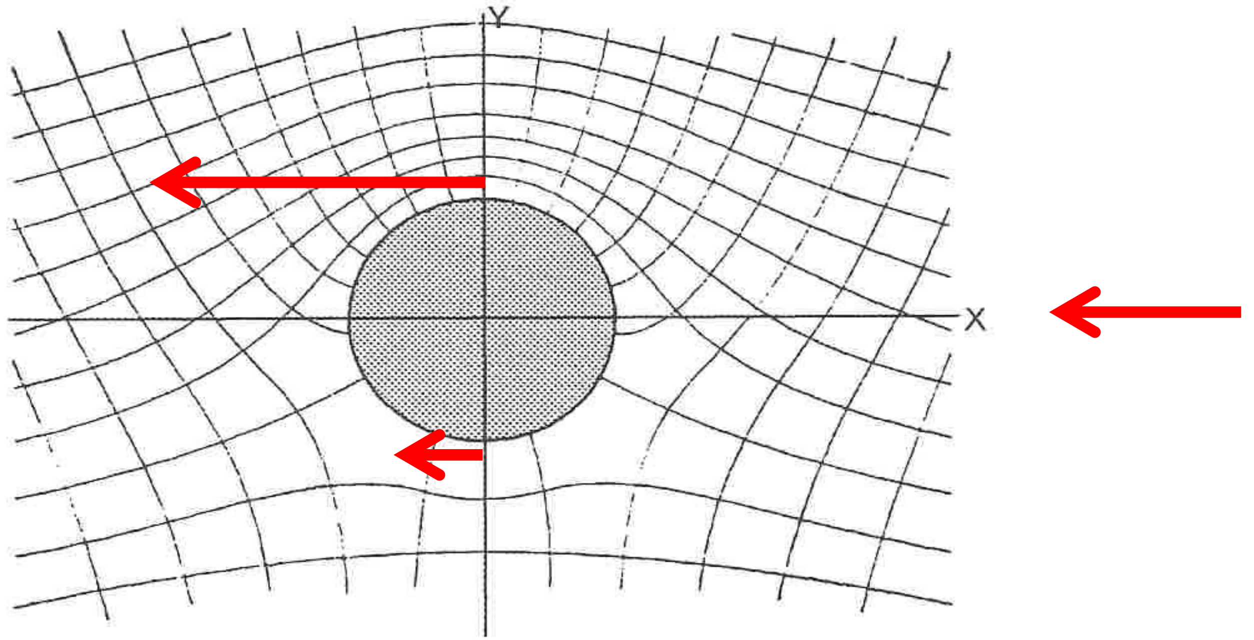
- Add circulation (or vortex flow element)
- Resulting velocity field:



Superposition

Circulation

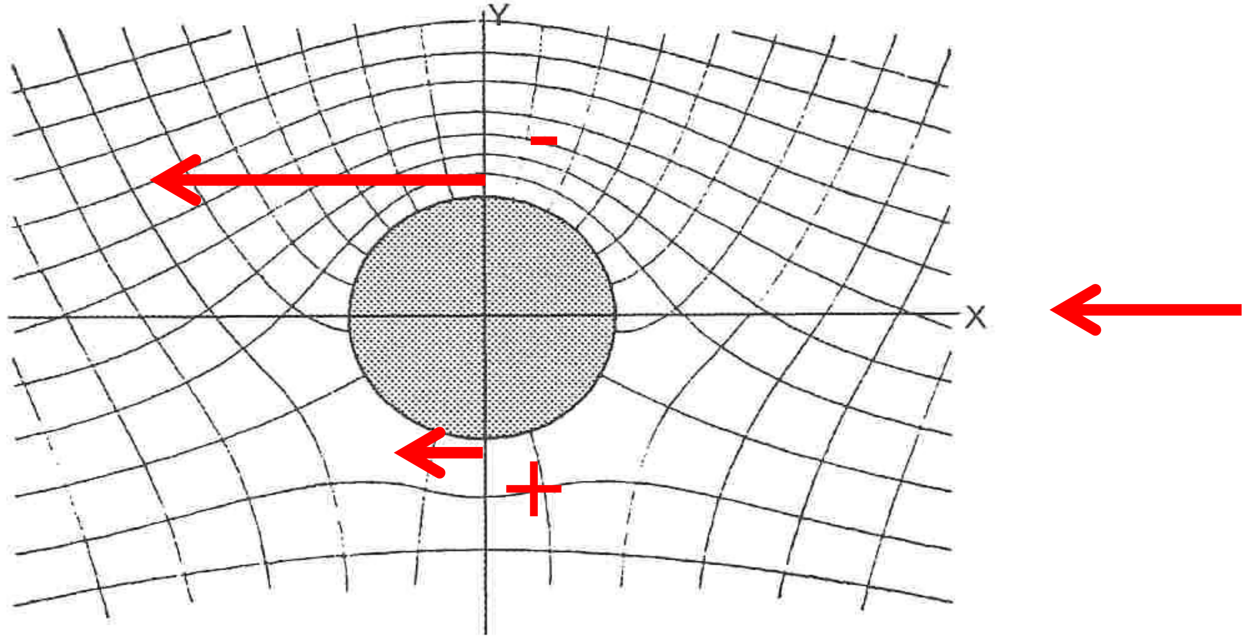
- Add circulation (or vortex flow element)
- Resulting velocity field:



Superposition

Circulation

- Add circulation (or vortex flow element)
- Resulting velocity field:

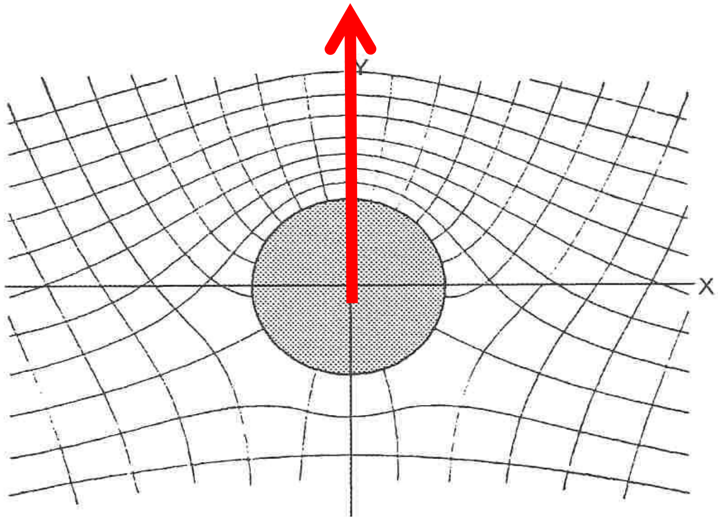


Superposition

Circulation

- Now integration of pressure yields a net force perpendicular to the undisturbed flow direction: the **lift** force

- However: still no net force in the flow direction: no **drag**



Sources images

All images are from the book *Offshore Hydromechanics* by Journée and Massie.