Offshore Hydromechanics Module 1

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7. Waves

Introduction

Topics of Module 1

• Problems of interest	Chapter 1
• Hydrostatics	Chapter 2
• Floating stability	Chapter 2
• Constant potential flows	Chapter 3
• Constant real flows	Chapter 4
• **Waves**	**Chapter 5**

Learning Objectives

Chapter 5

- To apply linear wave theory and to derive and apply potential flow theory to linear waves.
- To describe wave shoaling, reflection and diffraction.
- To describe basic nonlinear corrections to linear wave theory.
- To perform simple statistical analysis to irregular wave trains.
- To apply the concept of wave energy spectra and the relation between the time and the frequency domains.
- To describe wave climatology and wave prediction.

Waves

Introduction

- Sea:
  - Waves driven by local wind field
  - Short crested
  - Irregular
  - Unpredictable

- Swell:
  - Generated by wind (storms) far away
  - More regular (sine-like)
  - Long crested
  - Unidirectional
  - Longer waves

Waves

Introduction

- Deep water waves: short waves
  - (Almost) no influence sea floor

$$h/\lambda > 1/2$$

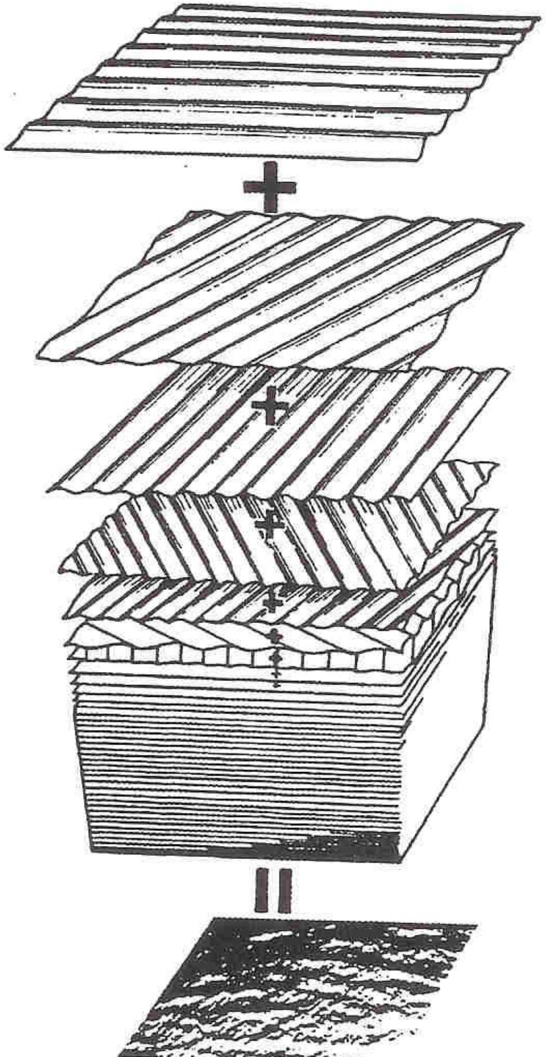
- Shallow water waves: long waves
  - Large influence by sea floor

$$h/\lambda < 1/20$$

Waves

Introduction

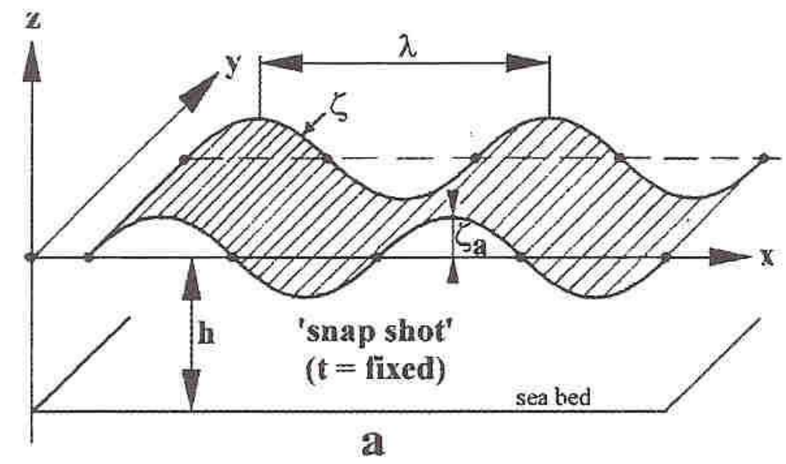
- Wind waves irregular
- Use superposition principle to decompose in regular sine waves
- (compare with Fourier Transform)



Regular waves

Definitions

- ζ wave elevation m
- λ wave length m
- ζ_a wave amplitude m
- H wave height m $H = 2\zeta_a$
- h water depth m
- T wave period s
- ω wave frequency rad/s $\omega = \frac{2\pi}{T}$
- k wave number rad/m $k = \frac{2\pi}{\lambda}$

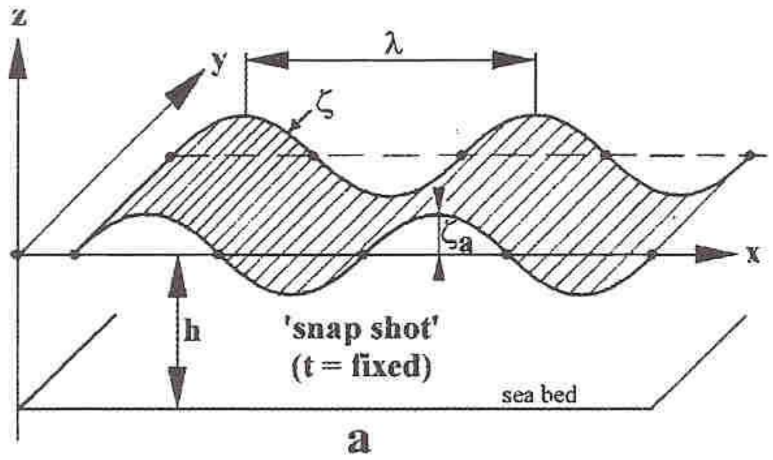


Regular waves

Definitions

- Wave speed (or better: phase velocity)

$$c = \frac{\lambda}{T} = \frac{\omega}{k}$$



- Wave profile (dependent on both **time** and **place**)

$$\zeta(x,t) = \zeta_a \cos(kx - \omega t)$$

- Due to minus sign before ωt-term: wave travels in positive x-direction
    - In case plus-sign: wave travels in the other direction

Regular waves

Potential theory

- Assumptions:
  - Small wave steepness
    - Ignoring nonlinear terms
    - Linear relation between wave **harmonic** signals:
      - Displacements
      - Velocities
      - Accelerations
      - Surface displacement
      - (pressures, etc)
      - **Wave potential**

Regular waves

Potential theory

1. Assume harmonic wave elevation:

$$\zeta = \zeta_a \cdot \cos(kx - \omega t)$$

2. Assume harmonic wave potential function:

$$\Phi_w(x, z, t) = P(z) \cdot \sin(kx - \omega t)$$

3. Use (boundary) conditions to find leading term *P(z)* (see book):
   - Continuity, Laplace equation
   - Sea bed boundary condition
   - Free surface dynamic boundary condition
   - Free surface kinematic boundary condition

Regular waves

Potential theory

- Resulting wave potential equation:

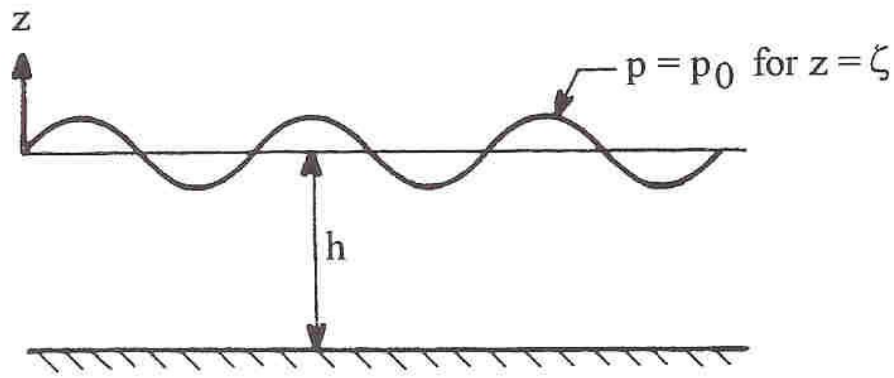
$$\Phi_w(x,z,t) = \zeta_a \frac{g}{\omega} \cdot \frac{\cosh k(h+z)}{\cosh kh} \cdot \sin(kx - \omega t)$$

- For deep water: $h \to \infty$

$$\Phi_w(x,z,t) = \zeta_a \frac{g}{\omega} \cdot e^{kz} \cdot \sin(kx - \omega t)$$

Regular waves

Potential theory - FS dynamic BC



Pressure at FS equals atmospheric pressure

- Bernoulli equation at FS:

$$\frac{\partial \Phi_w}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p_0}{\rho} + g\zeta = 0 \qquad at\ z = \zeta$$

- 2D and small wave steepness:

$$\frac{\partial \Phi_w}{\partial t} + \frac{p_0}{\rho} + g\zeta = 0 \qquad at\ z = 0$$

Neglected/ included in potential

Regular waves

Potential theory – FS kinematic BC

Velocity of water particles at FS equals velocity of FS (no leak condition)

- Wave profile:

$$\zeta = \zeta_a \cdot \cos(kx - \omega t) \qquad \Rightarrow \qquad \frac{\partial z}{\partial t} = \frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} \cdot \frac{\partial x}{\partial t} \qquad at\ z = \zeta$$

- Small wave steepness:

$$\frac{\partial z}{\partial t} = \frac{\partial \zeta}{\partial t} \qquad at\ z = \zeta$$

- Linearization (small wave steepness):

$$\frac{\partial \Phi_w}{\partial z} = \frac{\partial \zeta}{\partial t} \qquad at\ z = 0$$

Regular waves

Potential theory – FS combined BC

- Resulting kinematic FS BC:

$$\frac{\partial \Phi_w}{\partial z} = \frac{\partial \zeta}{\partial t} \qquad at\ z = 0$$

- Combining with dynamic FS BC:

$$\frac{\partial \Phi_w}{\partial t} + g\zeta = 0 \qquad at\ z = 0$$

$\frac{\partial}{\partial t}$

$$\frac{\partial^2 \Phi_w}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = 0$$

$$\frac{\partial^2 \Phi_w}{\partial t^2} + g\frac{\partial \Phi_w}{\partial z} = 0 \quad at\ z = 0$$

$$\frac{1}{g} \cdot \frac{\partial^2 \Phi_w}{\partial t^2} + \frac{\partial z}{\partial t} = 0 \quad at\ z = 0$$

Combined FS BC: Cauchy-Poisson condition

TUDelft

Regular waves

Potential theory – Dispersion relation

- Substitution of wave potential in CP condition yields:

$$\omega^2 = kg \cdot \tanh kh$$

- Deep water:

$$\omega^2 = kg \qquad k = \frac{2\pi}{\lambda}, \quad \omega = \frac{2\pi}{T} \quad \Rightarrow \quad \frac{4\pi^2}{T^2} = \frac{2\pi g}{\lambda} \quad \Rightarrow \quad \lambda = \frac{g}{2\pi}T^2 \approx 1.56T^2$$

- Shallow water:

$$\omega = k\sqrt{gh} \qquad k = \frac{2\pi}{\lambda}, \quad \omega = \frac{2\pi}{T} \quad \Rightarrow \quad \frac{2\pi}{T} = \frac{2\pi}{\lambda}\sqrt{gh} \quad \Rightarrow \quad \lambda = T\sqrt{gh}$$

Regular waves

Potential theory – phase velocity

- Using dispersion relation and wave celerity:

$$\omega^2 = kg \cdot \tanh kh \qquad c = \frac{\lambda}{T} = \frac{\omega}{k} \qquad \Rightarrow \qquad c = \sqrt{\frac{g}{k} \cdot \tanh kh}$$

- Deep water:

$$c = \sqrt{\frac{g}{k}} = \frac{g}{\omega}$$

- Shallow water:

$$c = \sqrt{gh}$$

$\sqrt{gh}$ critical velocity

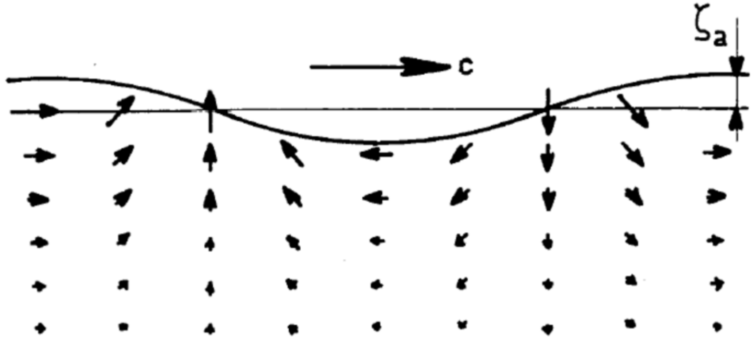
Regular waves

Potential theory – orbital velocity

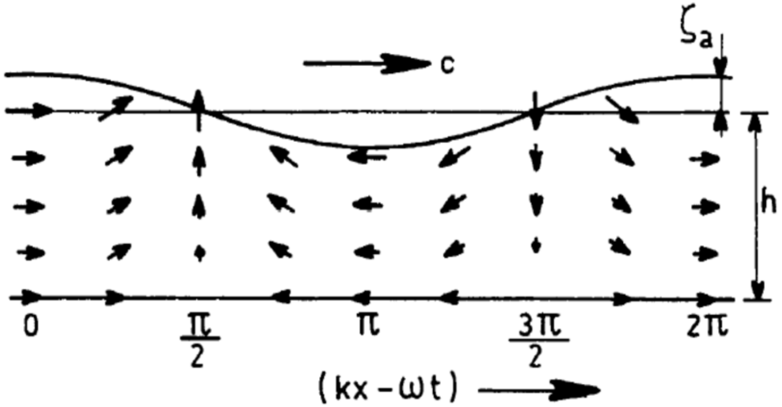
$$u = \frac{\partial \Phi_w}{\partial x} \quad w = \frac{\partial \Phi_w}{\partial z}$$

- Deep water:

$$V_o = \sqrt{u^2 + w^2} = \zeta_a \omega \cdot e^{kz}$$

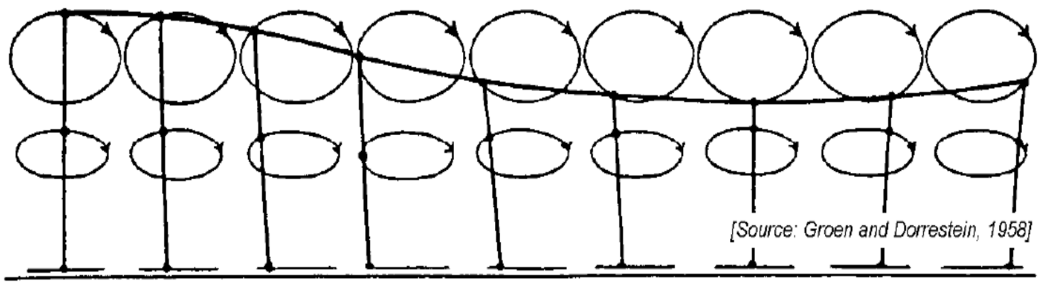


- Shallow water:



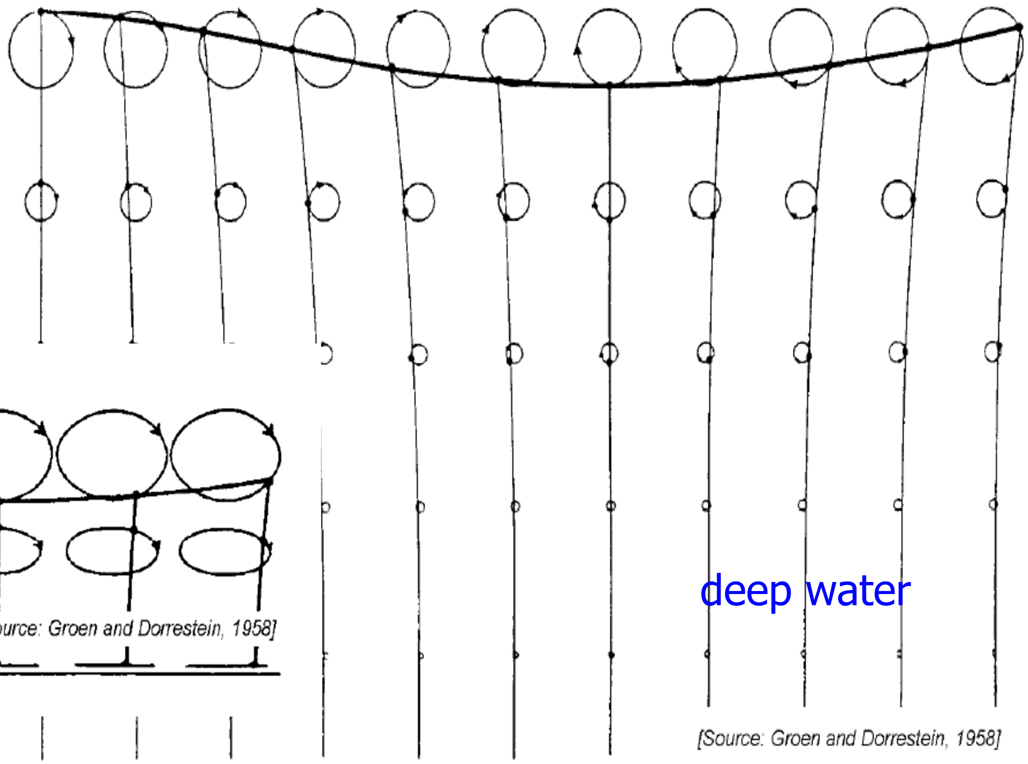
Regular waves

Potential theory – orbital trajectories



[Source: Groen and Dorrestein, 1958]

shallow water



deep water

[Source: Groen and Dorrestein, 1958]

Regular waves

Potential theory – wave pressure

- Use the (linearized) Bernoulli equation

$$\frac{\partial \Phi_w}{\partial t} + \frac{p}{\rho} + gz + \frac{1}{2}(u^2 + w^2) = 0$$

linearized

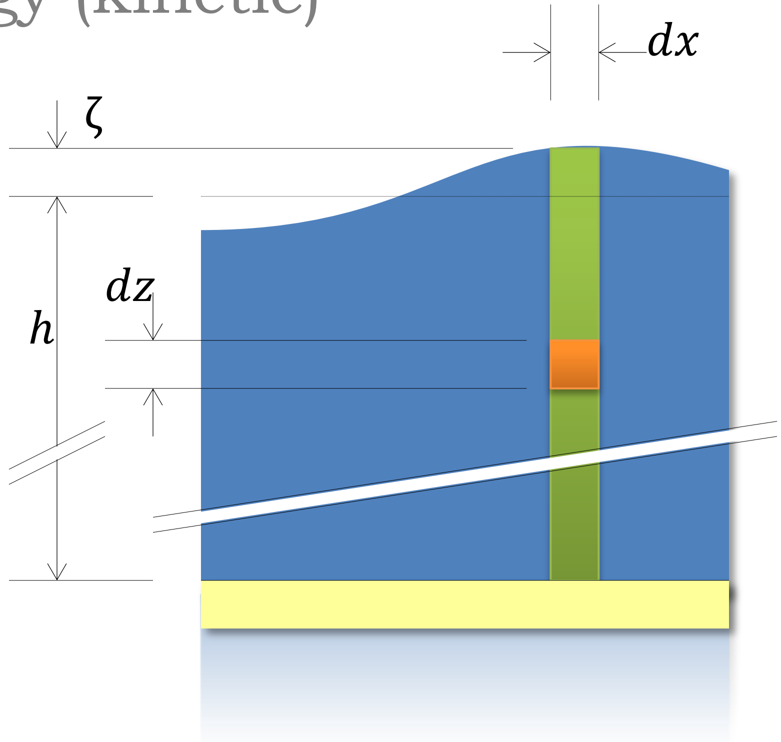
$$p = -\rho g z + \rho g \zeta_a \cdot \frac{\cosh k(h+z)}{\cosh kh} \cdot \cos(kx - \omega t)$$

- Deep water:

$$p = -\rho g z + \rho g \zeta_a \cdot e^{kz} \cdot \cos(kx - \omega t)$$

Regular waves

Potential theory – wave energy (kinetic)



$$K = \frac{1}{2} m V^2 = \int_{volume} \frac{1}{2}(u^2 + w^2)dm =$$

$$= \int_{-h}^{\zeta}\int_{0}^{\lambda} \frac{1}{2}\rho(u^2 + w^2)dxdz =$$

$$= \int_{-h}^{0}\int_{0}^{\lambda} \frac{1}{2}\rho(u^2 + w^2)dxdz$$

$$+\int_{0}^{\zeta}\int_{0}^{\lambda} \frac{1}{2}\rho(u^2 + w^2)dxdz =$$

small

$$K = \frac{1}{2} m V^2 = \int_{-h}^{0}\int_{0}^{\lambda} \frac{1}{2}\rho(u^2 + w^2)dxdz = \ldots = \frac{1}{4}\rho g \zeta_a^2 \lambda$$

Regular waves

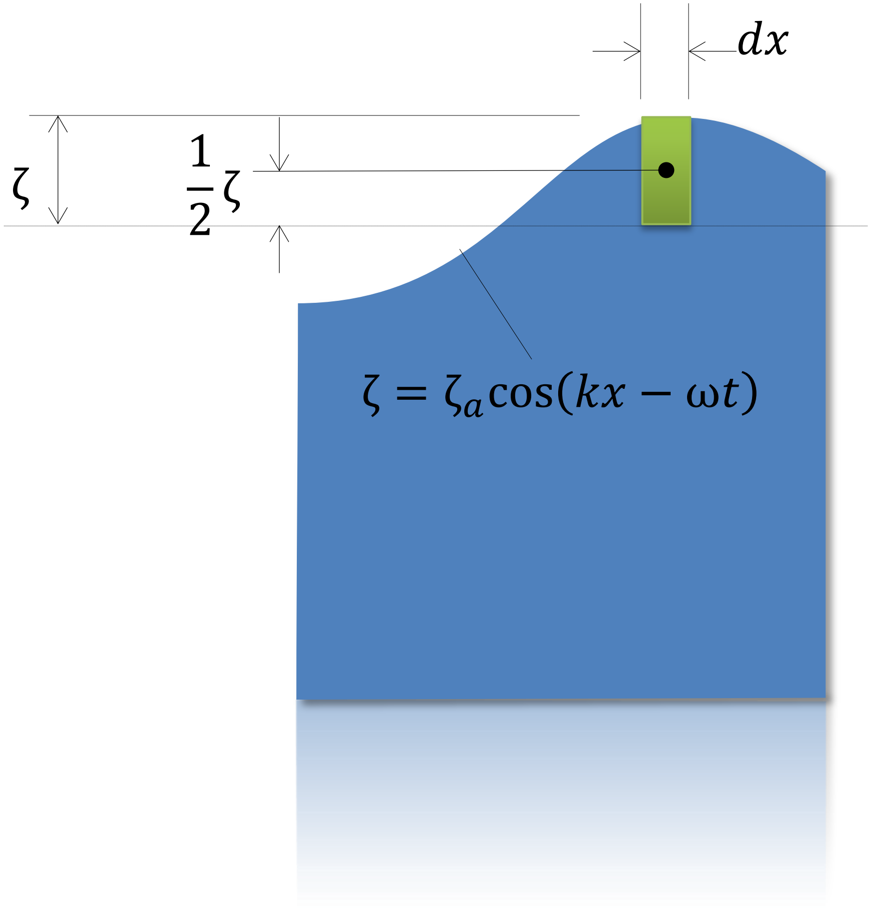
Potential theory – wave energy (potential)

$$P = mgh = \int_0^{\lambda} \left(\rho\zeta dx \cdot g \cdot \frac{1}{2}\zeta \right) =$$

$$= \int_0^{\lambda} \frac{1}{2}\rho g \zeta^2 dx =$$

$$= \frac{1}{2}\rho g \zeta_a^2 \int_0^{\lambda} \cos^2(kx - \omega t) dx =$$

$$= \frac{1}{2}\rho g \zeta_a^2 \cdot \frac{1}{2}\lambda$$



$$P = mgh = \ldots = \frac{1}{4}\rho g \zeta_a^2 \cdot \lambda$$

Regular waves

Potential theory – wave energy

- Two forms of energy:
  - Kinetic energy (velocity)

$$K = \frac{1}{2} m V^2 = \ldots = \frac{1}{4} \rho g \zeta_a^2 \cdot \lambda$$

per unit width

  - Potential energy (height)

$$P = mgh = \ldots = \frac{1}{4} \rho g \zeta_a^2 \cdot \lambda$$

per unit width

- Total energy:

$$E = K + P = \frac{1}{2} \rho g \zeta_a^2 = \frac{1}{8} \rho g H^2$$

per unit horizontal sea surface

Regular waves

Potential theory – wave energy transport

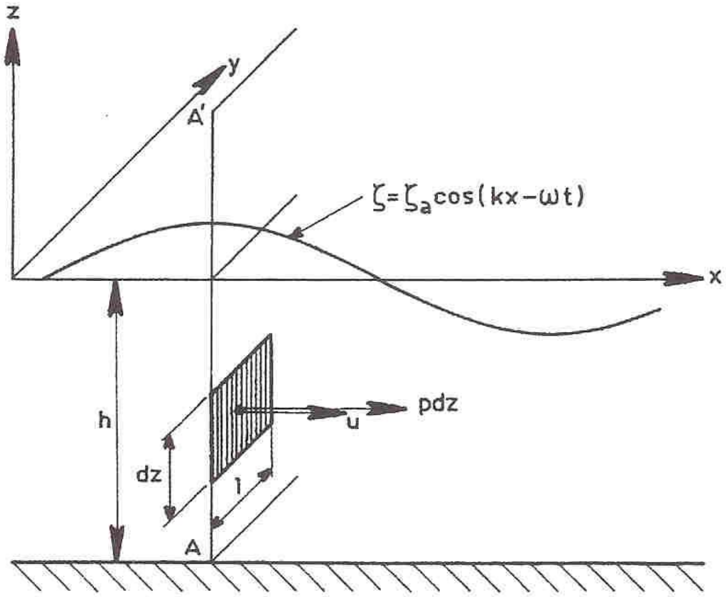
- Work = **force x distance**

$$dW = [p \cdot 1 \cdot dz] \cdot [u \cdot dt] = p \cdot u \cdot dz \cdot dt$$

- Average work per unit time:
  (over one period T): **Power**

$$P = \frac{1}{T} \int_t^{t+T} \int_{-h}^{0} p \cdot u \cdot dz \cdot dt$$

$$P = \ldots = \frac{1}{2} \rho g \zeta_a^2 \cdot \frac{c}{2} \cdot \left(1 + \frac{2kh}{\sinh 2kh}\right)$$



Regular waves

Potential theory – group ve

- Thus power:

$$P = \ldots = \frac{1}{2}\rho g \zeta_a^2 \cdot \frac{c}{2} \cdot \left(1 + \frac{2kh}{\sinh 2}\right.$$

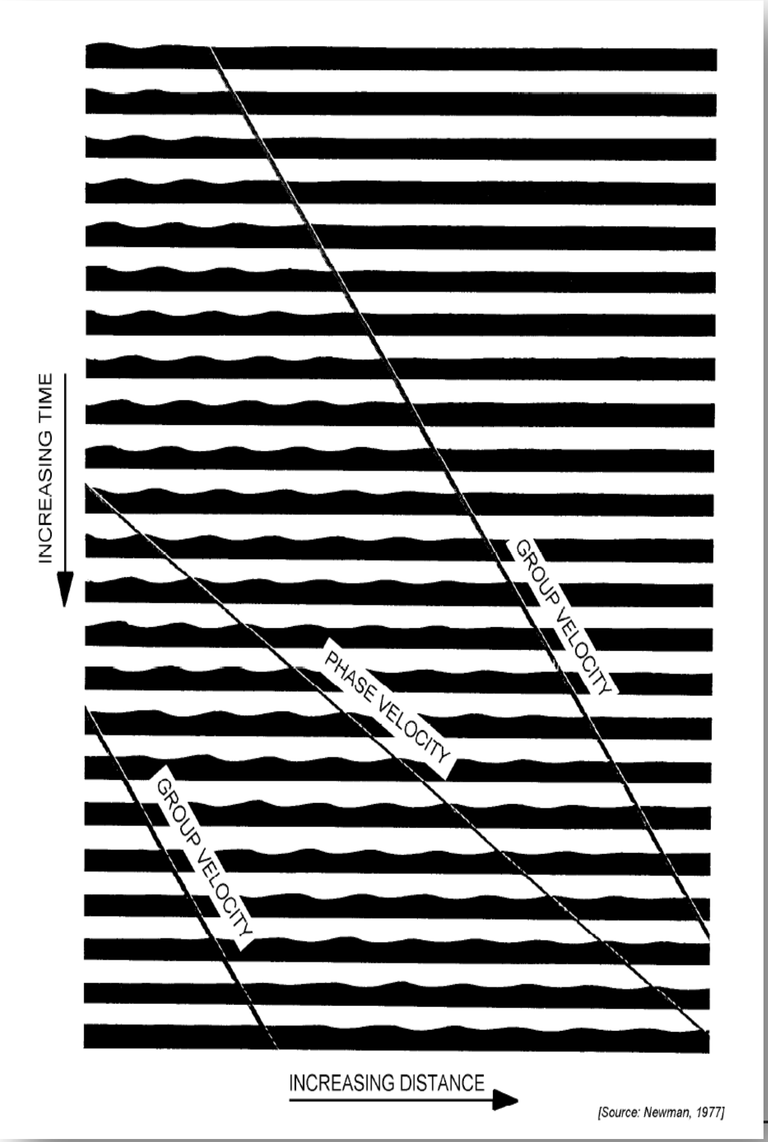
- Also Power = **energy x velocity**:

$$P = E \cdot c_g$$

- Now the group velocity becomes:

$$c_g = \frac{c}{2} \cdot \left(1 + \frac{2kh}{\sinh 2kh}\right)$$

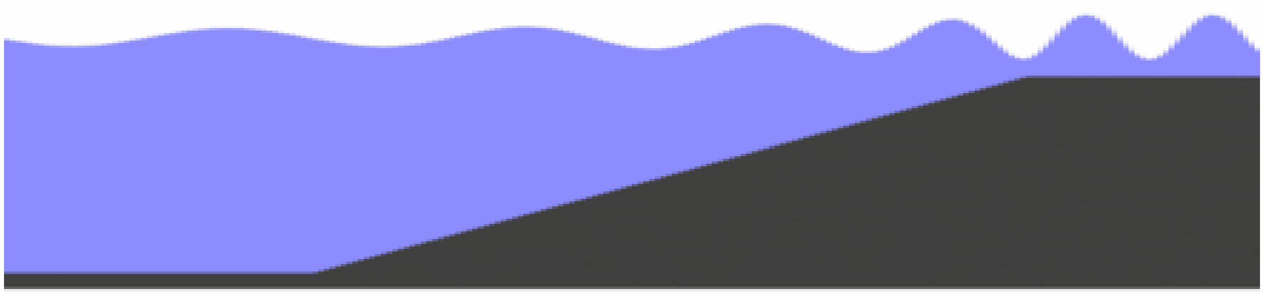
$$c_g = \frac{c}{2} \quad deep\ water \qquad c_g =$$



Regular waves

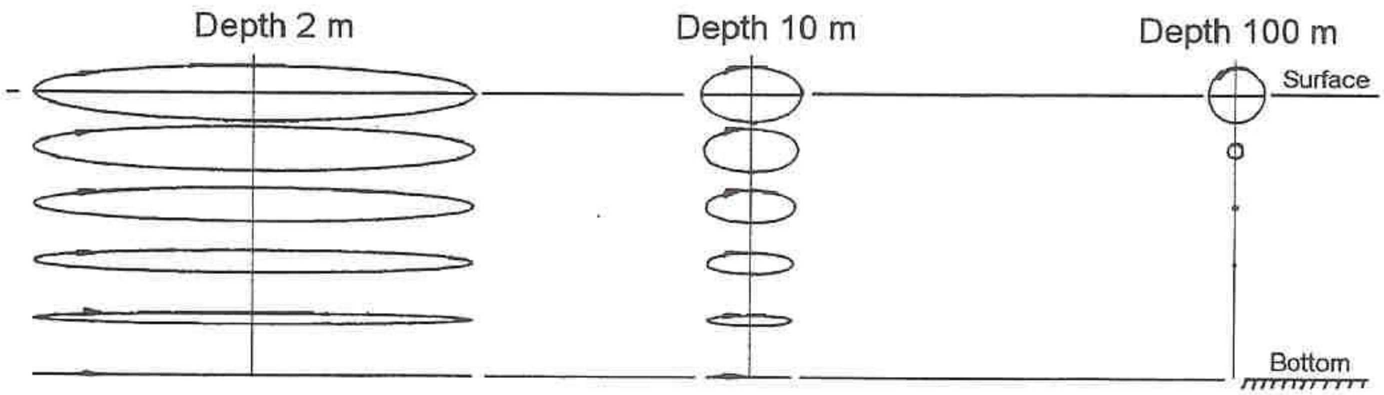
Shoaling

- When waves move from deep to shallow water:
  - Wave length decreases for fixed wave period
  - Lower celerity (wave velocity)
- Energy transport needs to remain constant:
  - Wave height increases near shore (higher energy density)



Regular waves

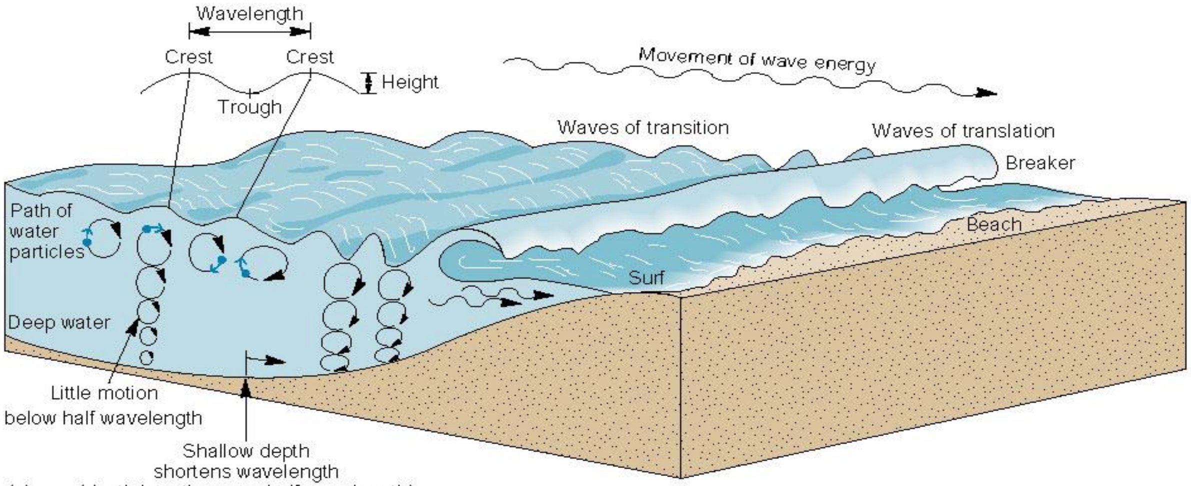
Shoaling



$\lambda = 100$

Regular waves

Shoaling



(a)

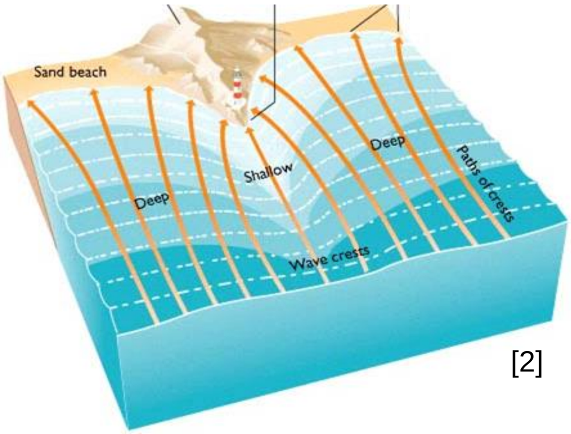
(b)

[1]

Regular waves

Refraction, reflection, diffraction

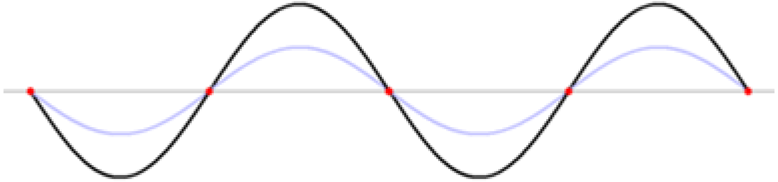
Refraction



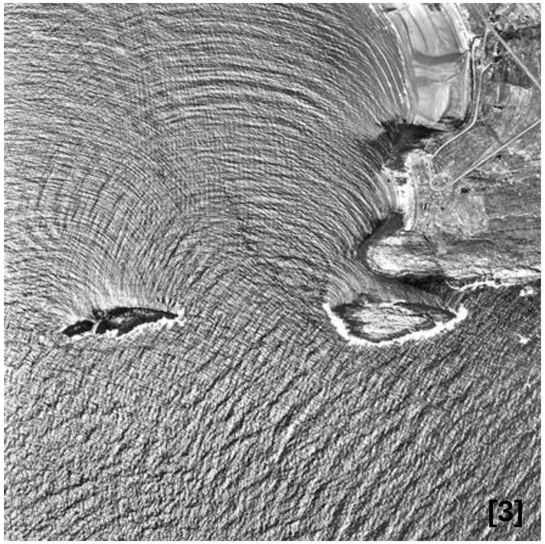
[2]

Reflection

Standing wave:



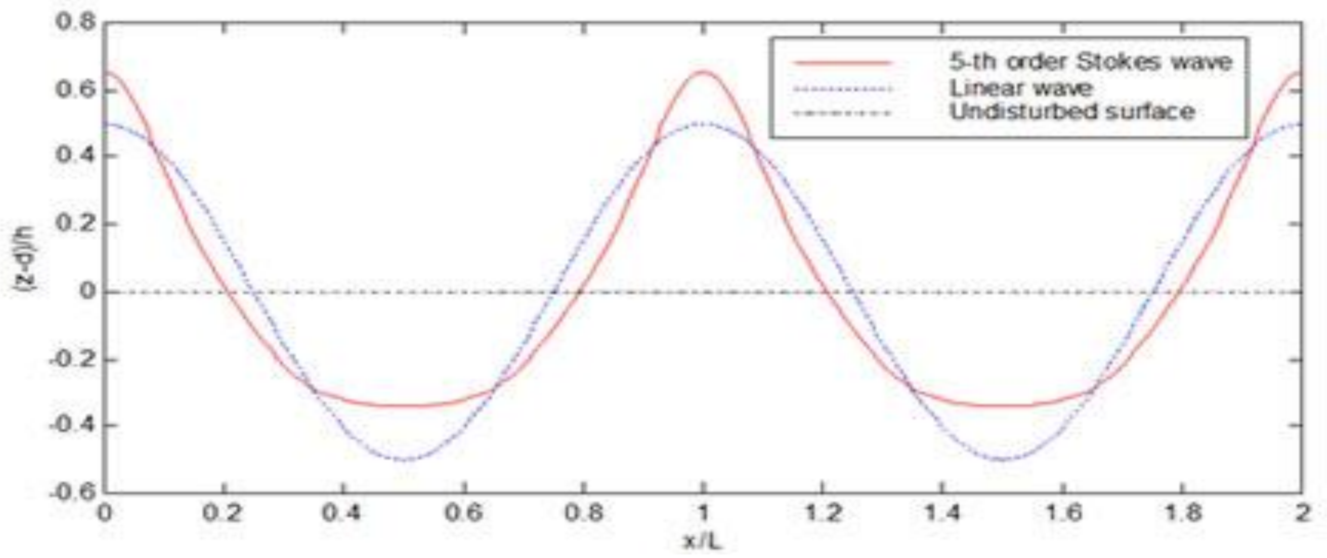
Diffraction



Regular waves

Limits to linear wave theory

- Waves in reality not sinusoidal
  - Use non-linear wave: stokes waves for instance:



- Small wave steepness: no detailed information above z=0

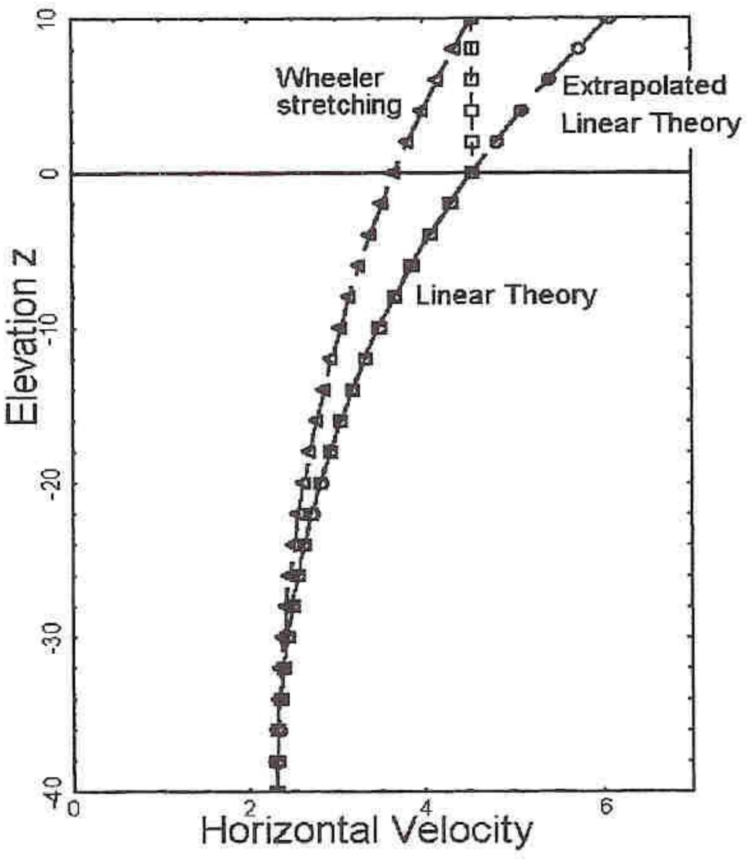
TU Delft

Regular waves

Wave pressure in the splash zone

- Linear wave theory:
  - No information above z = 0

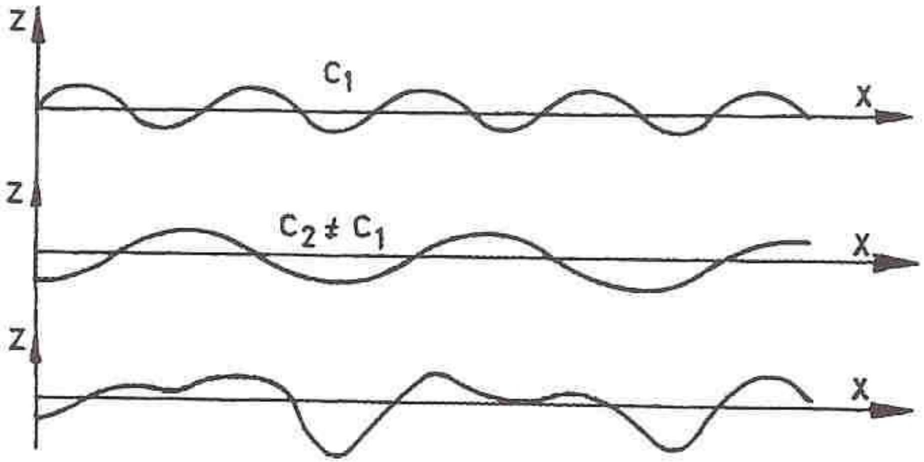
- Solution:
  - Wave profile stretching

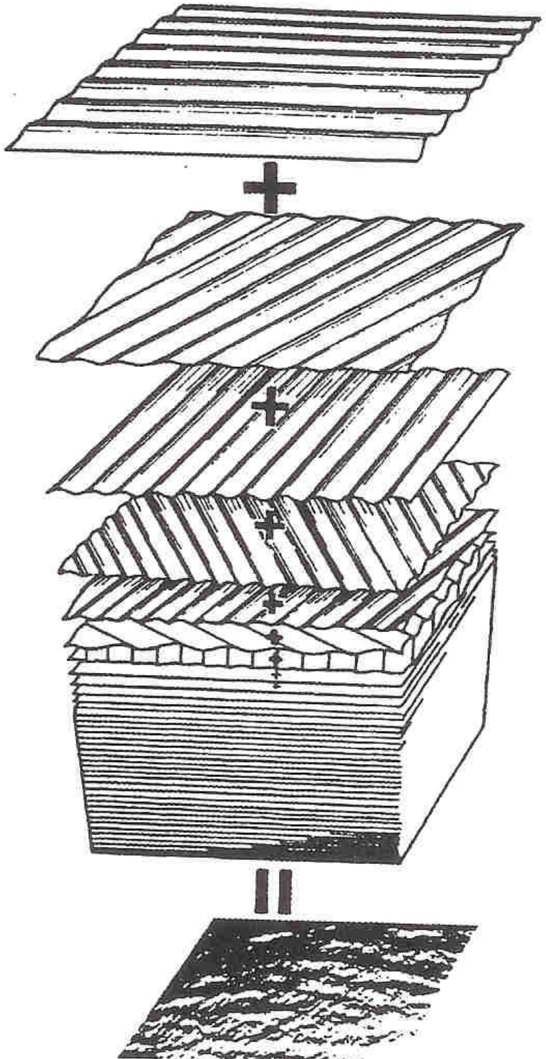


Irregular waves

Wave superposition

- Basic assumption:
  - Decompose irregular waves into a large number of regular wave components
  - (Fourier transform)





[Source: Pierson, Neuman and James]

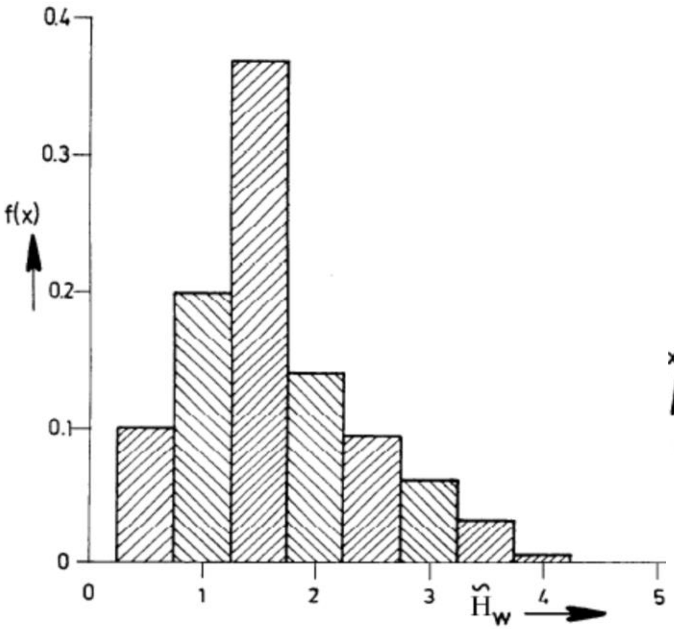
Irregular waves

Characterization of irregular sea state

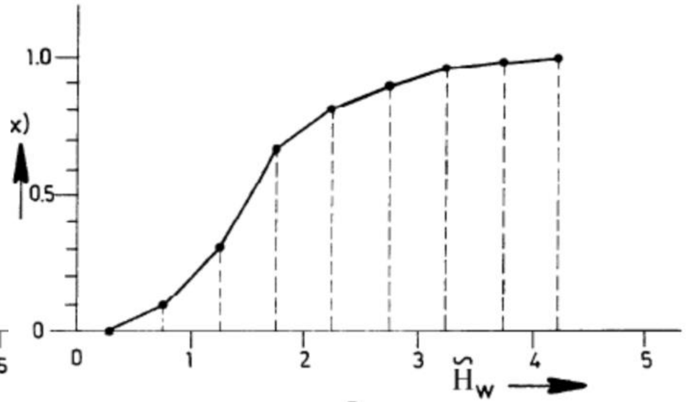
- Period: Average zero up crossing or average crest or trough period
- Significant wave height H_s or $H_{1/3}$
  - The average height of the one-third highest part of the observed waves
- Visually estimated wave height H_v approx. corresponds with significant wave height
- Mean wave height H or $H_{1/1}$

Irregular waves

Probability density distributions



a



b

$$P(\overline{H} > a) = \int_a^\infty f\,(x) \cdot dx$$

Irregular waves

Wave elevation statistics

- Standard deviation of the water level elevation signal $\zeta(t)$ and significant wave height

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{n=1}^{N}\zeta_n^2}$$

$$\zeta_{a1/3} = 2 \cdot \sigma$$

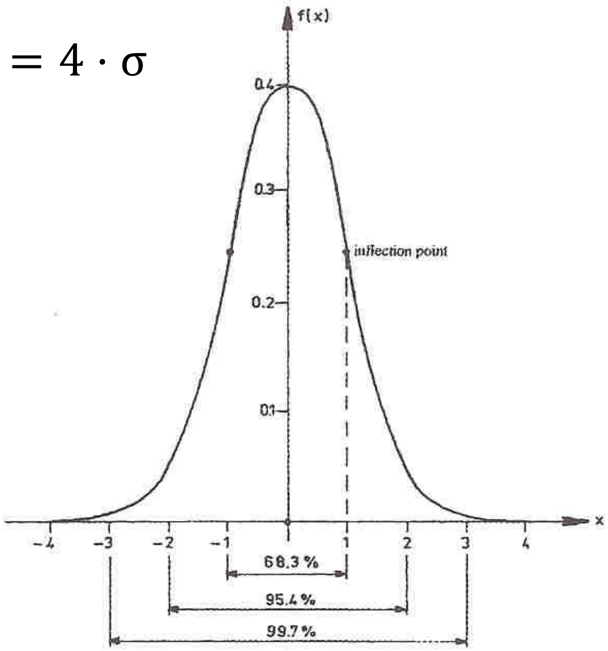
$$H_{1/3} = 4 \cdot \sigma$$

- Gaussian water level distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\left(\frac{x}{\sigma\sqrt{2}}\right)^2}$$

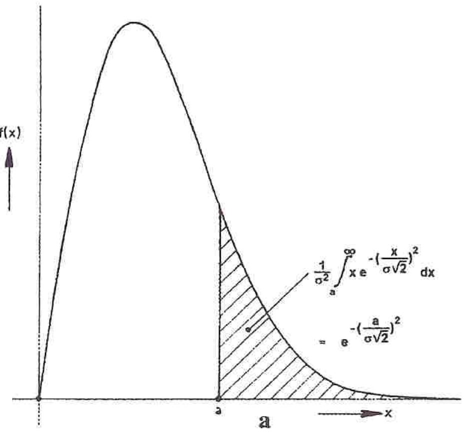
- Probability of exceedance

$$p(\zeta > a) = \int_a^\infty f(x) \cdot dx = \frac{1}{\sigma\sqrt{2\pi}} \int_a^\infty e^{-\left(\frac{x}{\sigma\sqrt{2}}\right)^2} dx$$



Irregular waves

Wave height statistics



- In case:
  - Wave elevation spectrum: narrow banded
  - Gaussian distributed
- Then: Rayleigh distributed wave **height** distribution

$$f(x) = \frac{x}{\sigma^2} \cdot e^{-\left(\frac{x}{\sigma\sqrt{2}}\right)^2}$$

- Probability of exceedance

$$p(\zeta > a) = \int_a^\infty f(x) \cdot dx = \frac{1}{\sigma^2}\int_a^\infty x \cdot e^{-\left(\frac{x}{\sigma\sqrt{2}}\right)^2} dx = e^{-\frac{a^2}{2\sigma^2}}$$

$$\zeta_{a1/3} = 2 \cdot \sigma$$

$$H_{1/3} = 4 \cdot \sigma$$

$$p(H_w > H) = e^{-\frac{1}{2}\left(\frac{\frac{H}{2}}{\frac{1}{4}H_{1/3}}\right)^2} = e^{-\frac{1}{2}\left(\frac{2H}{H_{1/3}}\right)^2} = e^{-2\left(\frac{H}{H_{1/3}}\right)^2}$$

Irregular waves

Wave height statistics

- Maximum wave height: choose **design criterion**:
  - The wave height that is exceeded once in every 1000 (storm) waves
  - It takes at least 3 hours for 1000 waves to pass by
  - By then storm should weaken
  - (chance of zero gives a design criterion of infinite wave height)

$$p(H_w > H_{max}) = e^{-2\left(\frac{H_{max}}{H_{1/3}}\right)^2} = \frac{1}{1000} \quad \Rightarrow \quad -2\left(\frac{H_{max}}{H_{1/3}}\right)^2 = \ln\frac{1}{1000}$$

$$\Rightarrow \quad \frac{H_{max}}{H_{1/3}} = \sqrt{-1/2 \cdot \ln\frac{1}{1000}} \quad \Rightarrow \quad H_{max} = 1.86 \cdot H_{1/3}$$

Irregular waves

Wave energy density spectrum

- Wave elevation in long-crested irregular sea:

$$\zeta(t) = \sum_{n=1}^{N} \zeta_{a_n} \cos(k_n x - \omega_n t + \varepsilon_n)$$

1. Apply Fourier transform to time trace of wave elevation
2. Use dispersion relation: relation between k and ω
3. Discard phase angle
   - (only statistical representation, not exact spacial and temporal reproduction)

- Then combinations of ζ_n and ω_n are obtained to represent the wave elevation

Irregular waves

Wave energy density spectrum

- More robust way:

1. Cut time signal in small pieces ('windows')
2. Fourier transform each window to obtain combinations of ζ_n and ω_n
3. Average the values of ζ_n over the windows (take mean square):

$$\overline{\zeta_{a_n}^2}$$

- Removes sensitivity to time shift in analysis
- Reduces 'precision', improve reliability
- Gives a smooth spectrum instead of 'grass'

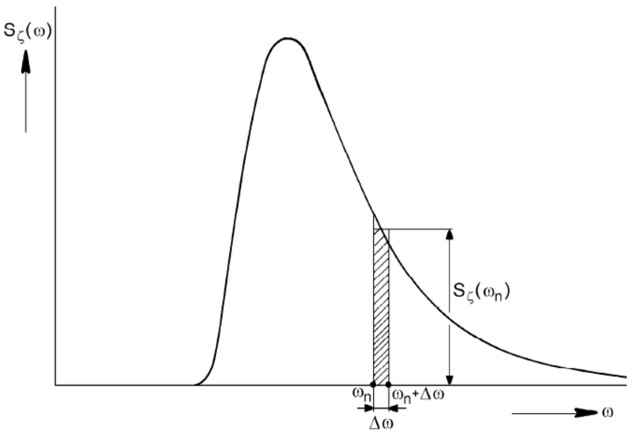
- Typically: measure 50 to 200 times largest expected wave period:
  - 15 to 20 minutes

Irregular waves

Wave energy density spectrum

- Now define a spectral function *S* as:

$$S_\zeta(\omega_n)\cdot\Delta\omega = \sum_{\omega_n}^{\omega_n+\Delta\omega} \frac{1}{2}\zeta_{a_n}^2(\omega_n)$$



  - Read as: the area under the *S* function for a narrow frequency band at ω is proportional to energy of waves at this frequency

$$E = \frac{1}{2}\rho g \zeta_a^2$$

  - Now let $\Delta\omega \to 0$:

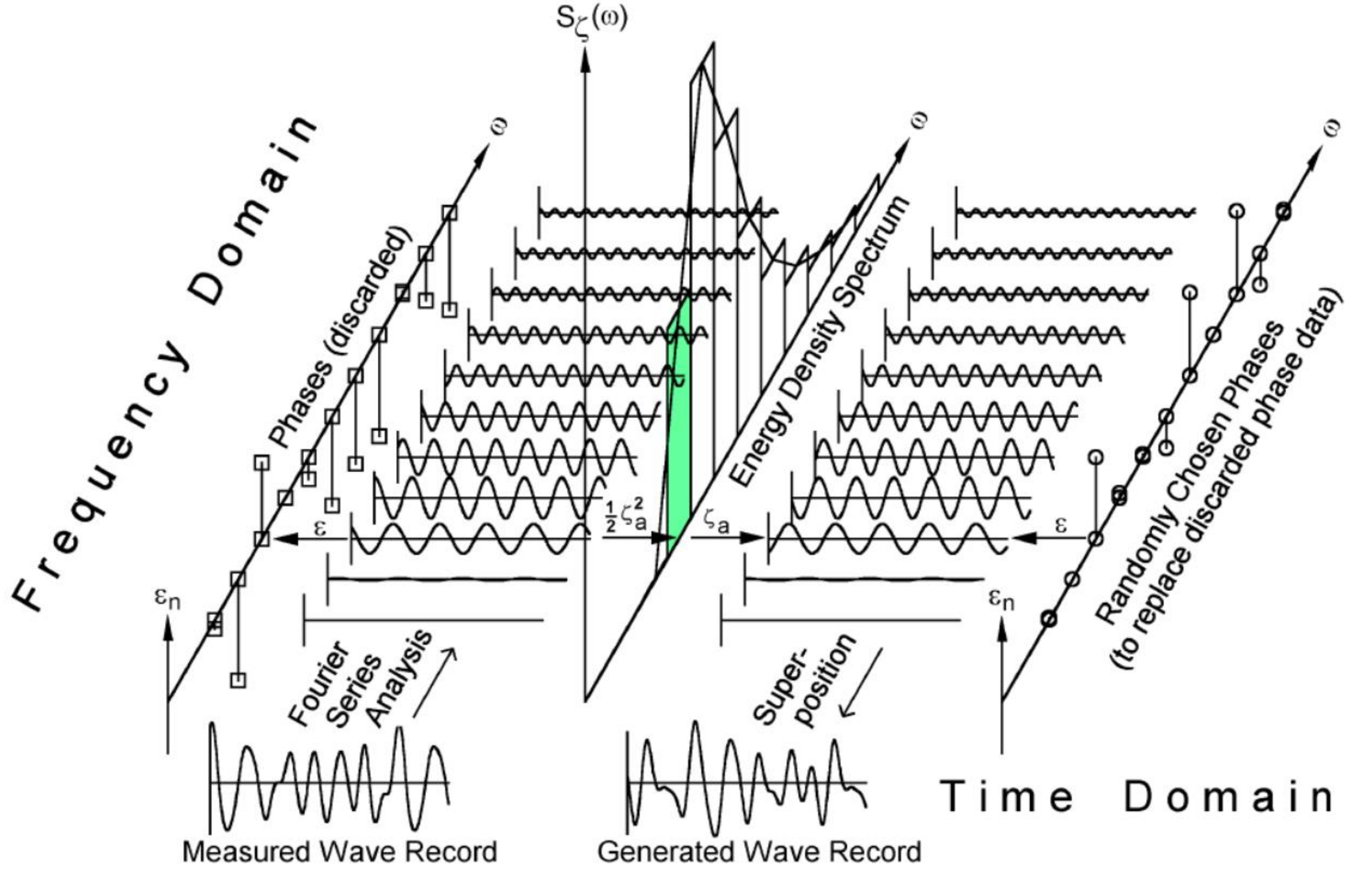
$$S_\zeta(\omega)\cdot d\omega = \frac{1}{2}\zeta_{a_n}^2(\omega)$$

  - Variance is area under *S*

$$\sigma_\zeta^2 = \int_0^\infty S_\zeta(\omega)\cdot d\omega$$

Irregular waves

Transformation to Time Series and back



Irregular waves

Wave energy density spectrum

- Mind the definition of S!

$$S_\zeta(\omega) \cdot d\omega = S_\zeta(f) \cdot df$$

- The amount of energy per frequency band is constant!!!

$$S_\zeta(f) = S_\zeta(\omega) \cdot \frac{d\omega}{df} \qquad \Rightarrow \qquad S_\zeta(f) = S_\zeta(\omega) \cdot 2\pi$$

$$\omega = 2\pi \cdot f$$

Irregular waves

Wave energy density spectrum – wave height and period

- Spectral moments: $$m_{n\zeta} = \int_0^\infty \omega^n \cdot S_\zeta(\omega) \cdot d\omega$$

- RMS wave elevation:

$$\sigma_\zeta = RMS = \sqrt{m_{0\zeta}}$$

- Significant wave amplitude:

$$\zeta_{a1/3} = 2 \cdot \sqrt{m_{0\zeta}}$$

- Significant wave height:

$$H_{1/3} = 4 \cdot \sqrt{m_{0\zeta}}$$

- Mean centroid wave period:

$$T_1 = 2\pi \cdot \frac{m_{0\zeta}}{m_{1\zeta}}$$

- Mean zero crossing period:

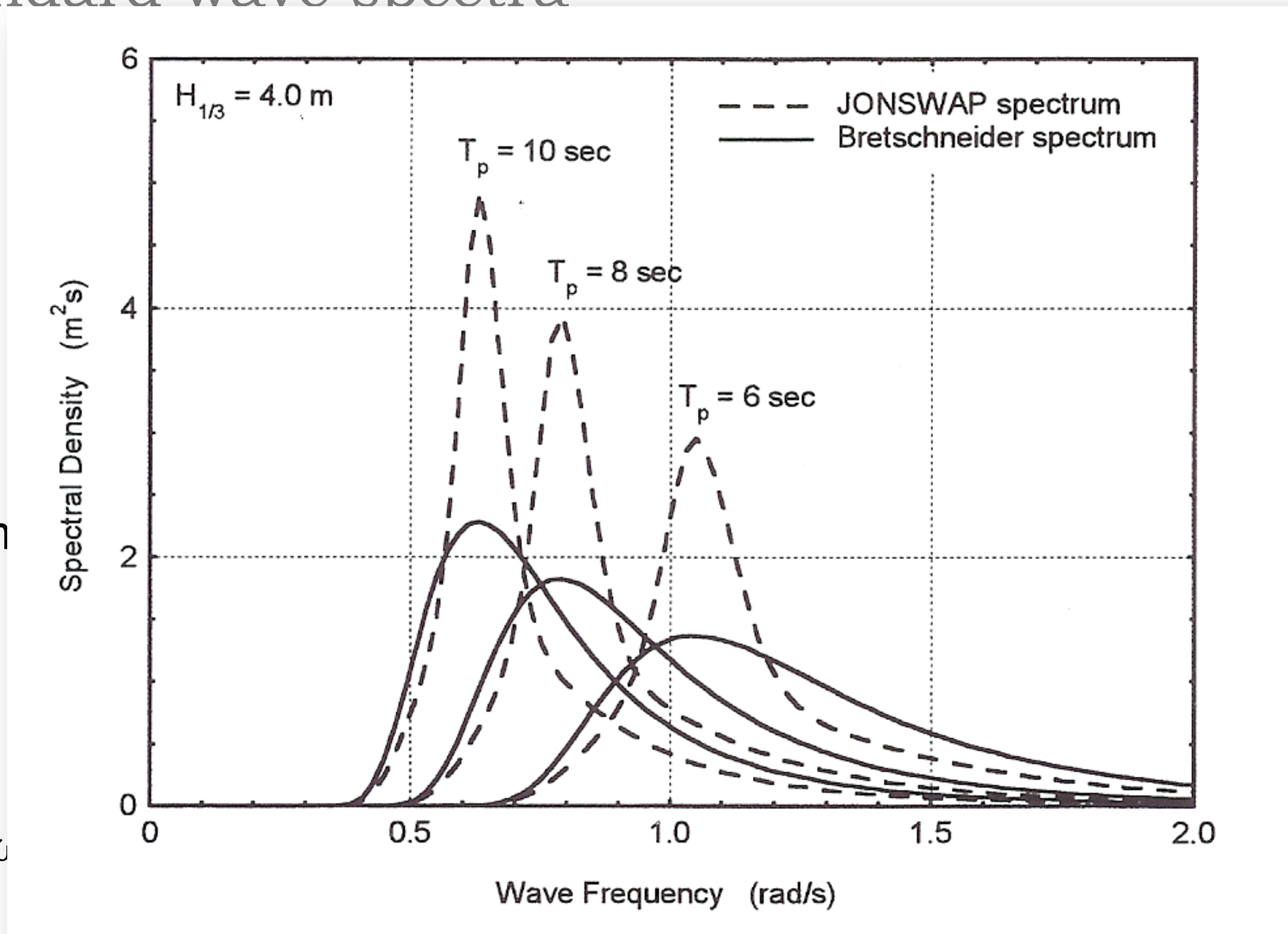
$$T_2 = 2\pi \cdot \sqrt{\frac{m_{0\zeta}}{m_{2\zeta}}}$$

Irregular waves

Standard wave spectra

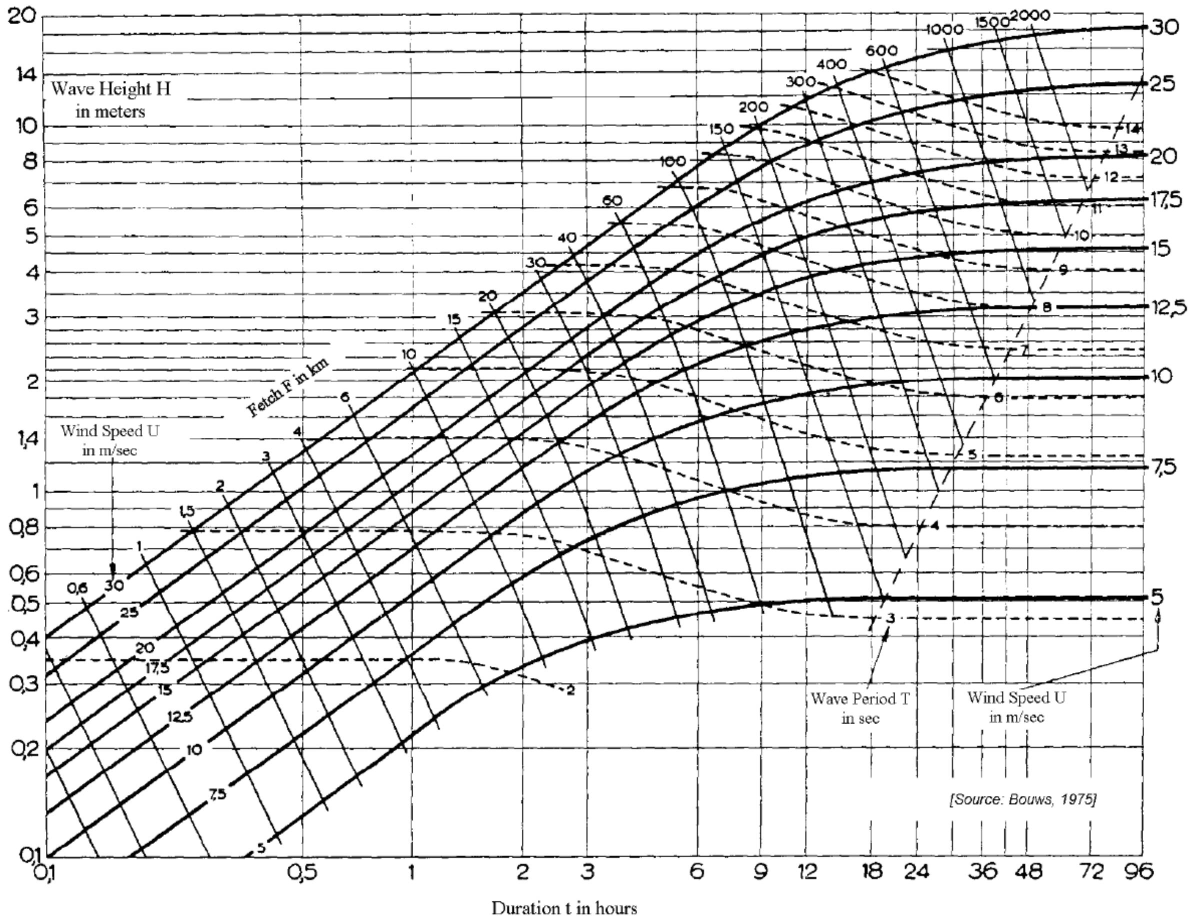
- For
  - ...sion
  -
- Con
  - ... $\omega^{-4}\Big)$
  - ... $- 1\Big)^2\Big]$

$S_\zeta(\omega$ ...



Irregular waves

Storm development



Irregular waves

Long term wave statistics

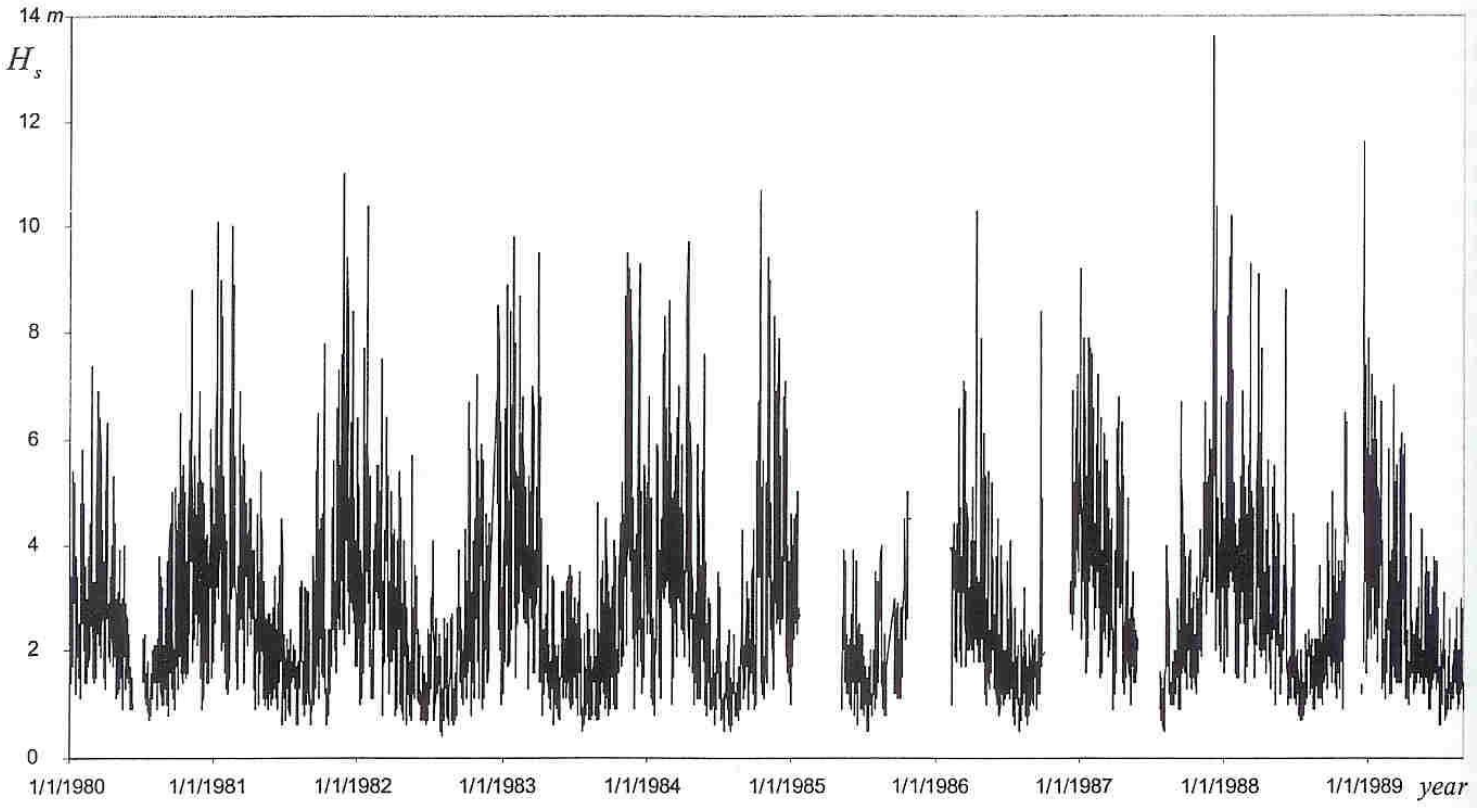


Fig. 4.17 The significant wave height $H_{m_0} = 4\sqrt{m_0}$ *over a 10-year period (1980-1989; NODC buoy 46005, position*

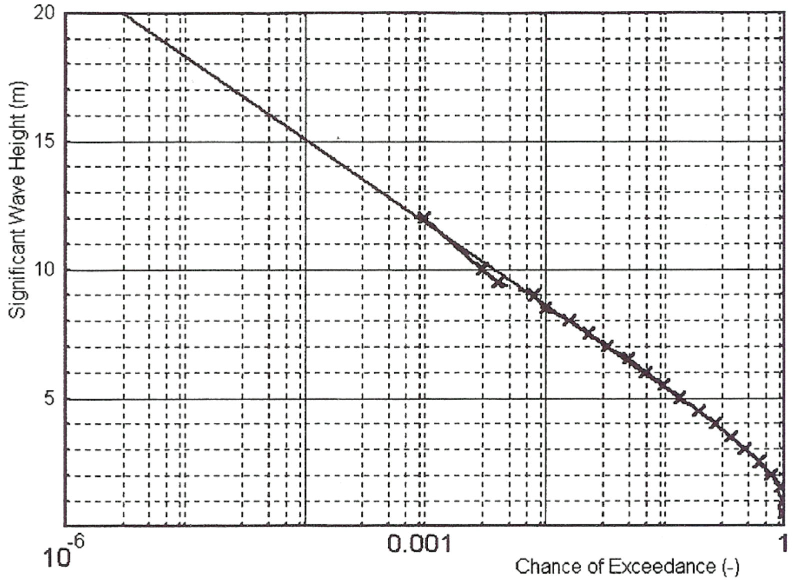
Irregular waves

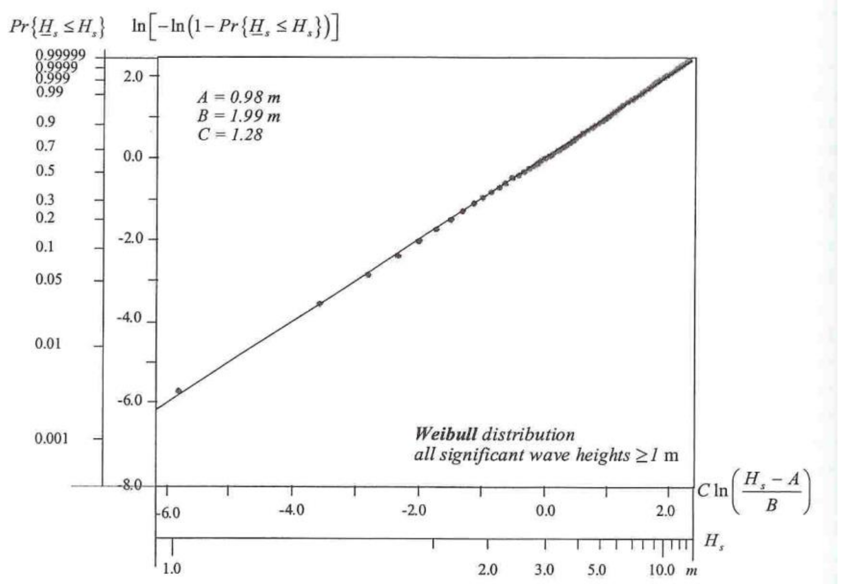
Scatter diagram

Winter Data of Areas 8, 9, 15 and 16 of the North Atlantic (Global Wave Statistics)											
	T_2 (s)										
H_s (m)	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5	11.5	12.5	13.5
14.5	0	0	0	0	2	30	154	362	466	370	202
13.5	0	0	0	0	3	33	145	293	322	219	101
12.5	0	0	0	0	7	72	289	539	548	345	149
11.5	0	0	0	0	17	160	585	996	931	543	217
10.5	0	0	0	1	41	363	1200	1852	1579	843	310
9.5	0	0	0	4	109	845	2485	3443	2648	1283	432
8.5	0	0	0	12	295	1996	5157	6323	4333	1882	572
7.5	0	0	0	41	818	4723	10537	11242	6755	2594	703
6.5	0	0	1	138	2273	10967	20620	18718	9665	3222	767
5.5	0	0	7	471	6187	24075	36940	27702	11969	3387	694
4.5	0	0	31	1586	15757	47072	56347	33539	11710	2731	471
3.5	0	0	148	5017	34720	74007	64809	28964	7804	1444	202
2.5	0	4	681	13441	56847	77259	45013	13962	2725	381	41
1.5	0	40	2699	23284	47839	34532	11554	2208	282	27	2
0.5	5	350	3314	8131	5858	1598	216	18	1	0	0

Irregular waves

Extrapolation to low probability of exceedance (design condition)





$$\log\{(P(H)\} = \frac{1}{a}H$$

Sources images

[1] Source: Greenfield Geography
[2] Waves, source: Revision World
[3] Diffraction in sea waves, source: unknown