Introduction to Aerospace Engineering

Lecture slides

Introduction to Aerospace Engineering AE1-102

Dept. Space Engineering

Astrodynamics & Space Missions (AS)

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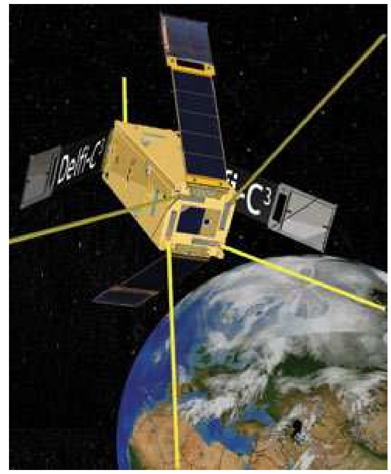


Part of the lecture material for this chapter originates from B.A.C. Ambrosius, R.J. Hamann, R. Scharroo, P.N.A.M. Visser and K.F. Wakker.

References to “”Introduction to Flight” by J.D. Anderson will be given in footnotes where relevant.

5 - 6

Orbital mechanics: satellite orbits (1)

This topic is (to a large extent) covered by Chapter 8 of “Introduction to Flight” by Anderson, although notations (see next sheet) and approach can be quite different.

General remarks

Two aspects are important to note when working with Anderson's "Introduction to Flight" and these lecture notes:

- The derivations in these sheets are done per unit of mass, whereas in the text book (p. 603 and further) this is not the case.
- Some parameter conventions are different (see table below).

parameter	notation in "Introduction to Flight"	customary notation
gravitational parameter	k^2	GM, or μ
angular momentum	h	H

Overview

- Fundamentals (equations of motion)
- Elliptical orbit
- Circular orbit
- Parabolic orbit
- Hyperbolic orbit

The gravity field overlaps with lectures 27 and 28 (”space environment”) of the course ae1-101, but is repeated for the relevant part here since it forms the basis of orbital dynamics.

Learning goals

The student should be able to:

- classify satellite orbits and describe them with Kepler elements
- describe and explain the planetary laws of Kepler
- describe and explain the laws of Newton
- compute relevant parameters (direction, range, velocity, energy, time) for the various types of Kepler orbits

Lecture material:

- these slides (incl. footnotes)

Anderson's "Introduction to Flight" (at least the chapters on orbital mechanics) is NOT part of the material to be studied for the exam; it is "just" reference material, for further reading.

Introduction

Why orbital mechanics ?

- Because the trajectory of a satellite is (primarily) determined by its initial position and velocity after launch
- Because it is necessary to know the position (and velocity) of a satellite at any instant of time
- Because the satellite orbit and its mission are intimately related

Satellites can perform remote-sensing (”observing from a distance”), with unparalleled coverage characteristics, and measure specific phenomena “in-situ”. If possible, the measurements have to be benchmarked/calibrated with “ground truth” observations.

LEO: typically at an altitude between 200 and 2000 km. GEO: at an altitude of about 36600 km, in equatorial plane.

Introduction (cnt'd)

Which questions can be addressed through orbital mechanics?

- What are the parameters with which one can describe a satellite orbit?
- What are typical values for a Low Earth Orbit?
- In what sense do they differ from those of an escape orbit?
- What are the requirements on a Geostationary Earth Orbit, and what are the consequences for the orbital parameters?
- What are the main differences between a LEO and a GEO, both from an orbit point of view and for the instrument?
- Where is my satellite at a specific moment in time?
- When can I download measurements from my satellite to my ground station?
- How much time do I have available for this?
- ……….

Some examples of relevant questions that you should be able to answer after having mastered the topics of these lectures.

Fundamentals

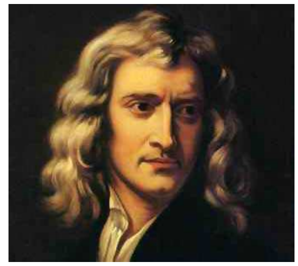
[Scienceweb, 2009]

Kepler's Laws of Planetary Motion:

1. The orbits of the planets are ellipses, with the Sun at one focus of the ellipse.
2. The line joining a planet to the Sun sweeps out equal areas in equal times as the planet travels around the ellipse.
3. The ratio of the squares of the revolutionary periods for two planets is equal to the ratio of the cubes of their semi-major axes.

The German Johannes Kepler (1571-1630) derived these empirical relations based on observations done by Tycho Brahe, a Danish astronomer. The mathematical foundation/explanation of these 3 laws were given by Sir Isaac Newton (next sheet).

Fundamentals (cnt'd)



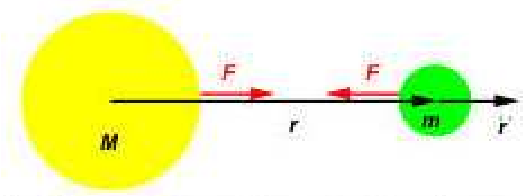
Newton's Laws of Motion:

1. In the absence of a force, a body either is at rest or moves in a straight line with constant speed.
2. A body experiencing a force **F** experiences an acceleration **a** related to **F** by **F** = m×**a**, where m is the mass of the body. Alternatively, the force is proportional to the time derivative of momentum.
3. Whenever a first body exerts a force **F** on a second body, the second body exerts a force −**F** on the first body. **F** and −**F** are equal in magnitude and opposite in direction.

Sir Isaac Newton (1643-1727), England.

Note: force **F** and acceleration **a** are written in bold, i.e. they are vectors (magnitude + direction).

Fundamentals (cnt'd)



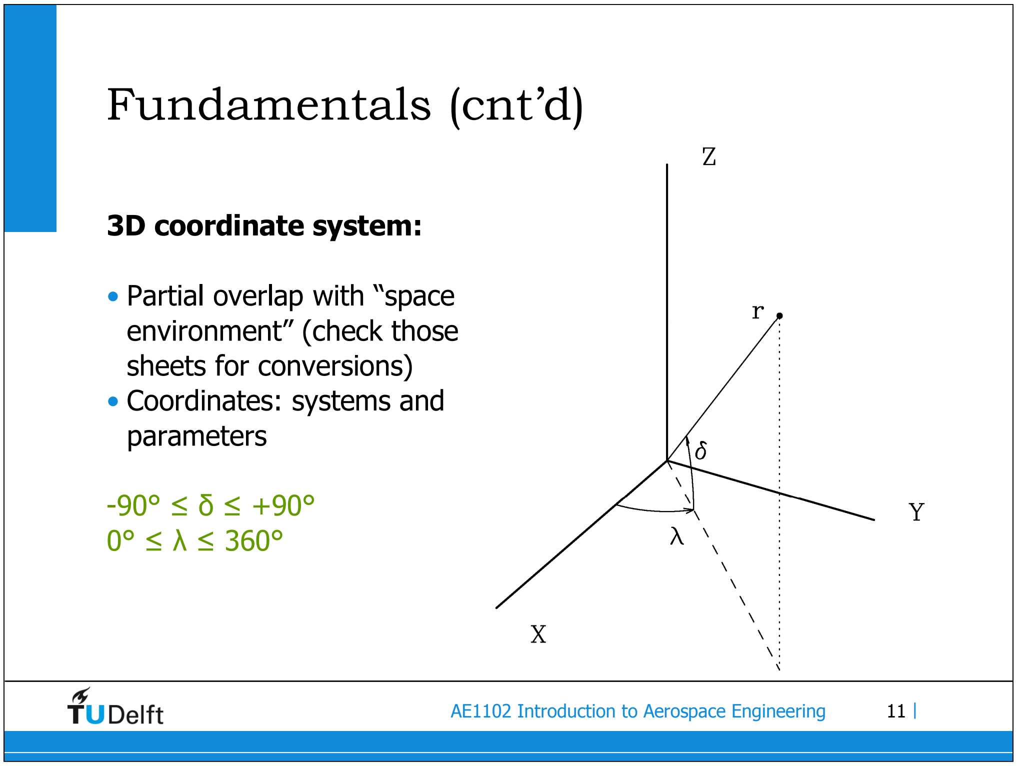
Newton's Law of Universal Gravitation:

Every point mass attracts every single other point mass by a force pointing along the line connecting both points. The force is directly proportional to the product of the two masses and inversely proportional to the square of the distance between the point masses:

$$F = G\frac{m_1\, m_2}{r^2}$$

Note 1: so, F has a magnitude and a direction → it should be written, treated as a vector.

Note 2: parameter "G" represents the universal gravitational constant; G = 6.6732 × 10^{-20} $km^3/kg/s^2$.



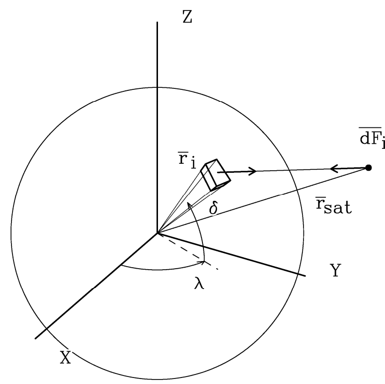
Selecting a proper reference system and a set of parameters that describe a position in 3 dimensions is crucial to quantify most of the phenomena treated in this chapter, and to determine what a satellite mission will experience. Option 1: cartesian coordinates, with components x, y and z. Option 2: polar coordinates, with components r (radius, measured w.r.t. the center-of-mass of the central object; not to be confused with the altitude over its surface), δ (latitude) and λ (longitude).

Fundamentals (cnt'd)

Gravitational attraction:

Elementary force: $$dF_i = \frac{G\, m_{sat}\, \rho\, dv}{r^2}$$

Total acceleration due to symmetrical Earth: $$\ddot{r} = -\frac{GM_{earth}}{r^2}$$



Parameter "G" is the universal gravitational constant (6.67259×10^{-11} $m^3/kg/s^2$), "m_{sat}" represents the mass of the satellite, "r" is the distance between the satellite and a mass element of the Earth (1st equation) or between the satellite and the center-of-mass of the Earth (2nd equation), "ρ" is the mass density of an element "dv" of the Earth [kg/m^3], "M_{earth}" is the total mass of the Earth (5.9737×10^{24} kg). The product of G and M_{earth} is commonly denoted as "μ" , which is called the gravitational parameter of the Earth ($=G\times M_{earth}=398600.44\times10^9$ m^3/s^2).

Gravitational acceleration for different “planets”

“planet”	mass [kg]	radius [km]	radial acceleration [m/s^2]	
			at surface	at h=1000 km
Sun	1.99×10^{30}	695990	274.15	273.36
Mercury	3.33×10^{23}	2432	3.76	3.47
Venus	4.87×10^{24}	6052	8.87	6.53
Earth	5.98×10^{24}	6378	9.80	7.33
Moon	7.35×10^{22}	1738	1.62	0.65
Mars	6.42×10^{23}	3402	3.70	2.21
Jupiter	1.90×10^{27}	70850	25.26	24.56
Saturn	5.69×10^{26}	60000	10.54	10.20
Uranus	8.74×10^{25}	25400	9.04	8.37
Neptune	1.03×10^{26}	25100	10.91	10.09

G = 6.6732×10^{-20} $km^3/kg/s^2$. Accelerations listed here are due to the central (*i.e.* main) term of the gravity field only.

Numerical example acceleration

Question:

Consider the Earth. What is the radial acceleration?

1. at sea surface
2. for an earth-observation satellite at 800 km altitude
3. for a GPS satellite at 20200 km altitude
4. for a geostationary satellite at 35800 km altitude

Answers: see footnotes below **(BUT TRY YOURSELF FIRST!)**

Answers **(DID YOU TRY?):**

1. 9.80 m/s^2
2. 7.74 m/s^2
3. 0.56 m/s^2
4. 0.22 m/s^2

Numerical example acceleration

Question:

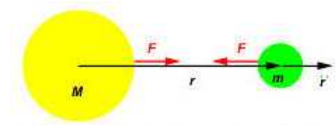
1. Consider the situation of the Earth, the Sun and a satellite somewhere on the line connecting the two main bodies. Where is the point where the attracting forces of the Earth and the Sun, acting on the satellite, are in equilibrium?
   Data: μ_{Earth} = 398600.4415 km^3/s^2, μ_{Sun} = 1.327178×10^{11} km^3/s^2, 1 AU (average distance Earth-Sun) = 149.6×10^6 km.
   Hint: trial-and-error.
2. The Moon orbits the Earth at a distance w.r.t. the center-of-mass of the Earth of about 384000 km. Still, the Sun does not pull it away from the Earth. Why?

Answers: see footnotes below **(BUT TRY YOURSELF FIRST!)**

Answers **(DID YOU TRY?):**

1. at a distance of 258811 km form the center of the Earth
2. two reasons: the Sun not only attracts the satellite in between, but also the Earth itself, so one needs to take the difference between the two; also, the centrifugal acceleration needs to be taken into account.

Fundamentals (cnt'd)



Gravitational attraction between 2 point masses or between 2 homogeneous spheres (with masses M and m):

$$F = G\frac{m\,M}{r^2} = M\,a_M = m\,a_m$$

Relative acceleration between mass m and mass M:

$$\ddot{r} = -a_M - a_m = -G\frac{m+M}{r^2}$$

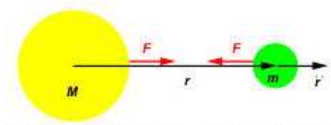
Or, with m << M (planet vs. Sun, or satellite vs. Earth):

$$\ddot{r} = -G\frac{M}{r^2} = -\frac{\mu}{r^2}$$

Note 1: m << M holds for most relevant combinations of bodies (sat-Earth, sat-Sun, planet-Sun), except for the Moon w.r.t. Earth.

Note 2: the parameter "μ" is called the gravitational parameter (of a specific body). Example: μ_{Earth} = 398600.4415 km^3/s^2 (relevant for the motion of satellites around the Earth), and μ_{Sun} = 1.328 × 10^{11} km^3/s^2 (relevant for motions of planets around the Sun, or spacecraft in heliocentric orbits).

Fundamentals (cnt'd)



Radial acceleration:

Scalar notation:

$$\ddot{r} = -\frac{\mu}{r^2}$$

Vector notation:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \quad \text{or} \quad \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = -\frac{\mu}{r^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

equation of motion for satellites and planets

Note: the vector **r** can easily be decomposed into its cartesian components x, y and z; the same can be done for the radial acceleration.

Equation of motion for (1) satellites orbiting around the Earth, (2) satellites orbiting around the Sun, and (3) planets orbiting around the Sun.

Fundamentals: conservation of angular momentum

1) Vectorial product of equation of motion with **r**:

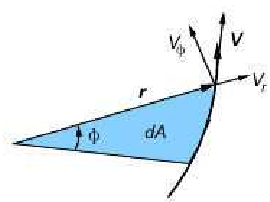
$$\mathbf{r} \times \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{r} = 0$$

so

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0$$

and

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{V} = \text{constant} = \mathbf{H}$$



- The motion is in one plane
- $H = rV_\varphi = r^2 (d\varphi/dt) = \text{constant}$
- Area law (second Law of Kepler): $dA/dt = \frac{1}{2} r\, r\, (d\varphi/dt) = \frac{1}{2} H$

Note: all parameters in **bold** represent vectors, all parameters in plain notation are scalars.

Fundamentals: conservation of energy

2) Scalar product of equation of motion with d**r**/dt:

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \frac{\mu}{r^3} \dot{\mathbf{r}} \cdot \mathbf{r} \quad = \quad 0$$

or

$$\frac{1}{2} \frac{d}{dt} (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) + \frac{1}{2} \frac{\mu}{r^3} \frac{d}{dt} (\mathbf{r} \cdot \mathbf{r}) \quad = \quad \frac{1}{2} \frac{d}{dt} (V^2) + \frac{1}{2} \frac{\mu}{r^3} \frac{d}{dt} (r^2) \quad = \quad 0$$

or

$$\frac{d}{dt} \left(\frac{1}{2} V^2 - \frac{\mu}{r} \right) \quad = \quad 0$$

Integration: $$\frac{V^2}{2} - \frac{\mu}{r} \quad = \quad \text{constant} \quad = \quad E$$

kinetic energy

potential energy
(incl. minus sign!)

Note: the step from (1/2) (μ/r^3) d(r^2)/dt to d($-\mu/r$)/dt is not a trivial one (if only for the change of sign….)

Fundamentals: orbit equation

3) Scalar product of equation of motion with **r**:

$$\mathbf{r}\cdot\ddot{\mathbf{r}}+\frac{\mu}{r^3}\mathbf{r}\cdot\mathbf{r} = 0$$

Therefore:

$$\frac{d}{dt}(\mathbf{r}\cdot\dot{\mathbf{r}})-(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})+\frac{\mu}{r} = 0$$

Note that $\mathbf{r}\cdot\dot{\mathbf{r}} = \mathbf{r}\cdot\mathbf{V} = rV_r$ and $\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \mathbf{V}\cdot\mathbf{V} = V^2$

so:

$$r\ddot{r}+\dot{r}^2-V^2+\frac{\mu}{r} = 0$$

Substitution of $V^2 = V_r^2+V_\varphi^2 = \dot{r}^2+(r\dot{\varphi})^2$

yields:

$$\ddot{r}-r\dot{\varphi}^2 = -\frac{\mu}{r^2}$$

Note 1: we managed to get rid of the vector notations, and are left with scalar parameters only.

Note 2: V_r is the magnitude of the radial velocity, V_φ is that of the tangential velocity (together forming the total velocity (vector) **V**).

Fundamentals: orbit equation (cnt'd)

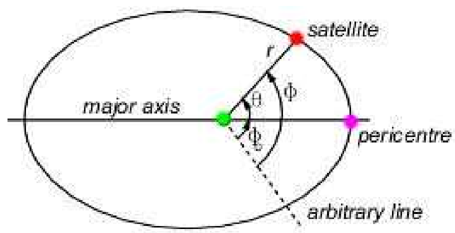
Combining equations $r^2\,\dot{\varphi} = H$ and $\ddot{r} - r\,\dot{\varphi}^2 = -\dfrac{\mu}{r^2}$

gives the equation for a conical section (1st Law of Kepler):

$$r = \frac{p}{1 + e\cos(\theta)}$$

where

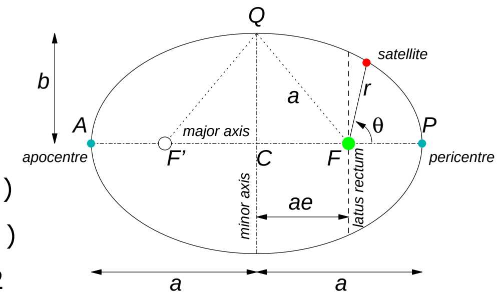
- $\theta = \varphi - \varphi_0$ = true anomaly
- φ_0 = arbitrary angle
- e = eccentricity
- $p = H^2/\mu$ = semi-latus rectum



Elliptical orbit

Orbital equation: $2a = r_{\theta=0} + r_{\theta=\pi} = \frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2} \quad \Rightarrow \quad p = a(1-e^2)$

So: $$r = \frac{a(1-e^2)}{1+e\cos(\theta)}$$



Other expressions:

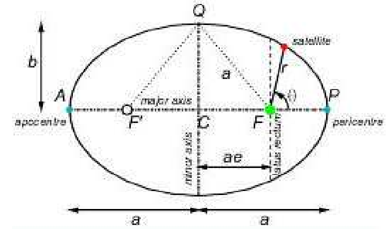
- pericenter distance $r_p = a(1 - e)$
- apocenter distance $r_a = a(1 + e)$
- semi-major axis $a = (r_a + r_p)/2$
- eccentricity $e = (r_a - r_p)/(r_a + r_p)$
- location of focal center $CF = a - r_p = a e$

The wording "pericenter" and "apocenter" is for a general central body. For orbits around Earth, we can also use "perigee" and "apogee", and for orbits around the Sun we use "perihelion" and "apohelion".

Elliptical orbit (cnt'd)

Example:

Satellite in orbit with pericenter at 200 km altitude and apocenter at 2000 km:



- $r_p = R_e + h_p = 6578$ km
- $r_a = R_e + h_a = 8378$ km
- $a = (r_a + r_p) / 2 = 7478$ km
- $e = (r_a - r_p) / (r_a + r_p) = 0.1204$

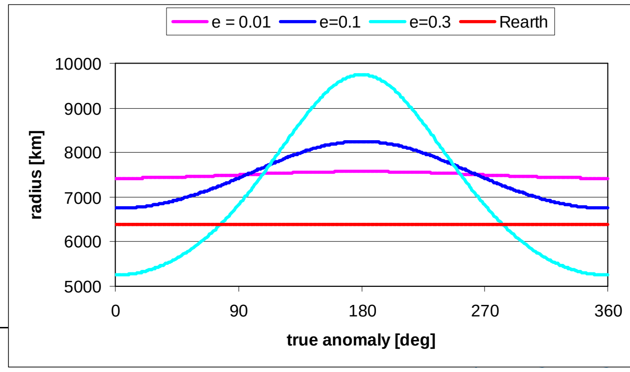
Note: the eccentricity can also be computed from the combination of pericenter radius and semi-major axis: $r_p=a(1-e)$ (or, for that matter, the combination of apocenter radius and semi-major axis: $r_a=a(1+e)$).

Note the difference between "radius" and "altitude" or "height" !!!

Elliptical orbit (cnt'd)

Example:

Satellite in orbit with semi-major axis of 7500 km and eccentricity of 0.01, 0.1 or 0.3:



Note: the variation in radial distance becomes larger for larger values of the eccentricity. For e=0.3 the pericenter value dips below the Earth radius ➔ physically impossible orbit.

Note the difference between "radius" and "altitude" or "height" !!!

Elliptical orbit: velocity and energy

conservation of angular momentum:

$$H = r_p V_p = r_a V_a \quad \Rightarrow \quad V_a = V_p \frac{r_p}{r_a}$$

conservation of energy:

$$E = \frac{1}{2} V_p^{\,2} - \frac{\mu}{r_p} = \frac{1}{2} V_a^{\,2} - \frac{\mu}{r_a}$$

substituting r_p, r_a, V_a yields:

$$V_p^{\,2} = \frac{\mu}{a} \left(\frac{1+e}{1-e} \right)$$

and

$$V_a^{\,2} = \frac{\mu}{a} \left(\frac{1-e}{1+e} \right)$$

Straightforward derivation of simple relations for the velocity at pericenter and apocenter.

Elliptical orbit: velocity and energy (cnt'd)

conservation of energy:

$$E = \frac{1}{2}V_p^{\,2} - \frac{\mu}{r_p} = -\frac{\mu}{2a} = \text{constant}$$

more general:

$$\frac{1}{2}V^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

so

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

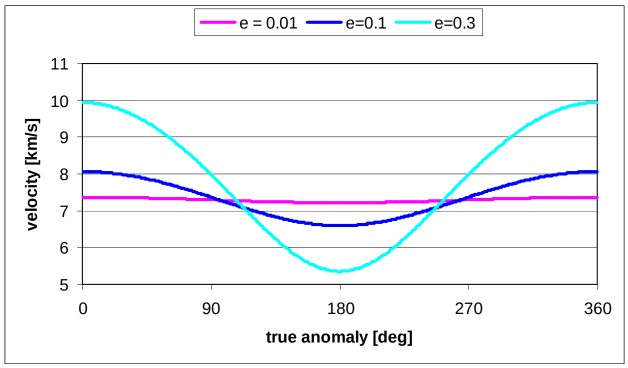
the "vis-viva" equation

The "vis-viva" equation gives an easy and direct relation between velocity and position (and semi-major axis). It does not say anything about the direction of the velocity. In turn, a satellite position and velocity (magnitude) determine the total amount of energy of the satellite, but can result in a zillion different orbits (with the same value for the semi-major axis, though).

Elliptical orbit: velocity and energy (cnt'd)

Example:

satellite in an orbit with semi-major axis of 7500 km and eccentricity of 0.01, 0.1 and 0.03:



Note: the orbit with a very low eccentricity hardly shows any variation in velocity, whereas for the orbit with highest velocity (e=0.3) the variation is almost a factor 2.

Elliptical orbit: orbital period

Area law (Kepler's 2nd equation):

$$\frac{dA}{dt} = \frac{1}{2} H = \text{constant} \quad \Rightarrow \quad A = \int_0^T \frac{dA}{dt} dt = \frac{1}{2} H\, T = \pi\, a\, b$$

Also: $b = a\sqrt{1-e^2}$ and $H = \sqrt{\mu\, p}$ and $p = a(1-e^2)$

Leads to Kepler's 3rd law:

$$T = 2\pi \sqrt{\frac{a^3}{\mu}}$$

Important conclusion: the orbital period in an elliptical orbit "T" is fully determined by the value of the semi-major axis "a" and the gravitational parameter "μ"; the shape of the orbit (as indicated by the eccentricity "e") does not play a role here!

Fundamentals (summary)

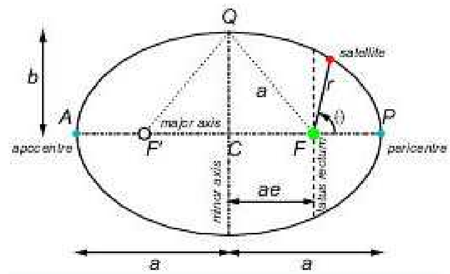
[Scienceweb, 2009]

Kepler's Laws of Planetary Motion revisited, now in mathematical formulation:

1. $$r = \frac{a\,(1-e^2)}{1+e\cos(\theta)}$$

2. $$\frac{dA}{dt} = \frac{1}{2}H = \frac{1}{2}\|\mathbf{r}\times\mathbf{V}\|$$

3. $$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$



See earlier sheet on Kepler.

Elliptical orbit: example

Numerical example 1:

Orbit around Earth, h_p = 300 km, h_a = 10000 km

Questions: a? e? V_p? V_a? T?

Answers: see footnotes below **(TRY FIRST !)**

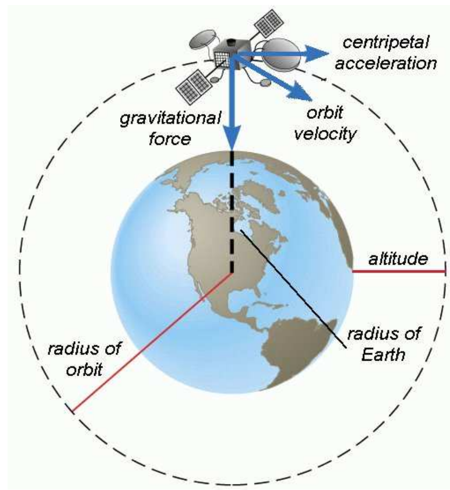
Answers: **(DID YOU TRY?)**

- $r_p = R_{earth} + h_p = 300 = 6678$ km
- $r_a = R_{earth} + h_a = 16378$ km
- $a = (r_p+r_a)/2 = 11528$ km
- $e = (r_a-r_p)/(r_a+r_p) = 0.421$
- $V_p = 9.209$ km/s
- $V_a = 3.755$ km/s
- $T = 2\pi\sqrt{(a^3/\mu)} = 12318.0$ s = 205.3 min

Circular orbit

Characteristics:

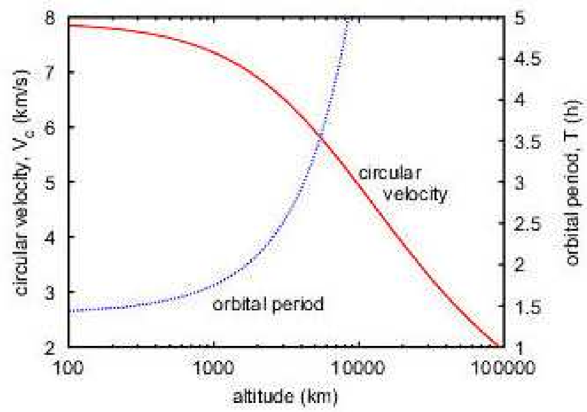
- $e = 0$
- $r_{min} = r_{max} = r$
- $a = r$
- $V = V_c = \sqrt{(\mu/a)}$
- $T = 2\pi\sqrt{(a^3/\mu)}$
- $E_{tot} = -\mu/2a < 0$



[Aerospaceweb, 2009]

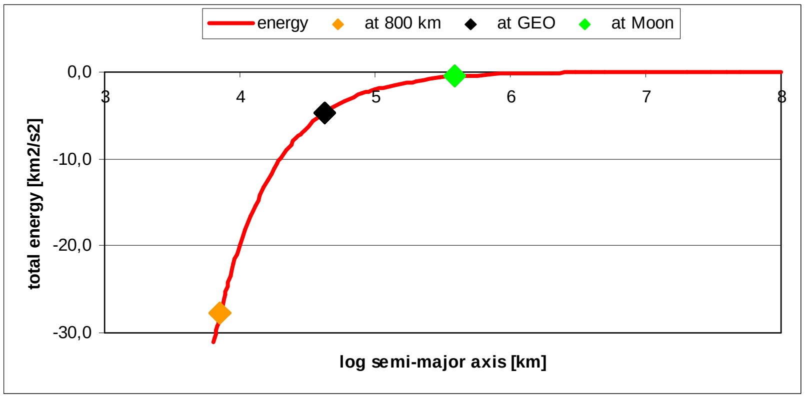
Some characteristics of circular orbits. The expressions can be easily verified by substituting e=0 in the general equations derived for an ellipse (with 0<e<1).

Circular orbit (cnt'd)



The orbital velocity at low altitudes is 7-7.9 km/s, but at higher altitudes it reduces quickly (notice log scale for altitude). The reverse happens with the orbital period. In the case of a circular orbit, the orbital period T and the velocity V are related to each other by the equation $T*V = 2\pi r = 2\pi a$. Do not confuse altitude (i.e. w.r.t. surface of central body) and radius (i.e. w.r.t. center of mass of central body).

Circular orbit (cnt'd)



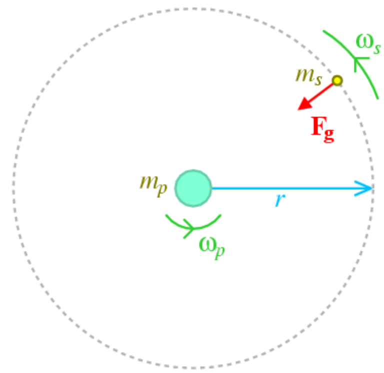
The energy required to get into a particular orbit initially quickly increases with the value of the semi-major axis, but then levels off. The step to go from 800 km altitude to geostationary altitude is much more difficult (energy-wise) that the step from the GEO to the Lunar orbit (let alone into parabolic/hyperbolic/escape orbit).

Geostationary orbit

Requirements:

- Stationary (*i.e.* non-moving) w.r.t. Earth surface
  - orbital period = 23 hour, 56 min and 4 sec
  - moves in equatorial plane
  - moves in eastward direction
- So: orbital elements:
  - e = 0
  - a = 42164.14 km
  - i = 0°
- And:
  - $h = a - R_e = 35786$ km
  - $V_c = 3.075$ km/s
  - $E = -4.727\ km^2/s^2$

[Sque, 2009]:



"geo" "stationary" as in "Earth" "fixed". The orbital period is related to the revolution of the Earth w.r.t. an inertial system, *i.e.* the stars -> use $23^h56^m4^s$ instead of our everyday-life 86400 s.

The value of "a" is derived from the expression for the orbital period. In reality, the effect of J_2 needs to be added, which causes the real altitude of the GEO to be some AAAA km higher.

Geostationary orbit (cnt'd)

Questions:

Consider an obsolete GEO satellite which is put into a graveyard orbit: 300 km above the standard GEO altitude.

1. What is the orbital period of this graveyard orbit?
2. If this graveyard orbit were to develop from perfectly circular to eccentric, what would be the maximum value of this eccentricity when the pericenter of this orbit were to touch the real GEO? Assume that the apocenter of this deformed graveyard orbit remains at GEO+300 km.

Answers: see footnotes below **BUT TRY FIRST !!**

Answers:

1) T = 24 uur, 11 minuten en 25.3 seconden.

2) e = 0.00354

Elliptical orbit: position vs. time

- Where is the satellite at a specific moment in time?
- When is the satellite at a specific position?

Why?

- to aim the antenna of a ground station
- to initiate an engine burn at the proper point in the orbit
- to perform certain measurements at specific locations
- to time-tag measurements
- to be able to rendez-vous
- …….

Elliptical orbit: position vs. time (cnt'd)

straightforward approach:

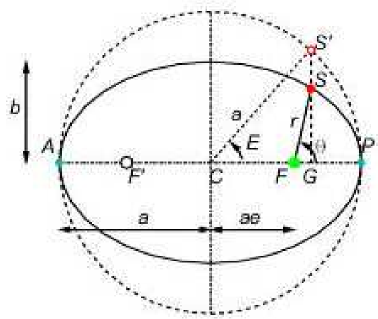
$$\frac{d\theta}{dt} = \frac{H}{r^2} \quad \Rightarrow \quad dt = \frac{r^2}{H} d\theta \quad \Rightarrow \quad \Delta t = \int dt = \int \frac{r^2}{H} d\theta$$

so

$$\Delta t = \sqrt{\frac{p^3}{\mu}} \int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2}$$

difficult relation $\rightarrow$

introduce new parameter E ("eccentric anomaly")



The straightforward approach is clear but leads to a difficult integral. Can be treated numerically, but then one might just as well give up the idea of using Kepler orbits and switch to numerical representations altogether. Do not confuse E ("eccentric anomaly") with E ("energy")!!

Elliptical orbit: position vs. time (cnt'd)

$$r\cos\theta = a\cos E - a\,e$$

ellipse:

$$\frac{GS}{GS'} = \frac{b}{a} = \sqrt{1-e^2}$$

here:

$$\frac{GS}{GS'} = \frac{r\sin\theta}{a\sin E}$$

or

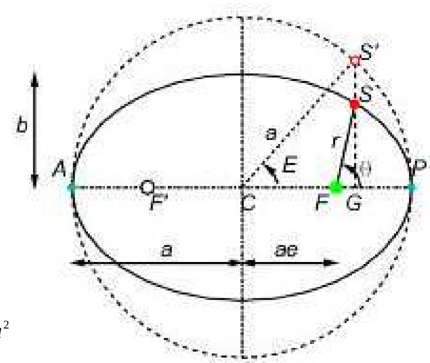
$$r\sin\theta = a\sqrt{1-e^2}\,\sin E$$

combining:

$$r^2 = (a\cos E - a\,e)^2 + (a\sqrt{1-e^2}\,\sin E)^2 = a^2$$

or (r>0):

$$r = a\,(1 - e\cos E)$$



S' and the eccentric anomaly E are related to a perfect circle with radius "a". E and θ are related to each other.

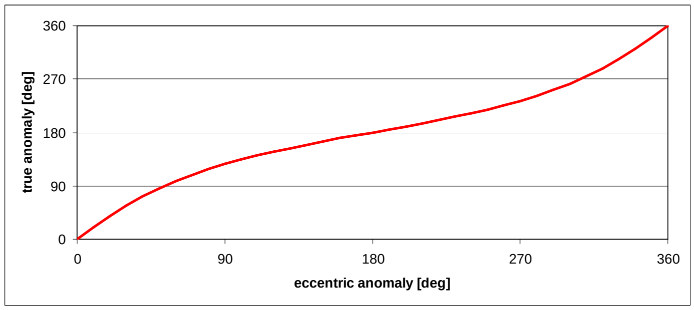
Elliptical orbit: position vs. time (cnt'd)

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = a(1-e\cos E)$$

from which

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

e.g. e=0.6 :



The relation between E and θ is unambiguous. The derivation of the relation between tan(θ/2) and tan(E/2) is tedious…..

Elliptical orbit: position vs. time (cnt'd)

Some numerical examples, for e=0.4 :

E [°]	E/2 [°]	θ/2 [°]	θ [°]
80	40	52.04	104.08
100	50	61.22	122.44
170	85	86.72	173.44
190	95	93.28	186.56
260	130	118.78	237.56
280	140	127.96	255.92
350	175	172.39	344.78

Verify!

Verify!

Elliptical orbit: position vs. time (cnt'd)

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad \text{so} \quad \dot{r} = \frac{\mu e \sin\theta}{\sqrt{\mu a(1-e^2)}}$$

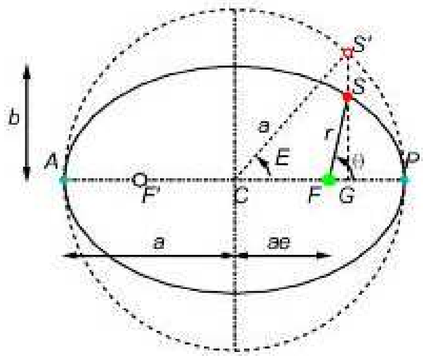
and

$$r = a(1-e\cos E) \quad \text{so} \quad \dot{r} = a e \dot{E} \sin E$$

(all derivatives taken after time t)

equating and using $\dfrac{r\sin\theta}{a\sin E} = \sqrt{1-e^2}$

(continued on next page)



2nd step in derivation of required relation.

Elliptical orbit: position vs. time (cnt'd)

$$\dot{E} = \sqrt{\frac{\mu}{a^3}} \frac{1}{1 - e\cos E}$$

or

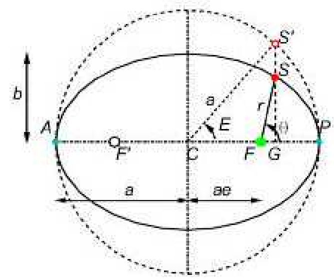
$$(1 - e\cos E)\,\dot{E} = \sqrt{\frac{\mu}{a^3}}$$

integration:

$$E - e\sin E = \sqrt{\frac{\mu}{a^3}}(t - t_p) = n(t - t_p) = M$$

or

$$M \quad = \quad E - e\sin E$$



"Kepler's Equation", where:

- t = current time [s]
- t_p = time of last passage pericenter [s]
- n = mean motion [rad/s]
- M = mean anomaly [rad]
- E = eccentric anomaly [rad]
- θ = true anomaly [rad]

Kepler's equation gives the relation between time (t, in [s]) and position (M and/or E, in [rad]). It holds for an ellipse, but other formulations also exist for hyperbola and parabola.

Elliptical orbit: position vs. time (cnt'd)

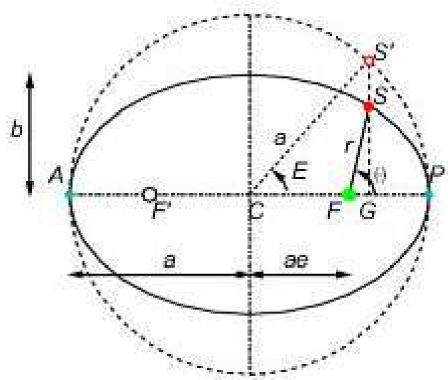
Example position → time:

Question:

- a = 7000 km
- e = 0.1
- μ = 398600 km^3/s^2
- θ = 35°
- $t-t_p$?

Answer:

- $n = 1.078007 \times 10^{-3}$ rad/s
- θ = 0.61087 rad
- E = 0.55565 rad
- M = 0.50290 rad
- $t-t_p$ = 466.5 s

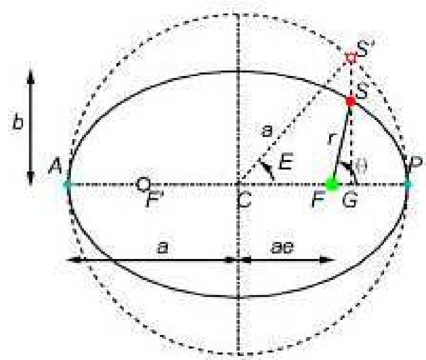


verify!

Straightforward application of recipe.

Elliptical orbit: position vs. time (cnt'd)

Example time → position (1-2):



Question

- a = 7000 km
- e = 0.1
- μ = 398600 km^3/s^2
- $t-t_p$ = 900 s
- θ ?

Straightforward application of recipe.

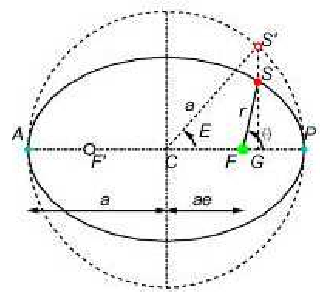
Elliptical orbit: position vs. time (cnt'd)

Example time → position (2-2):

Answer

- $n = 1.078007 \times 10^{-3}$ rad/s
- $M = 0.9702$ rad
- $E = 1.0573$ rad (iterate $E_{i+1} = M + e \sin(E_i)$)
- $\theta = 1.1468$ rad

verify!



iteration	E
0	0,970206
1	1,052707
2	1,057083
3	1,057299
4	1,057309
5	1,057310
6	1,057310

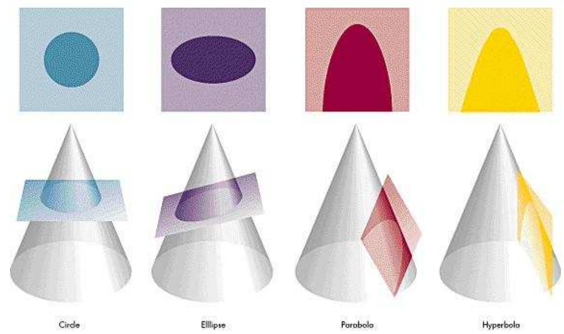
Straightforward application of recipe.

Fundamentals (cnt'd)

"Conical sections"

Orbit types:

- circle
- ellipse
- parabola
- hyperbola



[Cramer, 2009]

Summary of orbit types. See next page for summary of characteristics.

Parabolic orbit

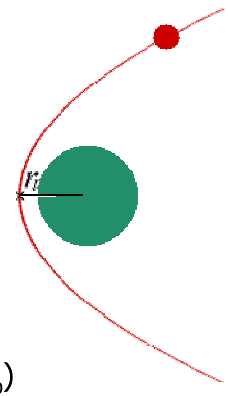
Characteristics:

$$r = \frac{p}{1 + e\cos(\theta)}$$

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

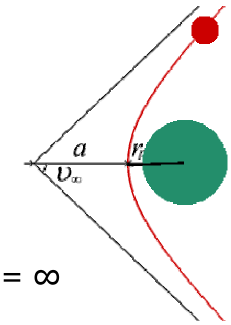
$$E = -\frac{\mu}{2a}$$

- e = 1
- $r_{min} = r_p$, $r_{max} = \infty$
- a = ∞
- $V_{rp} = V_{esc,rp} = \sqrt{(2\mu/r_p)}$
- $V_{min} = 0$
- $T_{pericenter \rightarrow inf} = \infty$
- $E_{tot} = 0$



Some characteristics of parabolic orbits. This is an "open" orbit, so there is no orbital period. The maximum distance (i.e. ∞) is achieved for θ=180°.

Hyperbolic orbit



Characteristics:

$$r = \frac{p}{1 + e\cos(\theta)}$$

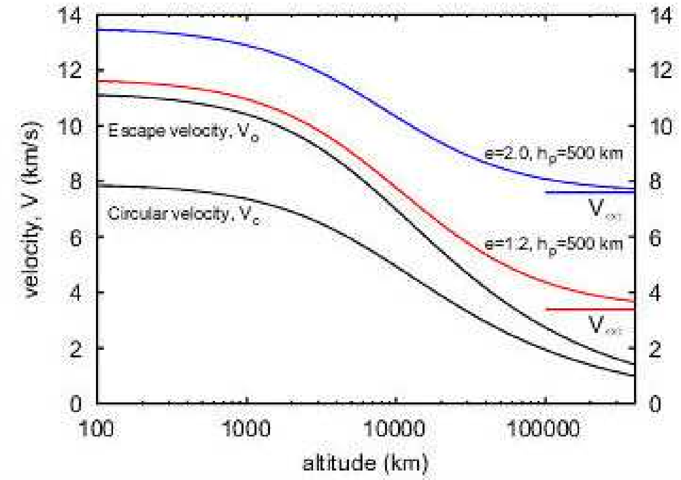
$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$E = -\frac{\mu}{2a}$$

- $e > 1$
- $r_{min} = r_p$, $r_{max} = \infty$
- $a < 0$
- $V_{rp} > V_{esc,rp}$
- $V^2 = V_{esc}^2 + V_{\infty}^2$
- $V_{\infty} > 0$
- $T_{pericenter \rightarrow inf} = \infty$
- $E_{tot} > 0$

Some characteristics of hyperbolic orbits. This is an “open” orbit, so there is no orbital period. The maximum distance (i.e. ∞) is achieved for a limiting value of θ, given by the zero crossing of the numerator of the equation for “r”: $1 + e \cos(\theta_{lim}) = 0$.

Hyperbolic orbit (cnt'd)



$V_{circ} = \sqrt{(\mu/r)}$; $V_{escape} = \sqrt{(2\mu/r)}$; $V_{hyperbola}{}^2 = V_{escape}{}^2 + V_\infty{}^2$ (so, at infinite distance $V_{hyperbola} = V_\infty$ as should be).

Hyperbola: position vs. time

in a similar fashion as for an elliptical orbit:

introduce hyperbolic anomaly F:

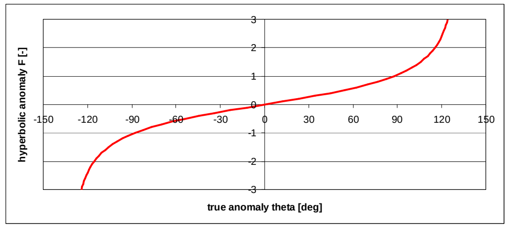
$$r = a\,(1 - e\cosh F\,)$$

without derivation:

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\,\tanh\frac{\mathrm{F}}{2}$$

relation between time and position:

$$e\sinh F - F = \sqrt{\frac{\mu}{-a^3}}\,\left(t - t_p\right) = M$$

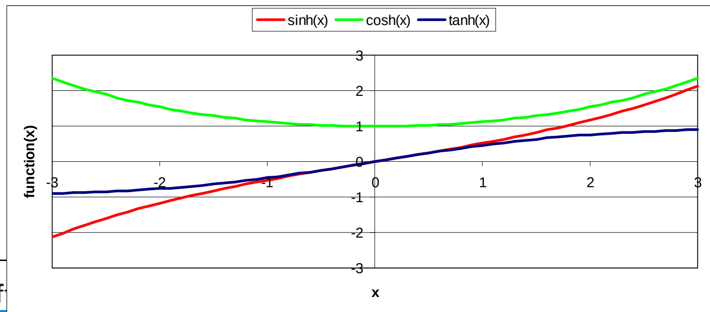


For a hyperbola, the same question arises. The time-position problem for the parabola is skipped because it too specific (e=1.000000000000000).

PLEASE NOTE: F is NOT an angle but a dimensionless parameter

Hyperbola: position vs. time (cnt'd)

$$\begin{aligned} \sinh(x) &= \frac{1}{2}\left(e^{x}-e^{-x}\right) & \quad \textit{also}: \quad \operatorname{asinh}(x) &= \ln\left(x+\sqrt{x^{2}+1}\right) \\ \cosh(x) &= \frac{1}{2}\left(e^{x}+e^{-x}\right) & \operatorname{acosh}(x) &= \ln\left(x+\sqrt{x^{2}-1}\right) \\ \tanh(x) &= \frac{\sinh(x)}{\cosh(x)} & \operatorname{atanh}(x) &= \frac{1}{2}\ln\frac{1+x}{1-x} \end{aligned}$$



General definitions of hyperbolic functions.

Hyperbola: position vs. time (cnt'd)

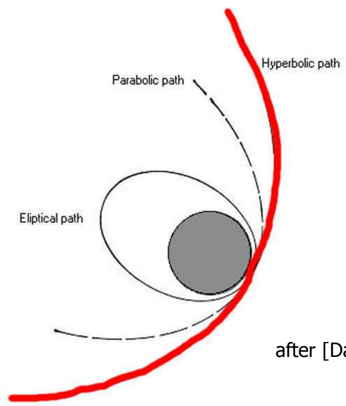
Example position → time:

Question:

- $a = -7000$ km
- $e = 2.1$
- $\mu = 398600$ km^3/s^2
- $\theta = 35°$
- $t-t_p$?

Answer:

- $n = 1.078007 \times 10^{-3}$ rad/s
- $F = 0.38$
- $M = 0.4375$ rad
- $t-t_p = 405.87$ s



after [Darling, 2009]

verify!

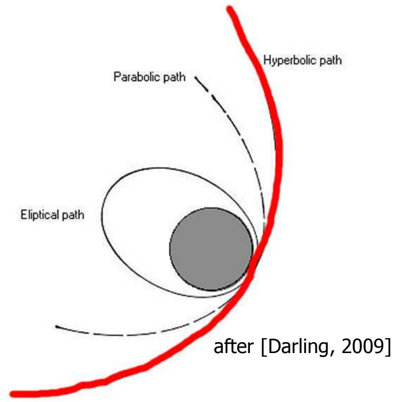
Straightforward application of recipe.

Hyperbola: position vs. time (cnt'd)

Example time → position:

Question:

- $a = -7000$ km
- $e = 2.1$
- $\mu = 398600$ km^3/s^2
- $t - t_p = 900$ s
- θ ?



after [Darling, 2009]

Continued on next slide

Hyperbola: position vs. time (cnt'd)

Example time → position (cnt'd):

Answer:

- $n = 1.078007 \times 10^{-3}$ rad/s
- $M = 0.9702$ rad
- $F = 0.7461$ (iterate $F_{k+1} = F_k - (e \sinh(F_k) - F_k - M) / (e \cosh(F_k) - 1)$

Iteration (k)	sinh(F)	cosh(F)	F_k
0			0 (first guess)
1	0	1	0,882006
2	1,000894	1,414846	0,755347
3	0,829252	1,299099	0,746161
4	0,817353	1,291536	0,746118
5	0,817297	1,291501	0,746118

- $\theta = 1.0790$ rad (*i.e.* 61.8220°)

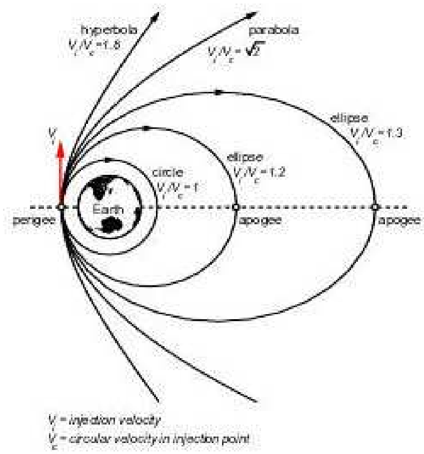
verify!

Straightforward application of recipe. The solution for the hyperbolic anomaly F is obtained by means of Newton-Raphson iteration (see formula); This is necessary because a direct iteration (like in the case of elliptical orbits) does not converge for an eccentricity greater than 1.

Orbits: Summary

Orbit types:

- circle
- ellipse
- parabola
- hyperbola



Summary of orbit types. See next page for summary of characteristics.

Orbits: Summary (Cnt'd)

	circle	ellipse	parabola	hyperbola
e	0	$0 < e < 1$	1	> 1
a	> 0	> 0	∞	< 0
r	a	$p / (1+e \cos(\theta))$	$p / (1+e \cos(\theta))$	$p / (1+e \cos(\theta))$
r_{min}	a	$a\,(1-e)$	$p/2$	$a\,(1-e)$
r_{max}	a	$a\,(1+e)$	∞	∞
V	$\sqrt{(\mu/r)}$	$< \sqrt{(2\mu/r)}$	$\sqrt{(2\mu/r)}$	$> \sqrt{(2\mu/r)}$
E_{tot}	< 0	< 0	0	➢0
θ,E,F	$E = \theta$	$\tan(E/2) = \sqrt{((1-e)/(1+e))}\,\tan(\theta/2)$	-	$\tanh(F/2) = \sqrt{((e-1)/(e+1))}\,\tan(\theta/2)$
M	$\sqrt{(\mu/a^3)}\,(t-\tau)$	$\sqrt{(\mu/a^3)}\,(t-\tau)$	$\sqrt{(\mu/p^3)}\,(t-\tau)$	$\sqrt{(\mu/-a^3)}\,(t-\tau)$
T	$M = E$	$M = E - e \sin(E)$	$2M = \tan(\theta/2) + ((\tan(\theta/2))^3)/3$	$M=e \sinh(F)-F$

Summary of orbit characteristics. After K.F. Wakker, lecture notes ae4-878

Example elliptical orbit

<u>Question:</u>

Consider a satellite at 800 km altitude above the Earth's surface, and velocity (perpendicular to the radius) of 8 km/s.

1. compute the semi-major axis
2. compute the eccentricity
3. compute the maximum radius of this orbit
4. compute the maximum altitude

<u>Answers:</u>

1. $0.5\ V^2 - \mu/r = -\ \mu/2a \rightarrow a = 8470$ km
2. $r_p = a(1\text{-}e) \rightarrow e = 0.153$
3. $r_a = a(1+e) \rightarrow r_a = 9766$ km
4. $h = r\text{-}R_e \rightarrow h_{max} = 3388$ km

Verify !!

Straightforward application of recipe. $R_e = 6378.137$ km, $\mu_{earth} = 398600.4415$ km^3/s^2.

Example elliptical orbit

<u>Question:</u>

Consider a satellite at 800 km altitude above the Earth's surface, and velocity (perpendicular to the radius) of 10 km/s.

1. compute the semi-major axis
2. compute the eccentricity
3. compute the maximum radius of this orbit
4. compute the maximum altitude
5. what would be the minimum velocity needed to escape from Earth?

Answers: see footnotes below **(BUT TRY YOURSELF FIRST!!)**

Answers: **(DID YOU TRY FIRST??)**

1. a = 36041 km
2. e = 0.801
3. r_a = 64910 km
4. h_{max} = 58532 km
5. V_{esc} = 10.538 km/s

Example hyperbolic orbit

Question:

Consider a satellite at 800 km altitude above the Earth's surface, with its velocity perpendicular to the radius.

1. compute the escape velocity
2. if the satellite would have a velocity 0.2 km/s larger than the escape velocity, what would the excess velocity V_∞ be?
3. idem, if $V = V_{esc} + 0.4$?
4. idem, if $V = V_{esc} + 0.6$?
5. idem, if $V = V_{esc} + 0.8$?
6. idem, if $V = V_{esc} + 1.0$?
7. what conclusion can you draw?

Answers: see footnotes below **(BUT TRY YOURSELF FIRST!!)**

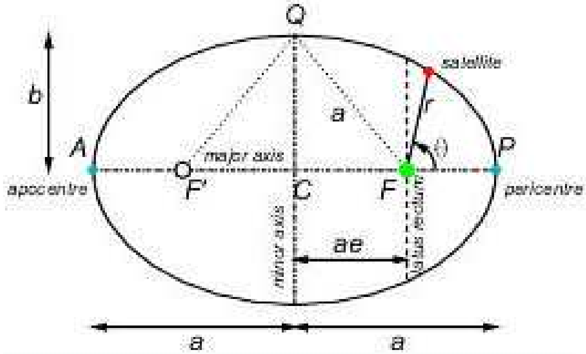
Answers: **(DID YOU TRY FIRST??)**

1. Vesc = 10.538 km/s
2. V_∞ = 2.063 km/s
3. V_∞ = 2.931 km/s
4. V_∞ = 3.606 km/s
5. V_∞ = 4.183 km/s
6. V_∞ = 4.699 km/s
7. a small increase in velocity at 800 km altitude pays off in a large value for the excess velocity.

Elliptical orbit: Summary

2-dimensional orbits: 3 Kepler elements:

- a – semi-major axis [m]
- e – eccentricity [-]
- t_p , τ – time of pericenter passage [s]

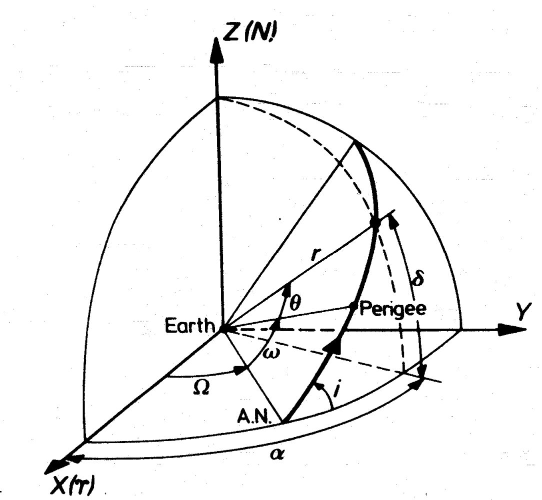


The time of passage of a well-defined point in the orbit (e.g. the pericenter) is indicated by “t_p” or, equivalently, “τ” (the Greek symbol tau). Knowing this value, one can relate the position in the orbit to absolute time (cf. following sheets).

Fundamentals (cnt'd)

3-dimensional orbits:
another 3 Kepler elements:

- i – inclination [deg]
- Ω - right ascension of ascending node [deg]
- ω – argument of pericenter [deg]



The inclination "i" is the angle between the orbital plane and a reference plane, such as the equatorial plane. It is measured at the ascending node, i.e. the location where the satellite transits from the Southern Hemisphere to the Northern Hemisphere, so by definition its value is between 0° and 180°. The parameters Ω and ω can take any value between 0° and 360°.