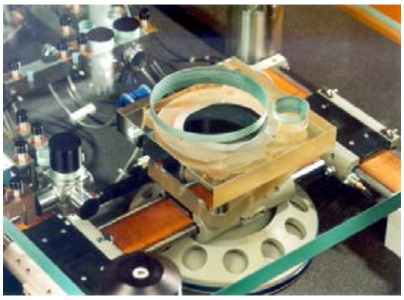
Mechatronic system design

Mechatronic system design wb2414-2013/2014
Course part 4



Dynamics of motion systems

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Mechatronic System Design

Contents

- Stiffness in Precision Engineering
  - Passive and active stiffness
- Compliance of (a combination of) dynamic elements
- Dynamic modelling of damped mass-spring systems.
- Transmissibility
- Coupled mass-spring systems
- Eigenmodes, eigenfrequencies and modeshapes
- Standard mechanical frequency responses

Stiffness of objects

	Well known objects
10^2 N/m	Soft pillow
10^4 N/m	Car suspension Soft couch
10^5 N/m	Table Bicycle
10^7 N/m	Office building
10^8 N/m	Concrete pillar
10^9 N/m	Steel train wheel on steel rail track

What is stiffness?

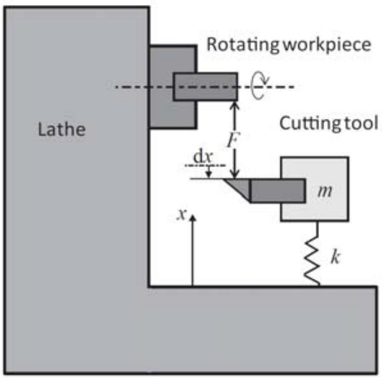
Hooke's law for force from spring:

$$F_s = -k\mathrm{d}x$$

Hooke-Newton law for external force:

$$F_r = F = k\mathrm{d}x$$

$$\mathrm{d}x = \frac{F}{k}$$

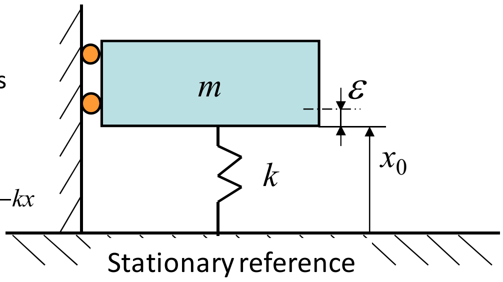


Where should you place the stiffness if possible?

➔ Take the shortest force loop

Natural frequency of the resonance of a mass-spring system

- At resonance the forces are in balance
- Deformation force (stiffness) plus acceleration force (mass) is zero.



$$F_a + F_d = m\frac{d^2x}{dt^2} + kx = 0 \quad \Rightarrow \quad m\frac{d^2x}{dt^2} = -kx$$

$$x = \hat{x}\sin(\omega_0 t) \;\Rightarrow$$

$$-m\hat{x}\omega_0^2 \sin(\omega_0 t) = -k\hat{x}\sin(\omega_0 t) \;\Rightarrow$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

The first natural frequency determines the sensitivity to harmonic vibrations!

- The maximum force needed to follow the acceleration:

$$\hat{F} = m\hat{a} = m\hat{x}_f\omega^2$$

- The maximum error due to this force:

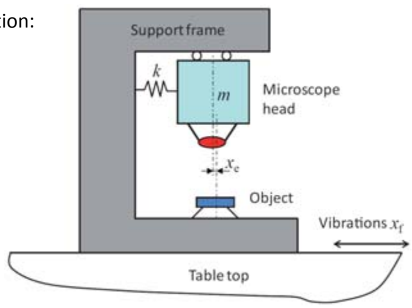
$$\hat{x}_e = \frac{\hat{F}}{k} = \frac{m.\hat{x}_f.\omega^2}{k}$$

- The natural frequency:

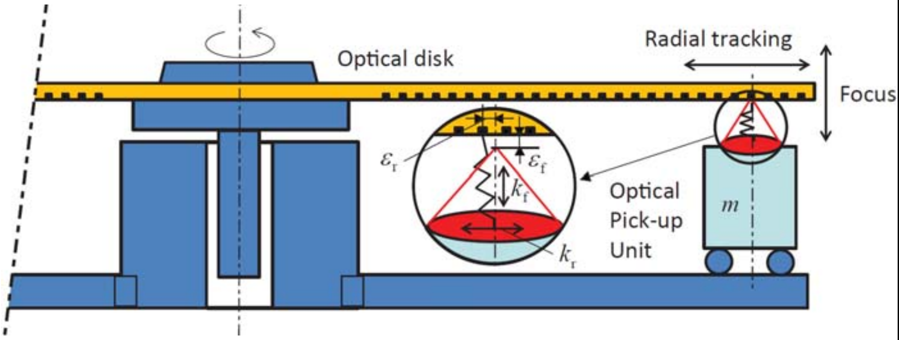
$$\frac{k}{m} = \omega_0^2$$

- Which results in:

$$\hat{x}_e = \hat{x}_f\frac{\omega^2}{\omega_0^2} \;\Rightarrow\; \frac{\omega_0}{\omega} = \frac{f_0}{f} = \sqrt{\frac{\hat{x}_f}{\hat{x}_e}} \;\Rightarrow\; f_0 \geq f\sqrt{\frac{\hat{x}_f}{\hat{x}_e}}$$



Active stiffness in a CD player, bandwidth

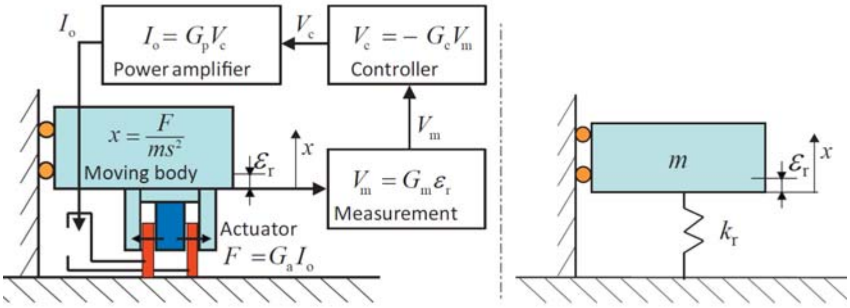


$$f_0 \ge f\sqrt{\frac{\hat{x}_r}{\varepsilon_r}} = 25\sqrt{\frac{200\cdot10^{-6}}{0,2\cdot10^{-6}}} = 800Hz$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k_r}{m}} \Rightarrow k_r = 4\pi^2 m f_0^2 = 2.5\cdot10^5 \text{ N/m}$$

- 200 µm radial vibrations at 25 Hz
- Mass lens: 10 $\cdot 10^{-3}$ kg
- Max radial error: 0,2 µm

Virtual stiffness



- Measure position
- Actuate with force proportional and opposite to the deviation (feedback!)

$$F = G_t \varepsilon_r = -G_m G_a G_p G_c \varepsilon_r$$

- Gives virtual spring stiffness $$k_r = G_t = \frac{F}{\varepsilon_r}$$

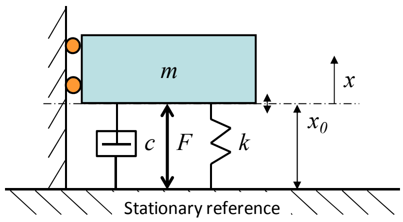
Contents

Stiffness and compliance

- Stiffness it the ability of a system to withstand a force by minimising the resulting motion/deformation
- Compliance is the opposite
- Both can be real, in phase with a periodic force, or complex, dynamic, frequency dependent, 90° out of phase with a periodic force.
- A spring has a real stiffness/compliance:

$$C_s = \frac{x}{F} = \frac{1}{k}$$

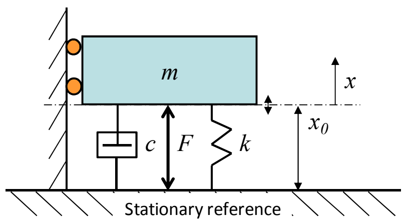
Compliance of (a combination of) dynamic elements



$$C_s = \frac{x}{F} = \frac{1}{k}$$

- k = stiffness of the spring
- c = damping coefficient of the damper
- m = mass of the body

Stiffness and compliance of a damper

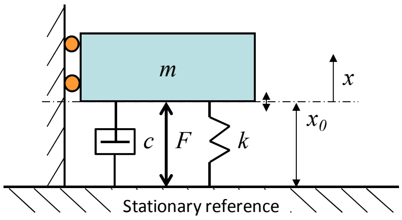


$$F(t) = c\frac{\mathrm{d}x}{\mathrm{d}t}, \quad F(s) = \mathcal{L}\{F(t)\} = scx$$

$$F(\omega) = \mathcal{F}\{F(t)\} = jc\omega x$$

$$C_\mathrm{d}(\omega) = \frac{1}{k_\mathrm{d}(\omega)} = \frac{x}{F} = \frac{1}{jc\omega}$$

Stiffness and compliance of a body

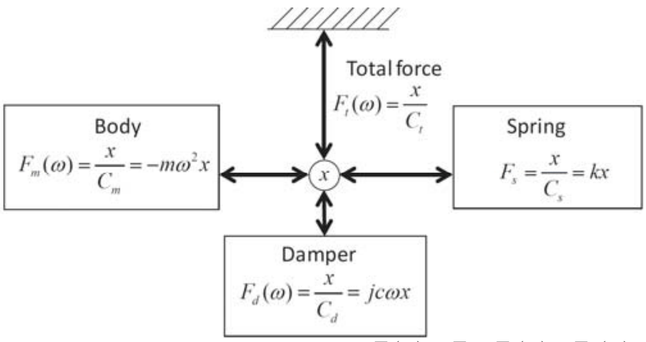


$$F(t) = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \quad F(s) = \mathcal{L}\{F(t)\} = ms^2x$$

$$F(\omega) = \mathcal{F}\{F(t)\} = -m\omega^2 x$$

$$C_\mathrm{m}(\omega) = \frac{1}{k_\mathrm{m}(\omega)} = \frac{x}{F} = -\frac{1}{m\omega^2}$$

Combined Compliance of body, spring and damper



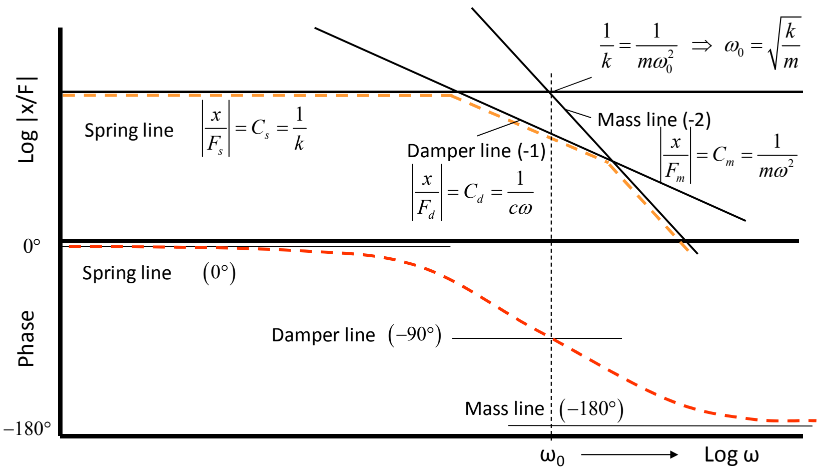
$$F_\mathrm{t}(\omega) = F_\mathrm{s} + F_\mathrm{d}(\omega) + F_\mathrm{m}(\omega) = x\left(\frac{1}{C_\mathrm{s}} + \frac{1}{C_\mathrm{d}} + \frac{1}{C_\mathrm{m}}\right) = \frac{x}{C_\mathrm{t}}$$

$$C_\mathrm{t}(\omega) = \frac{x}{F_\mathrm{t}}(\omega) = \frac{1}{\frac{1}{C_\mathrm{s}} + \frac{1}{C_\mathrm{d}} + \frac{1}{C_\mathrm{m}}}$$

Overview of the dynamic properties

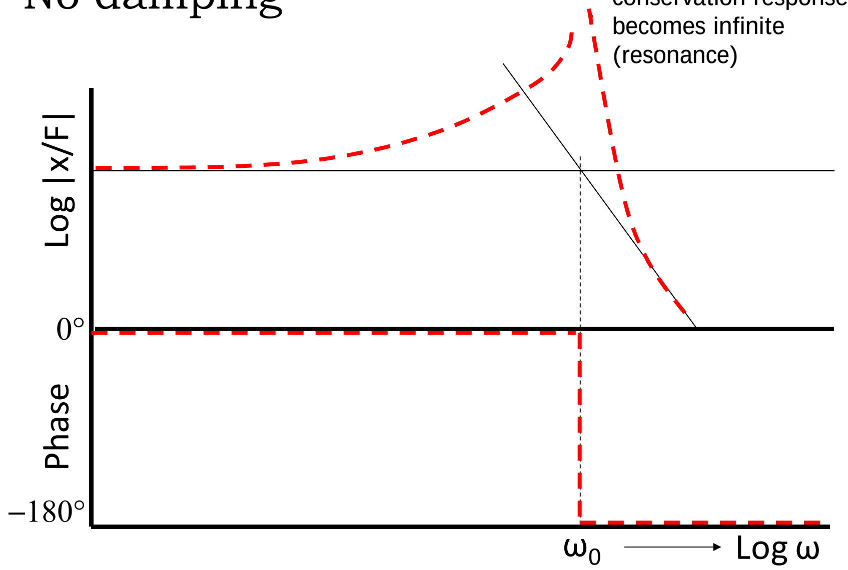
Item	Spring	Damper	Body
Variable	k	c	m
External force	$F_s(\omega) = kx$	$F_d(\omega) = jc\omega x$	$F_b(\omega) = -m\omega^2 x$
(dynamic) Stiffness	$k_s(\omega) = \frac{F_s}{x} = k$	$k_d(\omega) = \frac{F_d}{x}(\omega) = jc\omega$	$k_m(\omega) = \frac{F_b}{x}(\omega) = -m\omega^2$
(dynamic) Compliance	$C_s(\omega) = \frac{x}{F_s} = \frac{1}{k}$	$C_d(\omega) = \frac{x}{F_d}(\omega) = \frac{1}{jc\omega}$	$C_m(\omega) = \frac{x}{F_b}(\omega) = -\frac{1}{m\omega^2}$
Magnitude, Phase angle	$\frac{1}{k}$, 0°	$\frac{1}{c\omega}$, −90°	$\frac{1}{m\omega^2}$, −180°

The separate element responses in a bode plot

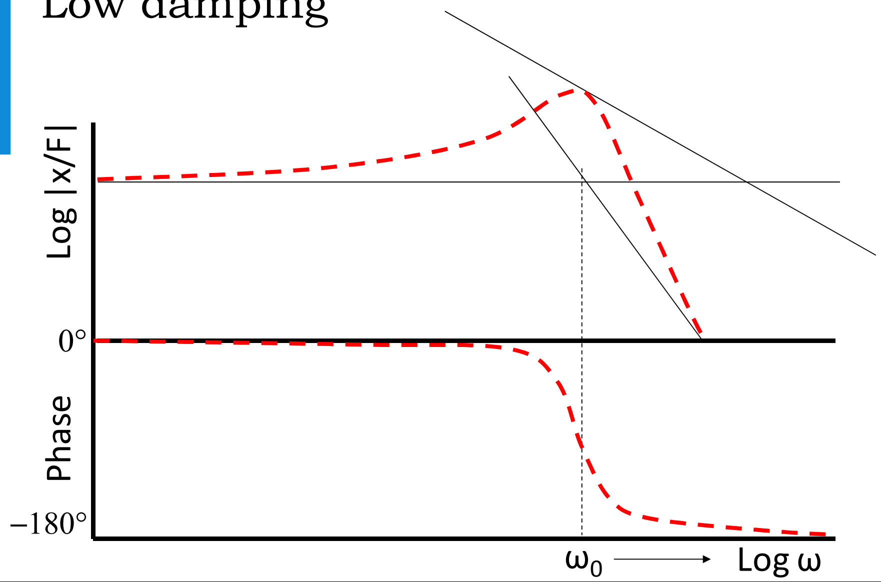


No damping

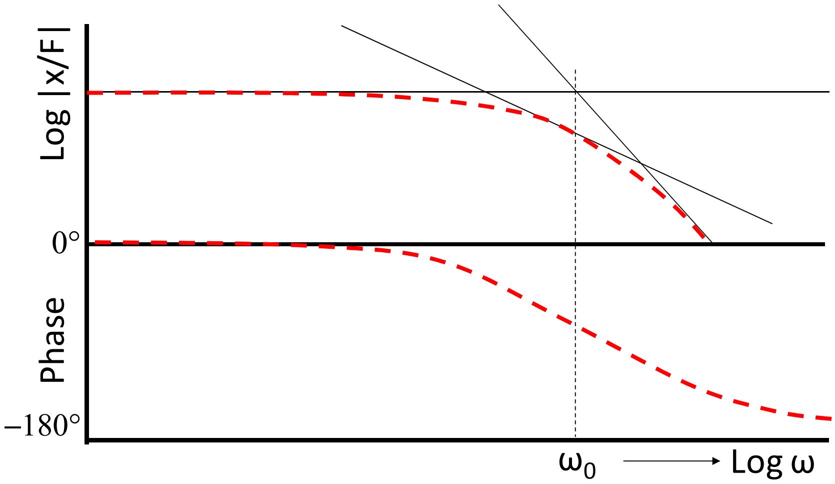
Due to energy conservation response becomes infinite (resonance)



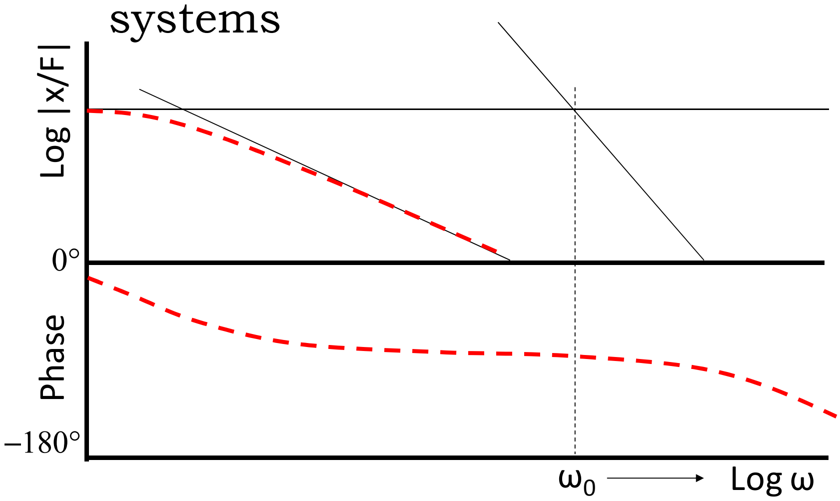
Low damping



High damping



Extreme damping, two first order systems



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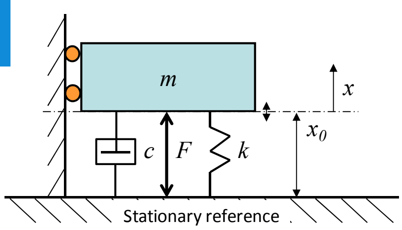
Start with second law of Newton

$F\!=\!m.a$



$$F(t) = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx)$$

Laplace gives:

$$F(s) = \mathcal{L}\{F(t)\} = x(ms^2 + cs + k)$$

$$C_t(s) = \frac{x_m}{F}(s) = \frac{1}{ms^2 + cs + k} = \frac{\frac{1}{k}}{\frac{m}{k}s^2 + \frac{cs}{k} + 1}$$

With only positive imaginary terms (Fourier):

$$\frac{1}{k} = C_s \qquad \omega_0 = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2\sqrt{km}}$$

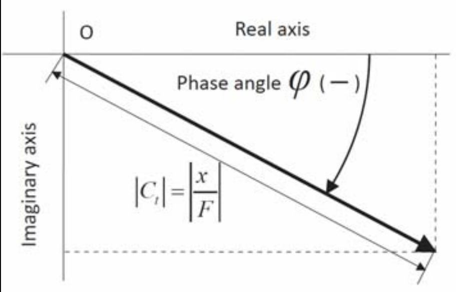
FRF

$$C_t(\omega) = \mathcal{F}\{F(t)\} = \frac{x}{F}(\omega) = \frac{C_s}{\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1}$$

The magnitude and the phase

Start with s=jω

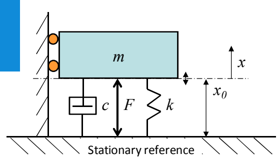
$$C_t(\omega) = \frac{x}{F} = \frac{C_s}{\frac{j^2\omega^2}{\omega_0^2} + j2\zeta\frac{\omega}{\omega_0} + 1} = \frac{C_s}{\underbrace{1 - \frac{\omega^2}{\omega_0^2}}_{\text{Real}} + \underbrace{j2\zeta\frac{\omega}{\omega_0}}_{\text{Imaginary}}} \quad [\text{m/N}]$$

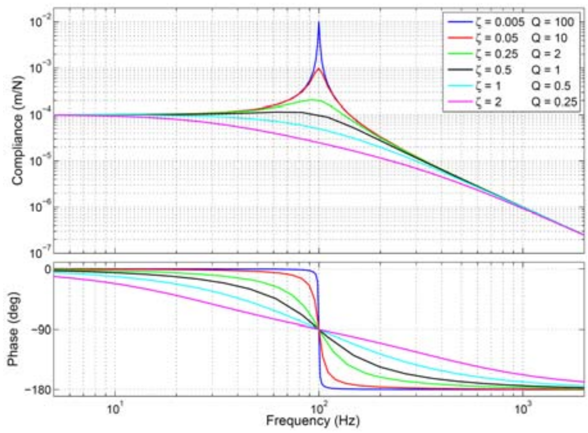


$$|C_t(\omega)| = \frac{C_s}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}}$$

And phase angle φ is – arctan (imaginary/real of the denominator)

Compliance Bode plot with damping





- $k = 10^4$ N/m
- $m = 0.25$ kg

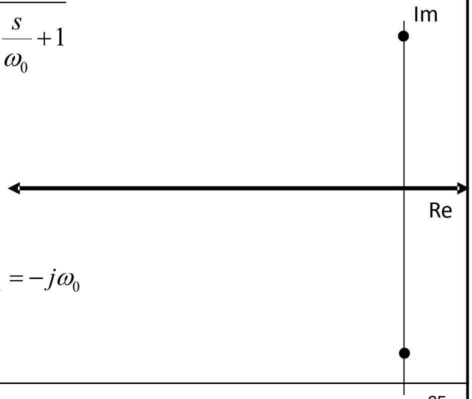
The damping ratio ζ is related to the pole location in the Laplace plane

$$s = \sigma + j\omega$$

Poles are those values of s where denominator of C_t is zero

$$C_t = \frac{x}{F} = \frac{\frac{1}{k}}{\frac{m}{k}s^2 + \frac{cs}{k} + 1} = \frac{C_s}{\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1}$$

$p_1 = -\sigma + j\omega_d$ and $p_2 = -\sigma - j\omega_d$



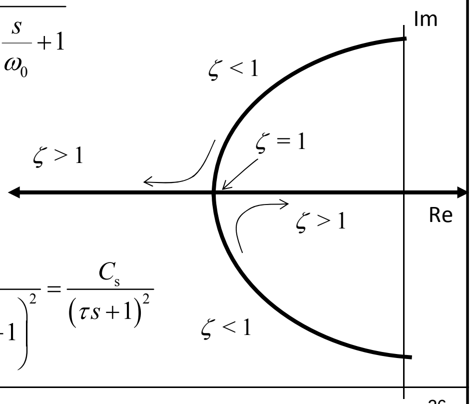
If $c = 0$ then $\zeta = 0$ no damping!

$C_t = \frac{C_s}{\frac{m}{k}s^2 + 1}$ and $p_1 = +j\omega_0$ and $p_2 = -j\omega_0$

Critical damping ratio $\zeta = 1$

$$C_t = \frac{x}{F} = \frac{\frac{1}{k}}{\frac{m}{k}s^2 + \frac{cs}{k} + 1} = \frac{C_s}{\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1}$$

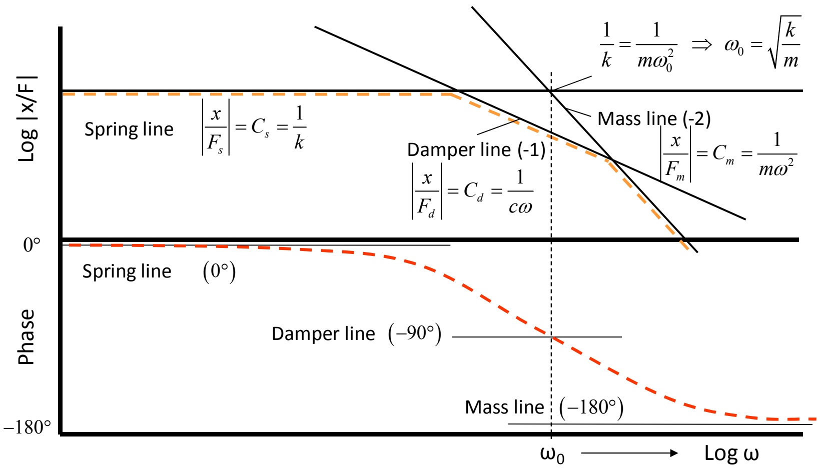
$p_1 = -\sigma + j\omega_d$ and $p_2 = -\sigma + j\omega_d$



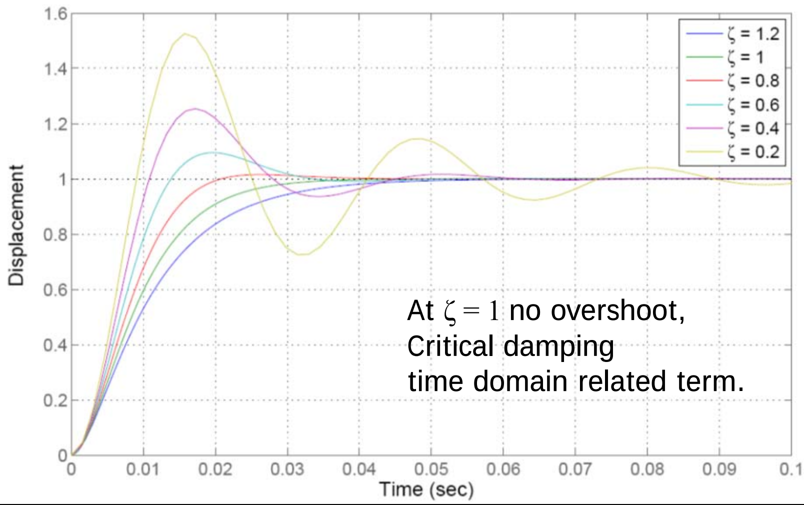
If $c = 2\sqrt{km}$ then $\zeta = 1$ and

$$C_t = \frac{\frac{1}{k}}{\frac{m}{k}s^2 + \frac{cs}{k} + 1} = \frac{C_s}{\frac{m}{k}s^2 + 2\sqrt{\frac{m}{k}}s + 1} = \frac{C_s}{\left(\sqrt{\frac{m}{k}}s + 1\right)^2} = \frac{C_s}{(\tau s + 1)^2}$$

Above critical damping the mass-spring system reduces to two first order systems



Effect of the damping ratio ζ on the stepresponse



At $\zeta = 1$ no overshoot,
Critical damping
time domain related term.

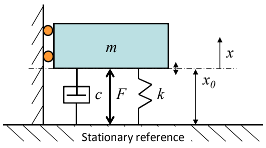
Energy at natural Frequency

Energy will be “trapped” in the system.
When excited in this frequency the amplitude will continue to rise.

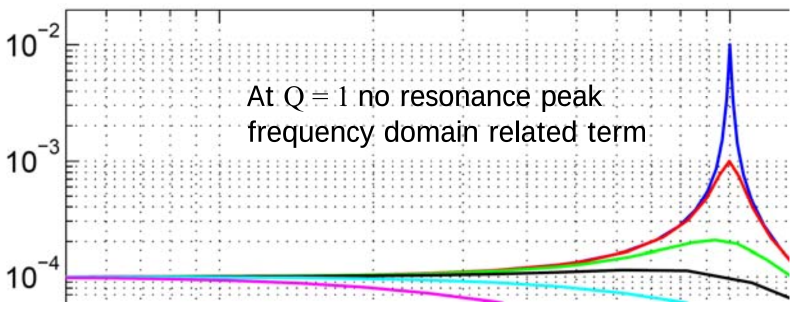
For electrical engineering (frequency domain) the quality factor Q is defined for this property as resonators are also useful. But in mechanical engineering the time domain related term “damping ratio” is more commonly used.

Let’s bridge the gap

Q=100 means peak level at 100 times spring-line level



- $k = 10^4$ N/m
- $m = 0.25$ kg



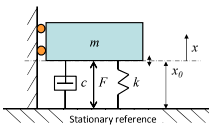
Relation Q and ζ

$$Q = \frac{1}{2\zeta} \qquad \zeta = \frac{1}{2Q}$$

Electrical vs mechanical, Time vs frequency domain

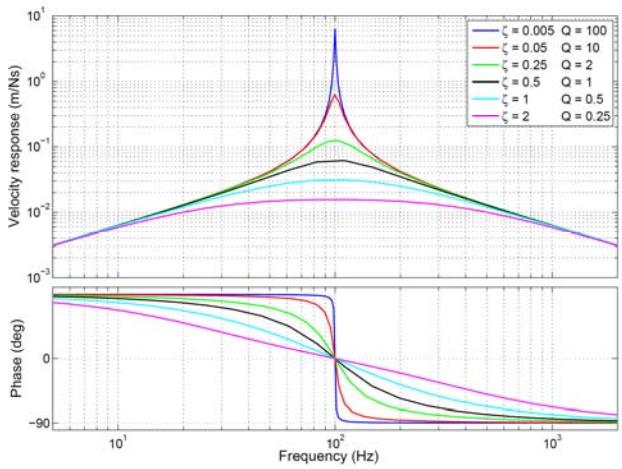
$$EE = \frac{1}{2ME} \qquad ME = \frac{1}{2EE}$$

Velocity response, kinetic energy



- $k = 10^4$ N/m
- $m = 0.25$ kg

$$\frac{v}{F} = \frac{{dx}/{dt}}{F} = \frac{sx}{F} = \frac{s \cdot C_s}{\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1}$$

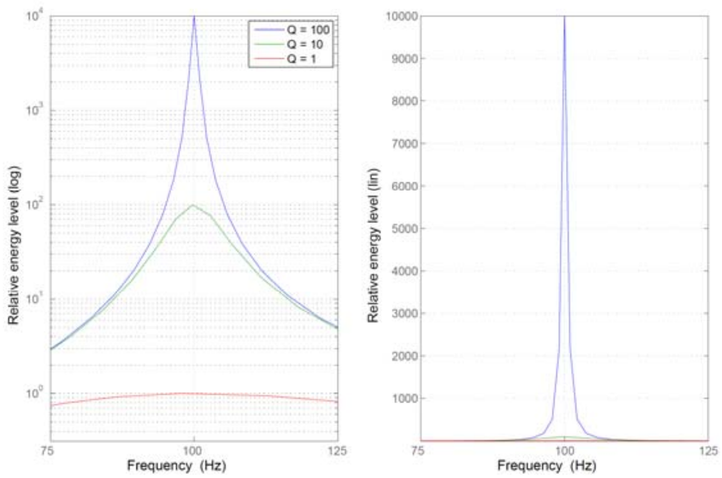


Resonance is energy storage, damping is velocity related

- Driving force is in phase with velocity at resonance
- Power is Force times speed
- Max energy transfer at resonance
- Damper is just the opposite
- Q tells something about the height of the resonance peak

$$Q = 2\pi \frac{\text{Maximum energy stored}}{\text{Energy lost per cycle}}$$

Q and energy



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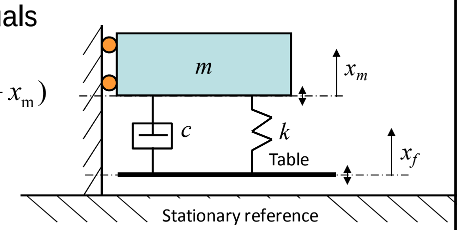
Transmissibility, transfer of motion through the support of a dynamic system

The force acting on the body equals

$$F_{\mathrm{t,m}}(t) = m\frac{\mathrm{d}^2 x_{\mathrm{m}}}{\mathrm{d}t^2} = c\frac{\mathrm{d}(x_{\mathrm{f}} - x_{\mathrm{m}})}{\mathrm{d}t} + k(x_{\mathrm{f}} - x_{\mathrm{m}})$$

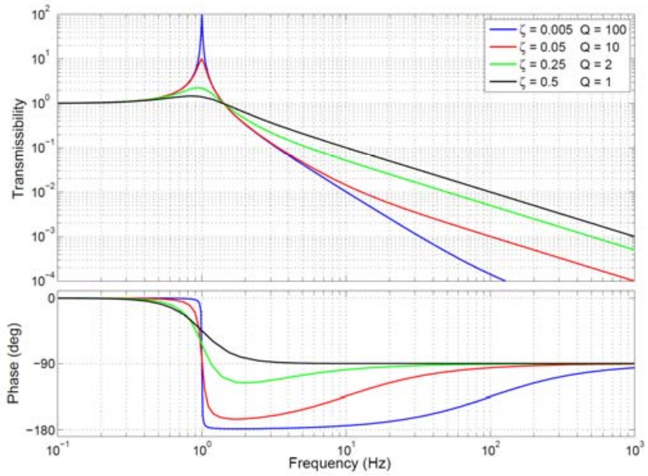


$$\Rightarrow \quad x_{\mathrm{m}}(ms^2 + cs + k) = x_{\mathrm{f}}(cs + k)$$

$$\frac{x_m}{x_f} = \frac{cs + k}{ms^2 + cs + k} = \frac{\frac{cs}{k} + 1}{\frac{m}{k}s^2 + \frac{cs}{k} + 1}$$

With: $\omega_0 = \sqrt{\frac{k}{m}}$ $\quad \zeta = \frac{c}{2\sqrt{km}}$ $\quad \Rightarrow \quad \frac{x_m}{x_f} = \frac{2\zeta\frac{s}{\omega_0} + 1}{\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1}$

Bode plot of transmissibility



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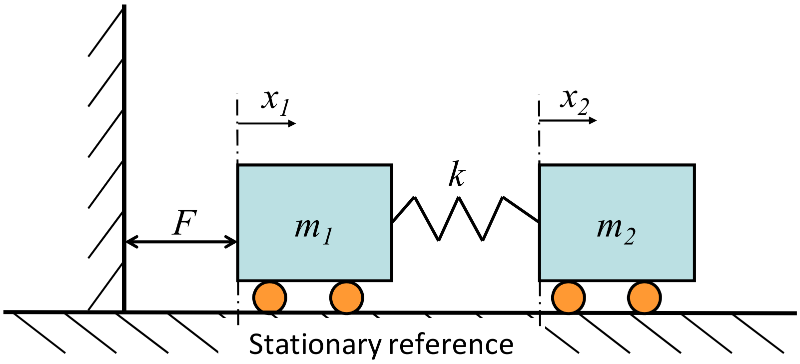
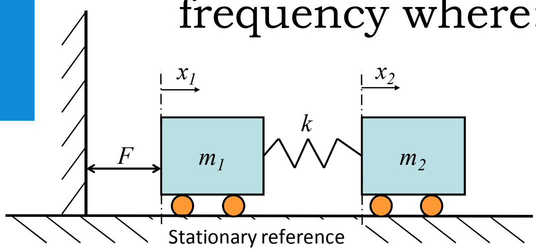
Coupled mass-spring systems



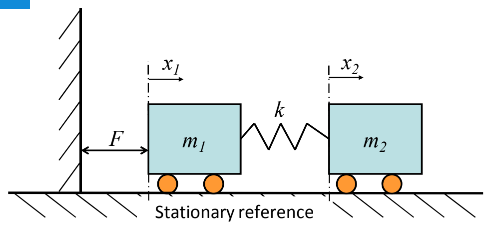
Elastically coupled Multi-body or finite-element non-rigid body dynamics

Which statement is true? There is a frequency where:



1. Only m_1 will resonate with k
2. Only m_2 will resonate with k
3. Only m_1 will stop moving
4. Only m_2 will stop moving

Equations of Motion



- First body

$$m_1 \frac{d^2 x_1(t)}{dt^2} = F(t) - k(x_1 - x_2)$$

$$m_1 s^2 x_1(s) = F - k(x_1 - x_2)$$

- Second body

$$m_2 \frac{d^2 x_2(t)}{dt^2} = k(x_1 - x_2)$$

$$m_2 s^2 x_2(s) = k(x_1 - x_2)$$

note: The domain (s or t) is only mentioned once!

Resulting equations in the Laplace domain

$$\frac{x_1}{F}(s) = \frac{m_2 s^2 + k}{m_1 m_2 s^4 + k\left(m_1 + m_2\right)s^2}$$

$$\frac{x_2}{F}(s) = \frac{k}{m_1 m_2 s^4 + k\left(m_1 + m_2\right)s^2}$$

Fourth order system

Low values of s at low frequencies:

$$\frac{x_1}{F}(s) = \frac{x_2}{F}(s) = \frac{1}{\left(m_1 + m_2\right)s^2}$$

High values of s at high frequencies:

$$\frac{x_1}{F}(s) = \frac{1}{m_1 s^2} \quad , \quad \frac{x_2}{F}(s) = \frac{k}{m_1 m_2 s^4}$$

At mid frequency the driven body shows a strange effect

$$\frac{x_1}{F} = \frac{m_2 s^2 + k}{m_1 m_2 s^4 + k\left(m_1 + m_2\right) s^2}$$

$$\frac{x_2}{F}(s) = \frac{k}{m_1 m_2 s^4 + k\left(m_1 + m_2\right) s^2}$$

Combination
of poles and zeros

when

$$s^2 = -\omega^2 = -\frac{k}{m_2}$$

then the compliance of x_1 shows a dip

m_1 = 0.1 m_2 , an actuated large mass by a lighter actuator.

LF: $\frac{x_1}{F}(s) = \frac{x_2}{F}(s) = \frac{1}{\left(m_1 + m_2\right) s^2}$

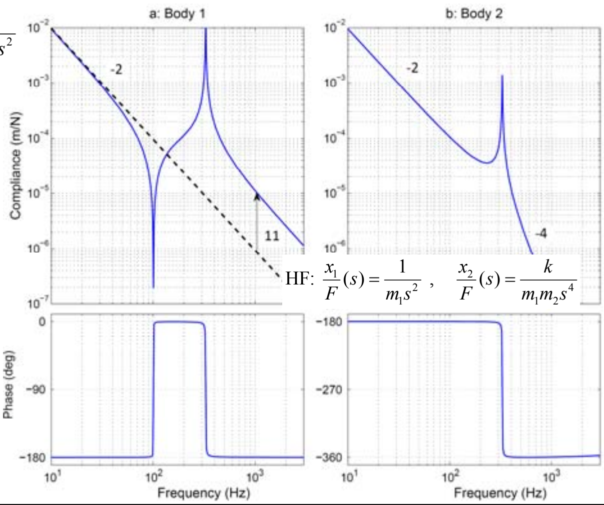
$$\frac{x_1}{F}(s) = \frac{m_2 s^2 + k}{m_1 m_2 s^4 + k\left(m_1 + m_2\right) s^2}$$

$$\frac{x_2}{F}(s) = \frac{k}{m_1 m_2 s^4 + k\left(m_1 + m_2\right) s^2}$$

$m_1 = 2.5 \cdot 10^{-3}$ kg

$m_2 = 25 \cdot 10^{-3}$ kg

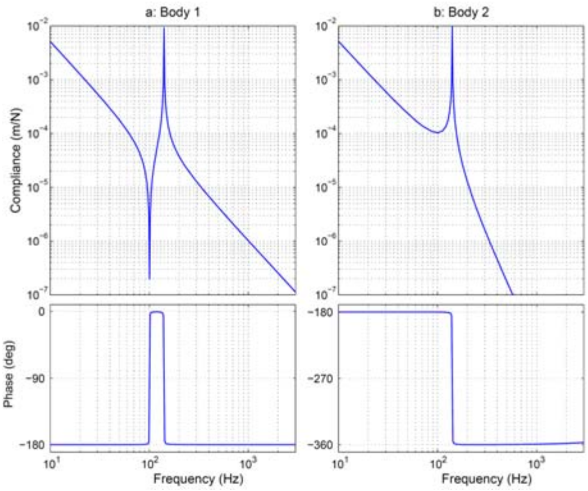
$k = 10^4$ N/m



$m_1 = m_2$, motor mass is optimised to the driven mass.

$$\frac{x_1}{F}(s) = \frac{m_2 s^2 + k}{m_1 m_2 s^4 + k(m_1 + m_2)s^2}$$

$$\frac{x_2}{F}(s) = \frac{k}{m_1 m_2 s^4 + k(m_1 + m_2)s^2}$$



$$m_1 = 2.5 \cdot 10^{-3} \quad \text{kg}$$

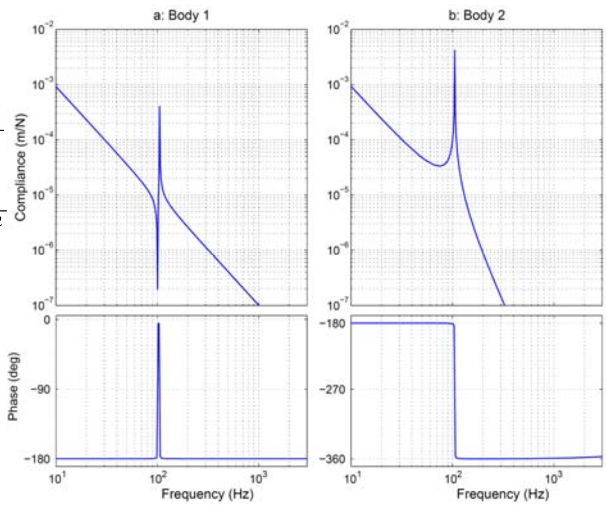
$$m_2 = 2.5 \cdot 10^{-3} \quad \text{kg}$$

$$k = 10^4 \quad \text{N/m}$$

$m_1 = 10\ m_2$, parasitic resonances

$$\frac{x_1}{F}(s) = \frac{m_2 s^2 + k}{m_1 m_2 s^4 + k(m_1 + m_2)s^2}$$

$$\frac{x_2}{F}(s) = \frac{k}{m_1 m_2 s^4 + k(m_1 + m_2)s^2}$$



$$m_1 = 0.25 \quad \text{kg}$$

$$m_2 = 25 \cdot 10^{-3} \quad \text{kg}$$

$$k = 10^4 \quad \text{N/m}$$

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Eigenmodes

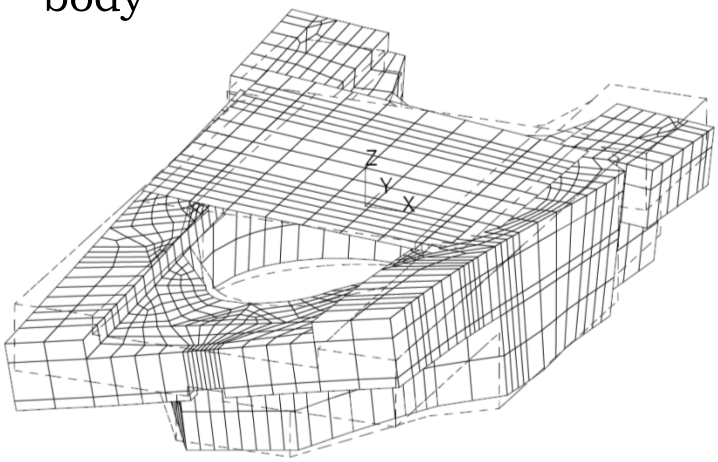
When excited, non-rigid bodies and complex mass-spring systems vibrate in different eigenmodes with two main properties:

- Eigenfrequency, the related natural (resonance) frequency.
- Mode-shape, the deformation that corresponds with the eigenmode, described in a multiple degree of freedom “shape-function ”.

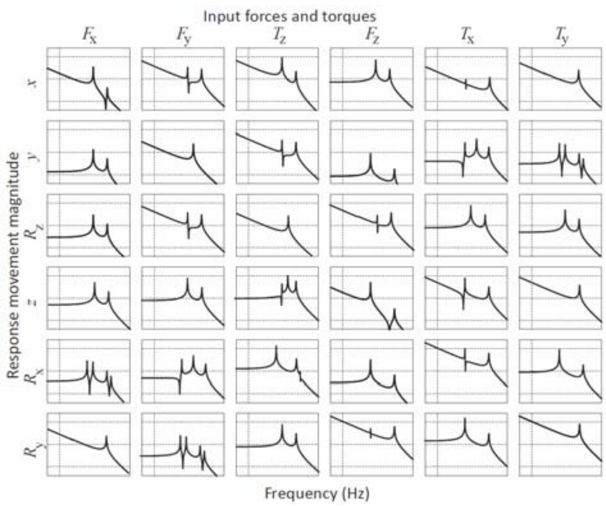
Modelling is done by discretisation of the system in multiple mass-spring systems.

- The shape function reduces to an eigenvector with one value for each body for the relative motion magnitude and direction (sign).

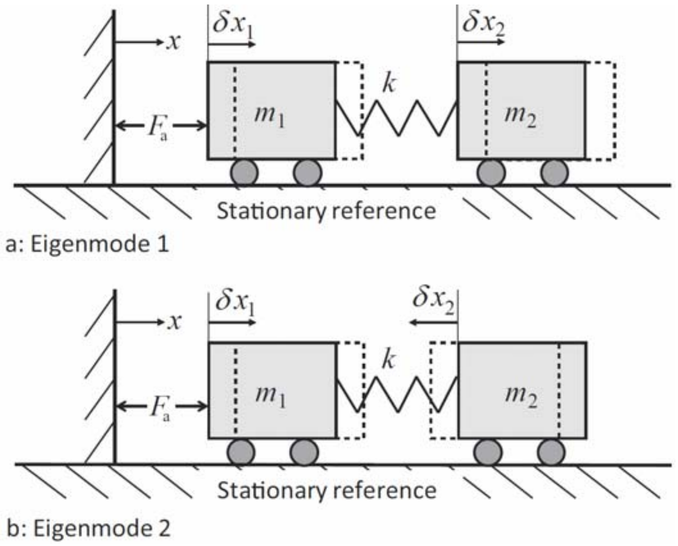
Mode-shape of complex non-rigid body



Eigenfrequencies of multiple eigenmodes in 6-DOF



Eigen modes of coupled bodies



$$\phi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Multiplicative expression shows two resonating eigenfrequencies

Starting with:

$$\frac{x_1}{F} = \frac{m_2 s^2 + k}{m_1 m_2 s^4 + k\left(m_1 + m_2\right)s^2} \qquad \frac{x_2}{F} = \frac{k}{m_1 m_2 s^4 + k\left(m_1 + m_2\right)s^2}$$

$$\frac{x_1}{F} = \frac{1}{\left(m_1 + m_2\right)s^2}\left(m_2 s^2 + k\right)\frac{1}{m_c s^2 + k}$$

with: $m_c = \frac{m_1 m_2}{m_1 + m_2}$

$$\frac{x_2}{F} = \frac{1}{\left(m_1 + m_2\right)s^2}\,\frac{k}{m_c s^2 + k}$$

$$\omega_1 = \sqrt{\frac{0}{m_1 + m_2}} = 0$$

$$\omega_2 = \sqrt{\frac{k}{M}}$$

Additive expression:

$$\frac{x_1}{F}=\frac{1}{\left(m_1+m_2\right)s^2}\left(m_2s^2+k\right)\frac{1}{m_cs^2+k} \qquad \text{with: } m_c=\frac{m_1m_2}{m_1+m_2}$$

$$\frac{x_2}{F}=\frac{1}{\left(m_1+m_2\right)s^2}\frac{k}{m_cs^2+k}$$

- Can be written as a combination of two eigenmodes with modal mass $\mathcal{M}_i$ and stiffness $\mathcal{K}_i$ using eigenvector ϕ_i :

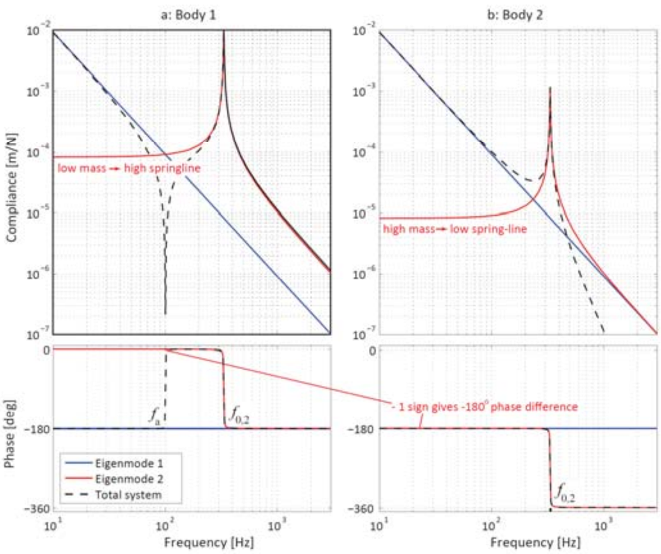
$$\frac{x_1}{F}=\frac{1}{(m_1+m_2)s^2}+\frac{\frac{m_2^2}{\left(m_1+m_2\right)^2}}{m_cs^2+k}=\frac{1}{\mathcal{M}_1s^2+\mathcal{K}_1}+\frac{\frac{m_2^2}{\left(m_1+m_2\right)^2}}{\mathcal{M}_2s^2+\mathcal{K}_2} \qquad \phi_1=\begin{bmatrix}1\\1\end{bmatrix}$$

$$\frac{x_2}{F}=\frac{1}{(m_1+m_2)s^2}+\frac{-\frac{m_1m_2}{\left(m_1+m_2\right)^2}}{m_cs^2+k}=\frac{1}{\mathcal{M}_1s^2+\mathcal{K}_1}+\frac{-\frac{m_1m_2}{\left(m_1+m_2\right)^2}}{\mathcal{M}_2s^2+\mathcal{K}_2} \qquad \phi_2=\begin{bmatrix}1\\-1\end{bmatrix}$$

Modes 1 and 2 combined

$$\frac{x_1}{F}=\frac{1}{\mathcal{M}_1s^2+\mathcal{K}_1}+\frac{\frac{m_2^2}{\left(m_1+m_2\right)^2}}{\mathcal{M}_2s^2+\mathcal{K}_2}$$

$$\frac{x_2}{F}=\frac{1}{\mathcal{M}_1s^2+\mathcal{K}_1}+\frac{-\frac{m_1m_2}{\left(m_1+m_2\right)^2}}{\mathcal{M}_2s^2+\mathcal{K}_2}$$



Theory on modal decomposition

- General equation of motion (vector/matrix) of finite element system:

$$M\ddot{x}(t) + Kx(t) = F(t)$$

- General transfer function:

$$Ms^2x(s) + Kx(s) = F(s)$$

- In absence of external force:

$$Ms^2x(s) + Kx(s) = 0$$

- Decoupled by eigenvalue problem, where ϕ_i is the eigenvector:

$$\left[K - \omega_{0,i}^2 M\right]\phi_i = 0$$

Modal mass and stiffness, orthogonality of eigenmodes.

- Modal mass:

$$\phi_i^T M \phi_j = 0 \quad (i \neq j)$$

$$\phi_i^T M \phi_i = \mathcal{M}_i$$

- Modal Stiffness:

$$\phi_i^T K \phi_j = 0 \quad (i \neq j)$$

$$\phi_i^T K \phi_i = \omega_{0,i}^2 \mathcal{M}_i = \mathcal{K}_i$$

Scaling

- Length of eigenvector is not defined (only the direction)

$$\phi_i^T M \phi_i = \mathcal{M}_i$$

- Three scaling methods are often applied:

$$1: \quad |\phi_i| = 1$$
$$2: \quad \max(\phi_i) = 1$$
$$3: \quad \mathcal{M}_i = 1$$

Modal coordinates

- When q_i equals the motion of eigenmode i then the total displacement vector $x(t)$ will be:

$$x(t) = q_1(t)\phi_1 + q_2(t)\phi_2 + \ldots q_n(t)\phi_n$$

where q_i is called the modal coordinate

- This gives the following displacement for DOF $x_k(t)$

$$x_k(t) = q_1(t)\phi_{1,k} + q_2(t)\phi_{2,k} + \ldots q_n(t)\phi_{n,k}$$

Full set of uncoupled equations

$x(t) = \Phi q(t)$

$$\text{with} : \Phi = \left[\phi_1 \; \phi_2 \; \ldots \; \phi_n\right] \text{ and } q(t) = \begin{bmatrix} q_1(t) \\ \vdots \\ q_i(t) \\ \vdots \\ q_n(t) \end{bmatrix}$$

applied to : $Ms^2x(s) + Kx(s) = F(s)$

gives with pre-multiplication with Φ^T :

$\Phi^T M \Phi \ddot{q}(t) + \Phi^T K \Phi q(t) = \Phi^T F(t)$

$$\begin{bmatrix} \mathcal{M}_1 & & & & \\ & \ddots & & & \\ & & \mathcal{M}_i & & \\ & & & \ddots & \\ & & & & \mathcal{M}_n \end{bmatrix} \begin{bmatrix} \ddot{q}_1(t) \\ \vdots \\ \ddot{q}_i(t) \\ \vdots \\ \ddot{q}_n(t) \end{bmatrix} + \begin{bmatrix} \mathcal{K}_1 & & & & \\ & \ddots & & & \\ & & \mathcal{K}_i & & \\ & & & \ddots & \\ & & & & \mathcal{K}_n \end{bmatrix} \begin{bmatrix} q_1(t) \\ \vdots \\ q_i(t) \\ \vdots \\ q_n(t) \end{bmatrix} = \begin{bmatrix} \phi_1^T F(t) \\ \vdots \\ \phi_i^T F(t) \\ \vdots \\ \phi_n^T F(t) \end{bmatrix}$$

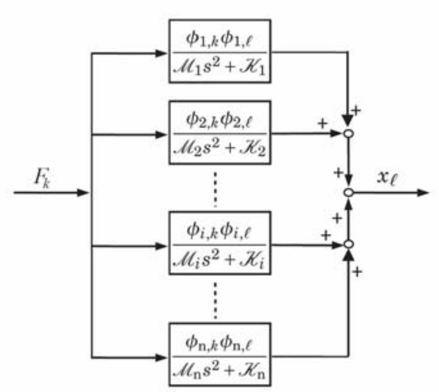
This can be transformed into a combined transfer function:

- The response of DOF x_ℓ by eigenmode i on force F_k equals:

$$\left(\frac{x_\ell}{F_k}\right)_i (s) = \frac{\phi_{i,k}\phi_{i,\ell}}{\mathcal{M}_i s^2 + \mathcal{K}_i}$$

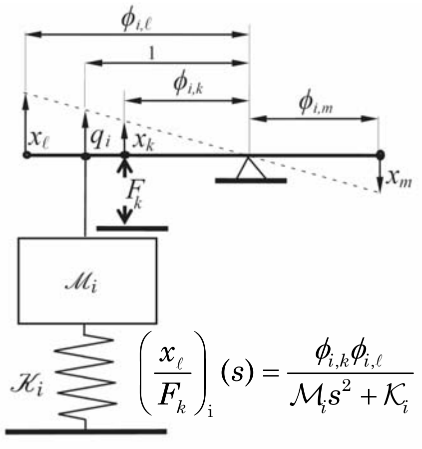
- The total response equals:

$$\frac{x_\ell}{F_k}(s) = \sum_{i=1}^{n} \left(\frac{x_\ell}{F_k}\right)_i = \sum_{i=1}^{n} \frac{\phi_{i,k}\phi_{i,\ell}}{\mathcal{M}_i s^2 + \mathcal{K}_i}$$

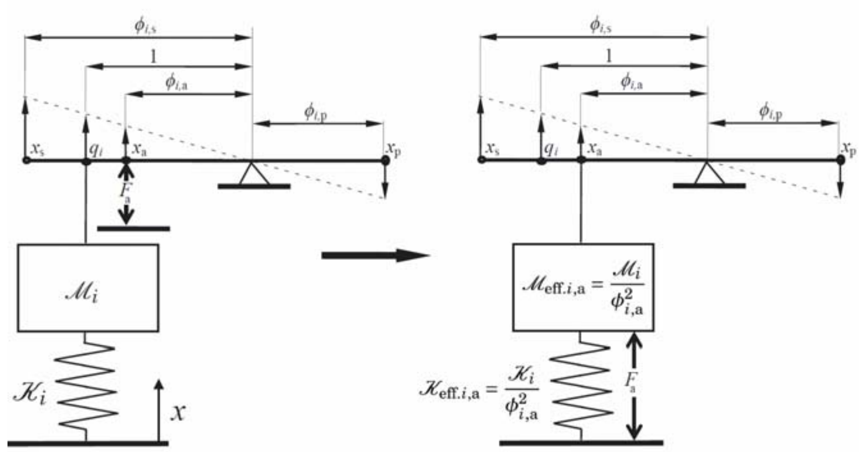


Graphical representation

- The modal coordinate is represented by the angle of the lever or the displacement at distance 1.
- Modal mass and stiffness are connected to the modal coordinate
- The eigenvectors determine the ratio of “controllability” of actuator and “observability” of sensor.

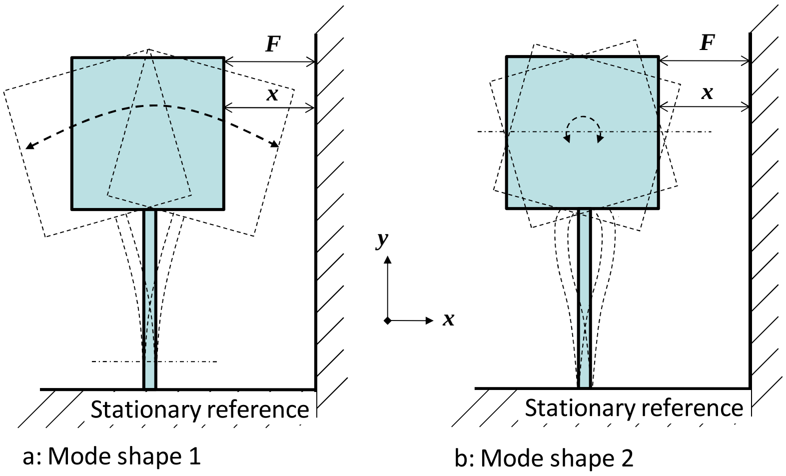


Effective modal values

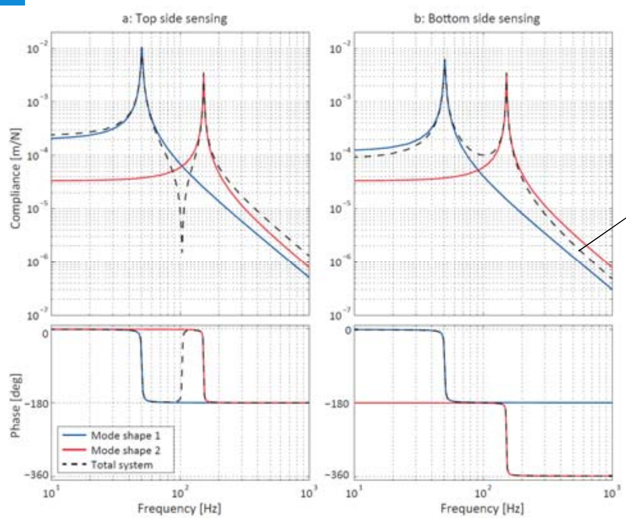


As perceived at the actuator

Mass on leafspring, two eigenmodes

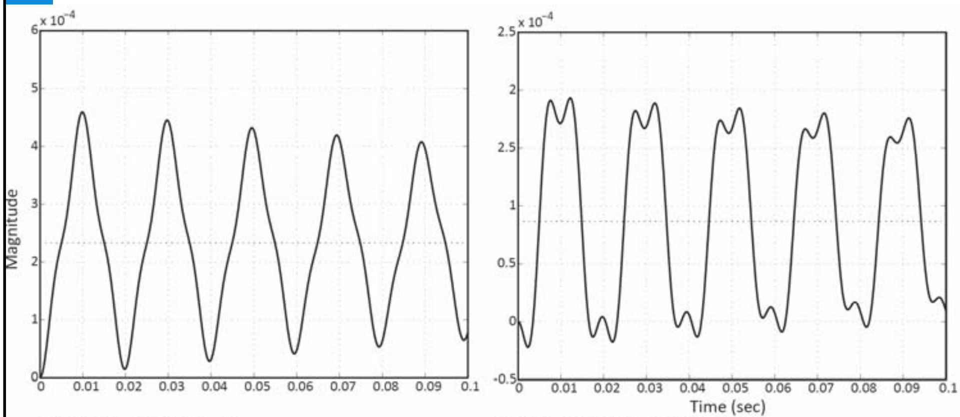


Modes 1 and 2 combined by measuring non co-local with the force!



-2 slope with -360° phase is non-minimum phase system!

Non-minimum phase step response

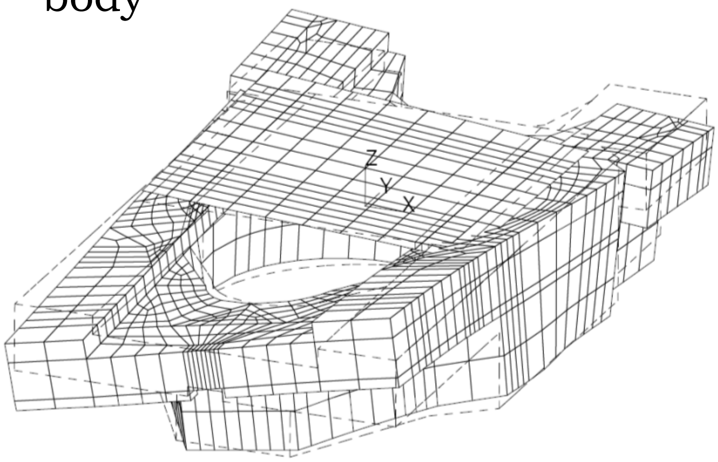


a: Collocated sensing (top) b: Non-collocated sensing (bottom)

Summary of terms

- Eigenvector:
  - Vector with terms that represent the relative motion amplitude of each element in the discretised structure. It is the discretised “shape function”.
  - Scaling is at will, as long as the scaling is equal for all properties of that eigenmode.
- Modal mass and stiffness: $\phi_i^T M \phi_i = \mathcal{M}_i$ $\phi_i^T K \phi_i = \mathcal{K}_i$
  - relate to the mass and stiffness matrix by double multiplication with the eigenvector.
  - Scaling of the eigenvector also scales the modal mass and stiffness (squared).
- Modal coordinates:
  - Represent the motion of the eigenmode at the location of the modal mass and stiffness.
  - It is used to determine the motion of all elements with the eigenvector and as such it also depends on the chosen scaling.
- Effective modal mass and stiffness:
  - Is the modal mass and stiffness as perceived at the actuator position. It is calculated from the modal mass and stiffness by dividing by the corresponding eigenvector term squared, hence the chosen scaling no longer plays a role.

Mode-shape of complex non-rigid body



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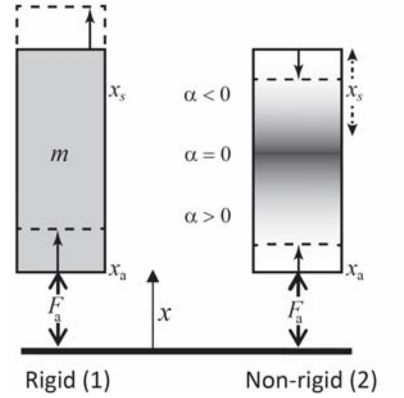
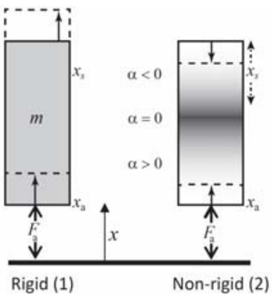
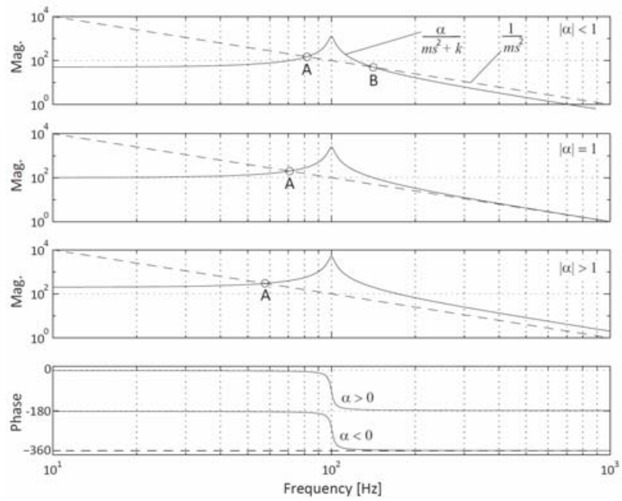
Typical combination of two eigenmodes, one rigid and one non-rigid.



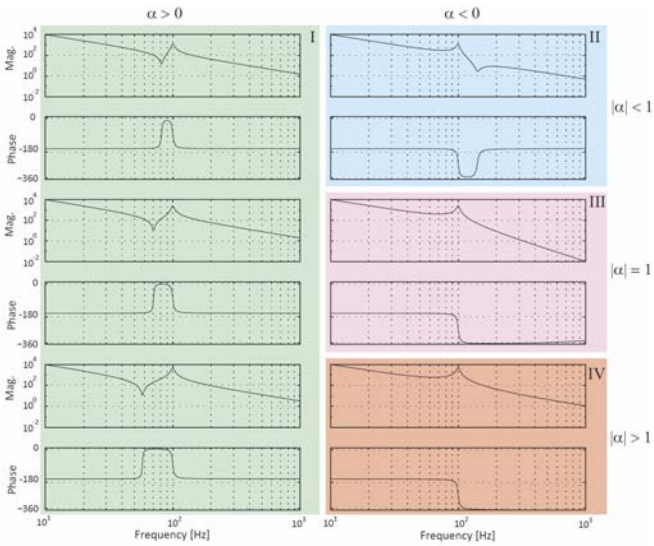
- Actuated at x_a.
- Sensed at different locations.

Different combinations of two eigenmodes





4 types of responses result



Eigenfrequencies of multiple eigenmodes in 6-DOF