Flight and Orbital Mechanics

Lecture slides

Flight and Orbital Mechanics

Lecture hours 3, 4 – Minimum time to climb

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Semester 1 - 2012

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- Introduction
- Summary previous lecture
- Performance diagram
- Optimal climb
- Solution low speed aircraft
- Energy height
- Solution high speed aircraft (M>1)
- Example exam question
- Summary

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Introduction

Question

What is the most efficient way (minimum time) to go from take-off at sea-level to Mach 1.5 at 15,000 m?

A. Climb at airspeed for max γ. At 15,000 m accelerate to Mach 2

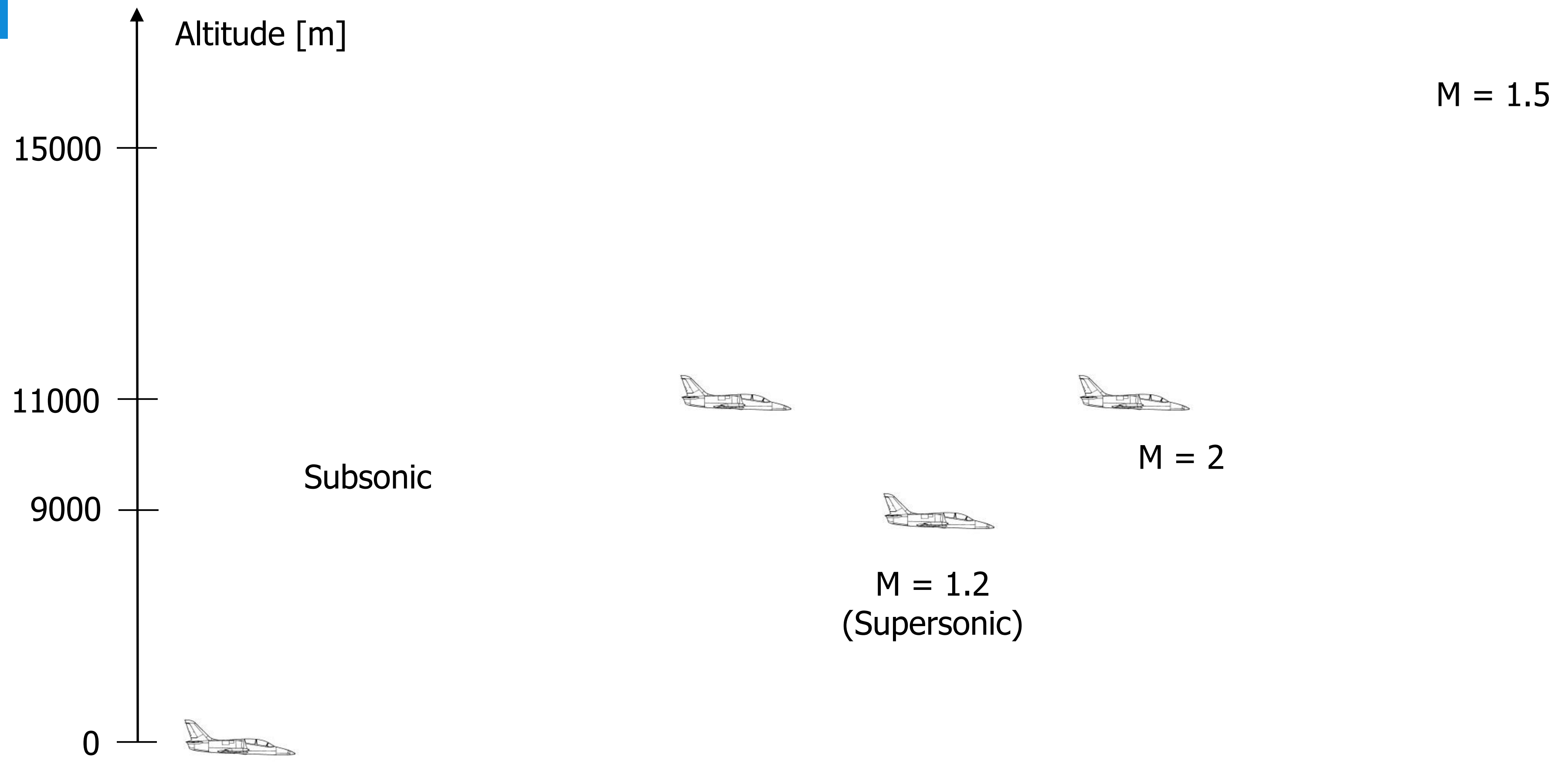
B. Climb at airspeed for max RC. At 15,000 m accelerate to Mach 2

√ **C. Climb at airspeed for max RC to 11,000 m. Descent and accelerate to Mach 1.2 at 9,000 m, climb and accelerate to Mach 2 at 11,000 m. Decelerated climb to 15,000 m, Mach 1.5**

D. Accelerate at sea-level to Mach 1.5, climb to 15,000 m at airspeed for max RC

Introduction

Solution



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Summary previous lecture

- A typical climb (civil subsonic aircraft) is performed at **constant indicated airspeed and** at a constant power setting**.** Therefore, the **true airspeed** is actually **increasing**. Since airspeed is not constant, it is an **unsteady climbing** flight

- The climb is almost a straight line. It is therefore a **quasi-rectilinear** flight

- Corresponding equations of motion:

$$\boxed{\begin{aligned} &T - D - W\sin\gamma = \frac{W}{g}\frac{dV}{dt} \\ &L = W \end{aligned}}$$

Summary previous lecture

- The rate of climb in an unsteady climb is smaller than in a steady climb because part of the excess energy is used to accelerate

$$\boxed{\frac{RC}{RC_{st}} = \frac{1}{1 + \frac{V}{g}\frac{dV}{dh}}; \qquad \frac{dV}{dh} > 0}$$

- You must be able to derive this ratio from the equations of motion
- You must be able to calculate this ratio (see example exam question)
- For more background information: read Ruijgrok – Elements of airplane performance section 14.2

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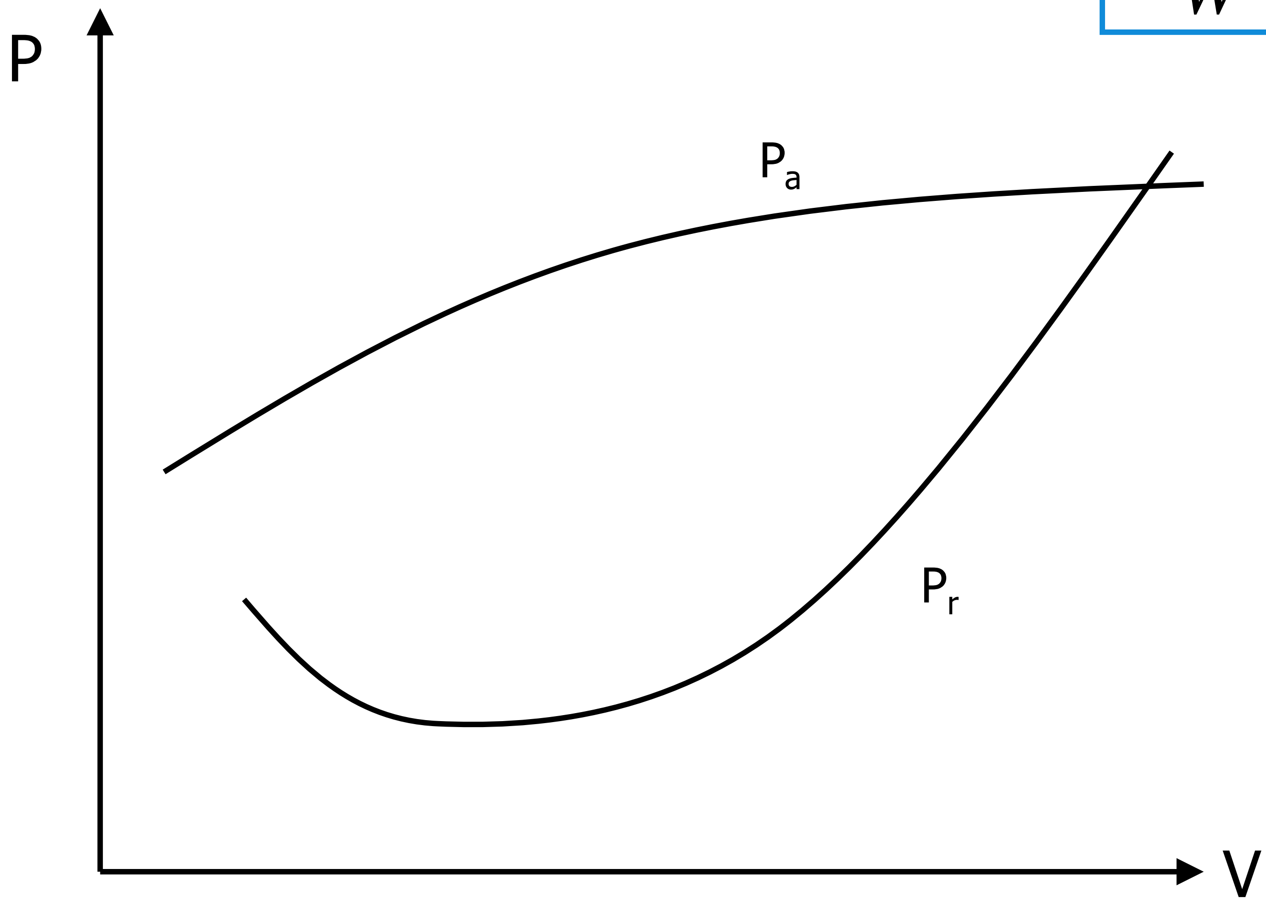
Performance diagram

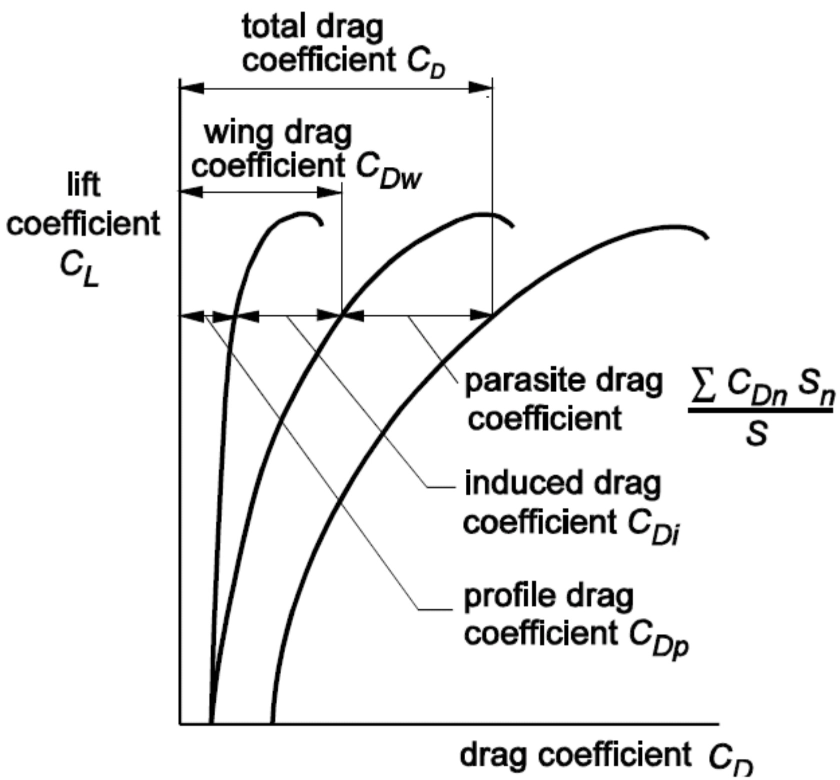
$$\boxed{\frac{P_a - P_r}{W} = RC_{steady}}$$

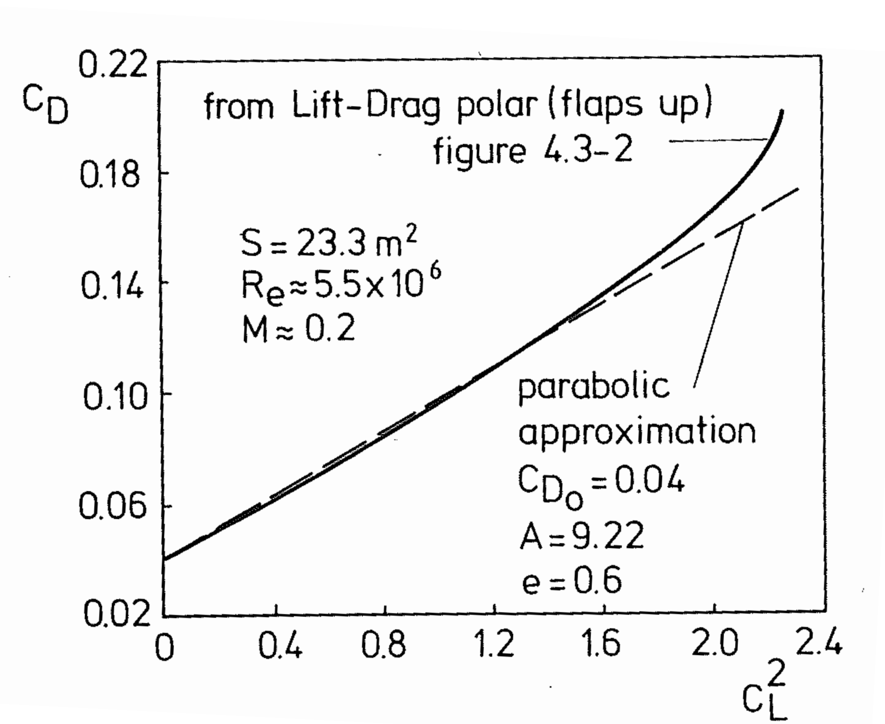


Aerodynamic forces (year 1)

Lift – Drag polar

$$C_D = C_{D_0} + \frac{C_L^2}{\pi A e}$$





Aerodynamic forces (year 1)

Lift

$$L = W$$

$$C_L \tfrac{1}{2} \rho V^2 S = W$$

$$C_L = \frac{W}{S} \frac{2}{\rho} \frac{1}{V^2}$$

Drag

$$C_D = C_{D_0} + \frac{C_L^2}{\pi A e}$$

$$D = C_D \tfrac{1}{2} \rho V^2 S$$

$$D = C_{D_0} \tfrac{1}{2} \rho V^2 S + \frac{C_L^2}{\pi A e} \tfrac{1}{2} \rho V^2 S$$

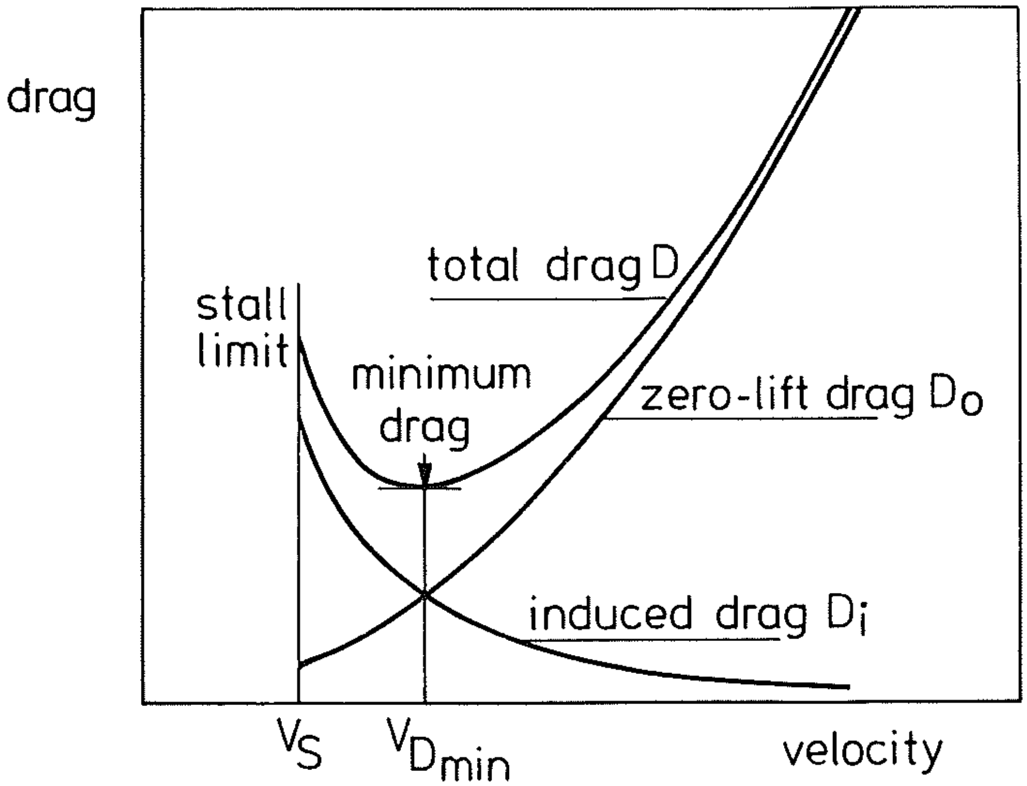
$$D = C_{D_0} \tfrac{1}{2} \rho V^2 S + \frac{W^2}{S^2} \frac{4}{\rho^2} \frac{1}{V^4} \frac{1}{\pi A e} \tfrac{1}{2} \rho V^2 S$$

$$D = C_{D_0} \tfrac{1}{2} \rho V^2 S + \frac{W^2}{\pi A e \tfrac{1}{2} \rho V^2 S}$$

So, one part of the drag <u>decreases</u> (!) with airspeed ($1/V^2$) and one part <u>increases</u> with airspeed (V^2)

Aerodynamic forces (year 1)

Drag as a function of airspeed



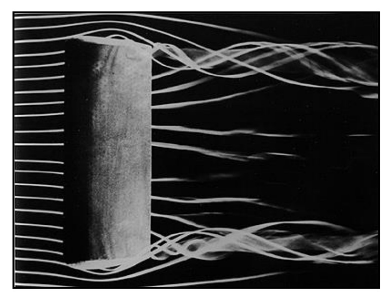
Aircraft are quite unique in the sense that drag increases when airspeed decreases!

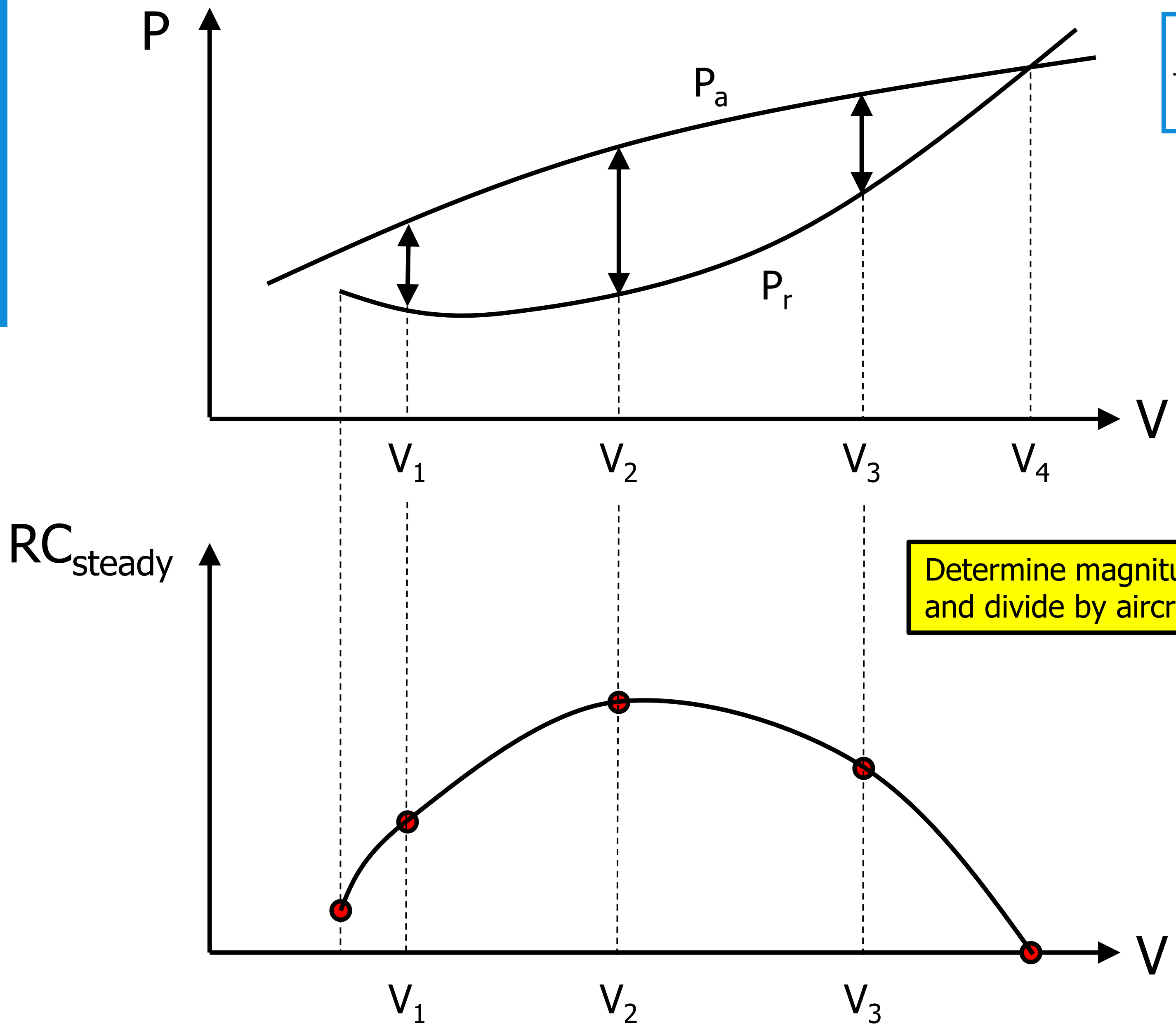
$$D = D_0 + D_i$$

$$D_0 = C_{D_0} \tfrac{1}{2} \rho V^2 S$$

$$D_i = \frac{W^2}{\pi A e \tfrac{1}{2} \rho V^2 S}$$


STARFIGHTERS



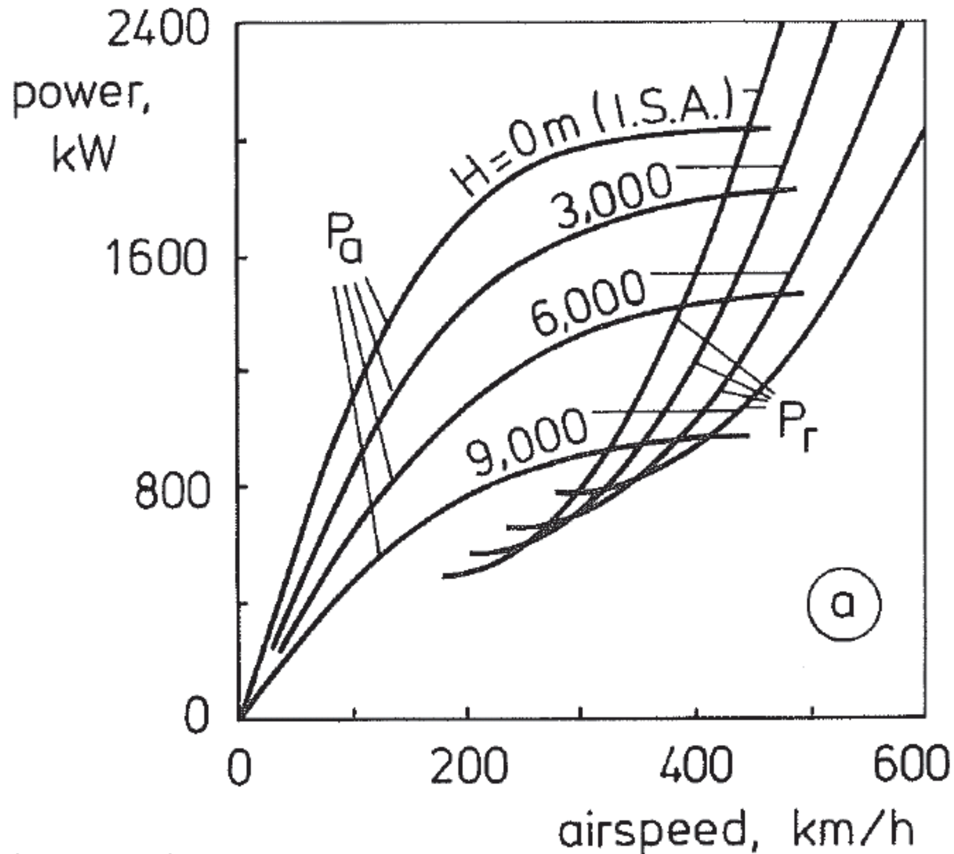


$$\frac{P_a - P_r}{W} = RC_{steady}$$

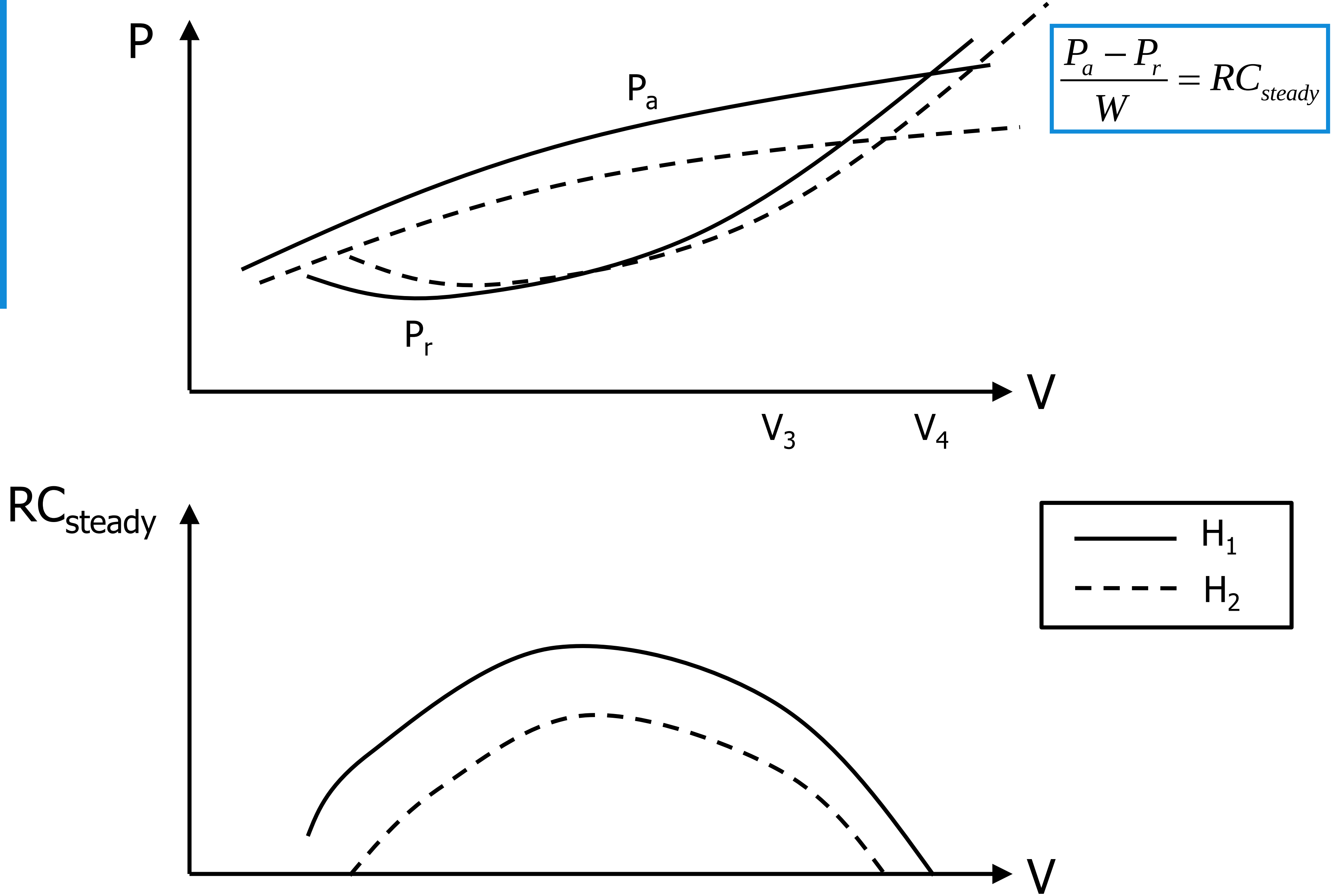
Determine magnitude of difference and divide by aircraft weight (W)

Performance diagram

How does it change with altitude?

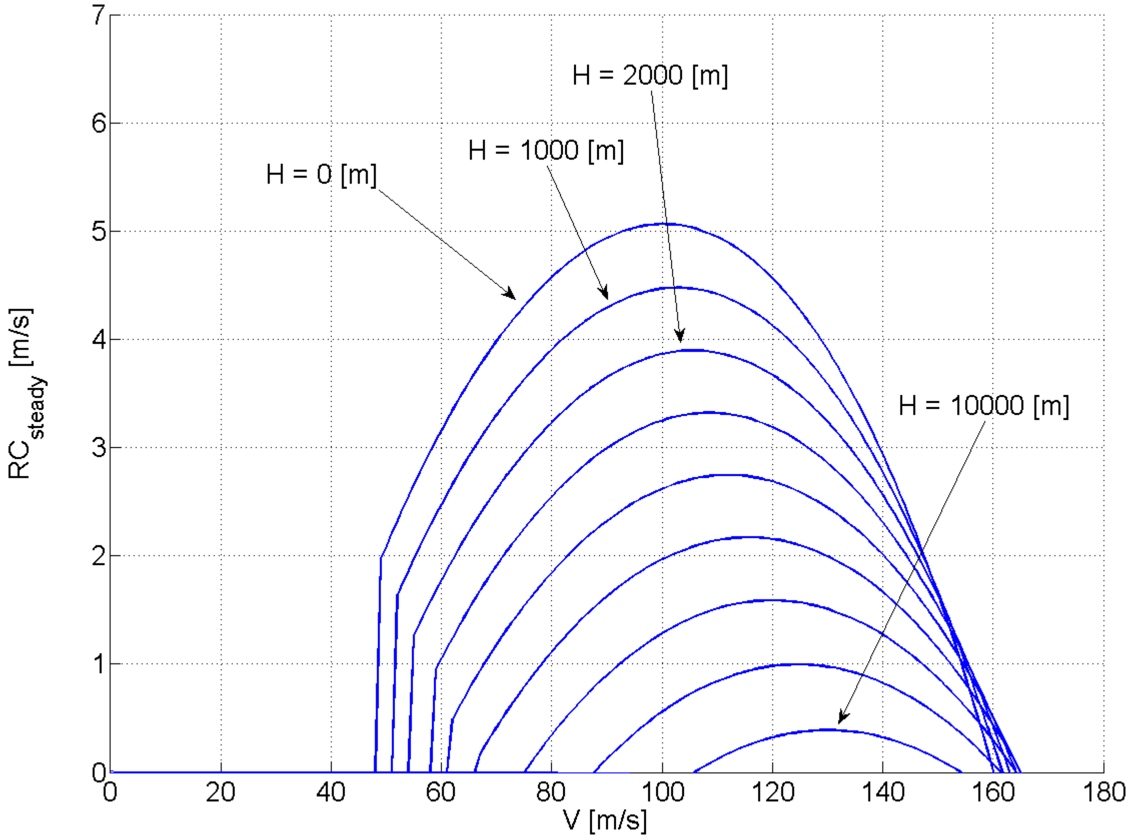


Note: this sheet is from the first year lecture


P
P_a
P_r
V
V_3
V_4
$$\frac{P_a - P_r}{W} = RC_{steady}$$
RC_steady
H_1
H_2
V

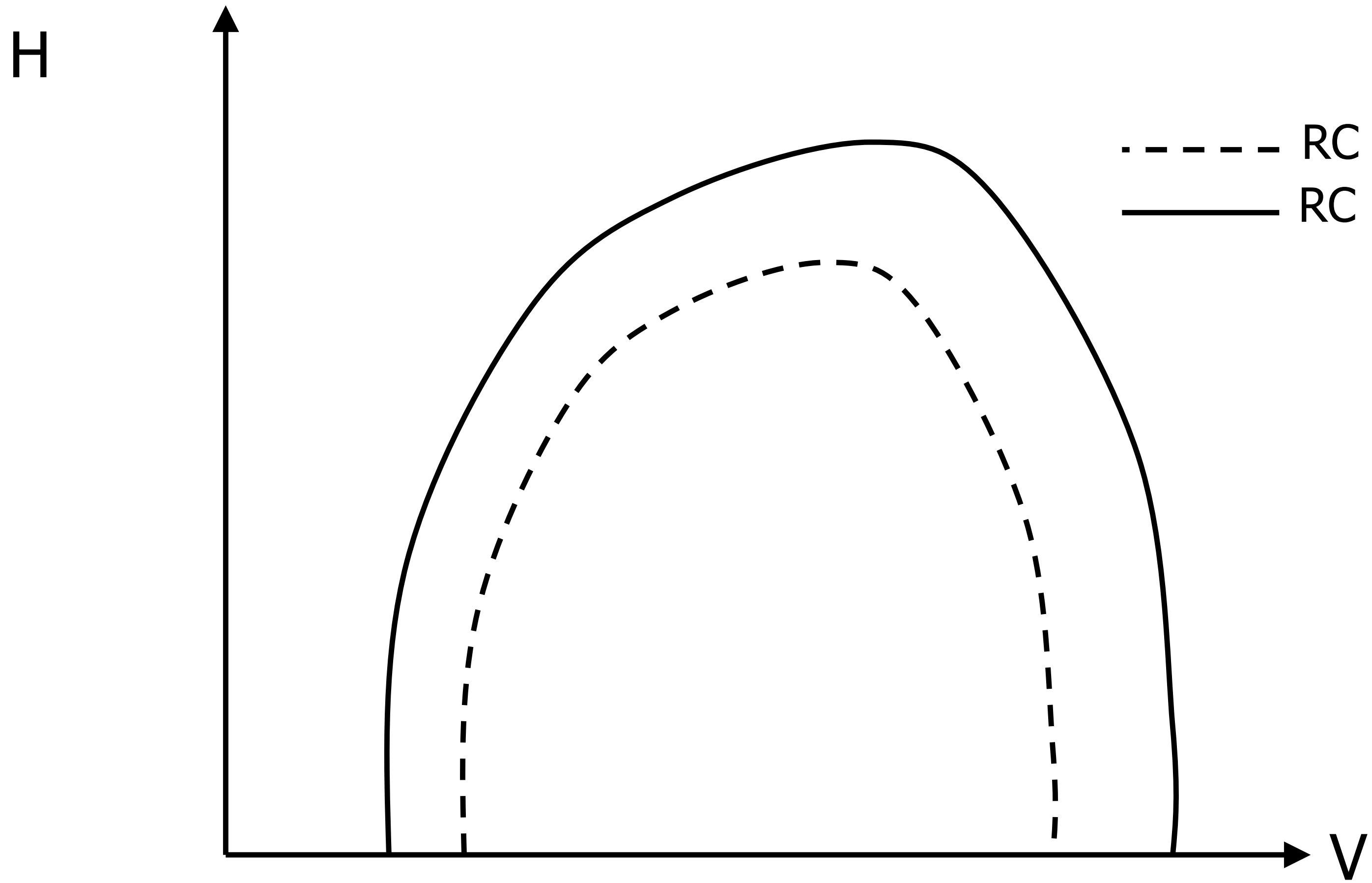
Performance diagram

Rate of climb as a function of airspeed and altitude



Performance diagram

Rate of climb as a function of airspeed and altitude



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Optimal climb

- During climb at constant indicated airspeed, the aircraft is accelerating. V(H) is fixed
- Easy flight technique for the pilot
- But is this optimal???
- What is optimal?

Optimal climb

What is optimal?

- Minimum time to climb (time)
- Minimum fuel consumed during climb (fuel)
- Distance covered during climb (distance)
- Noise during climb (noise)

Optimal climb

What is optimal?

- **Minimum time to climb (time)**
- Minimum fuel consumed during climb (fuel)
- Distance covered during climb (distance)
- Noise during climb (noise)

We will focus on the first option, but the other three are optimal as well

Optimal climb

- Objective: to find the true airspeed as a function of altitude that yields the minimum time to climb

$$RC = \frac{dH}{dt} \Leftrightarrow dt = \frac{dH}{RC}$$

$$t = \int dt = \int_{H_1}^{H_2} \frac{dH}{RC}$$

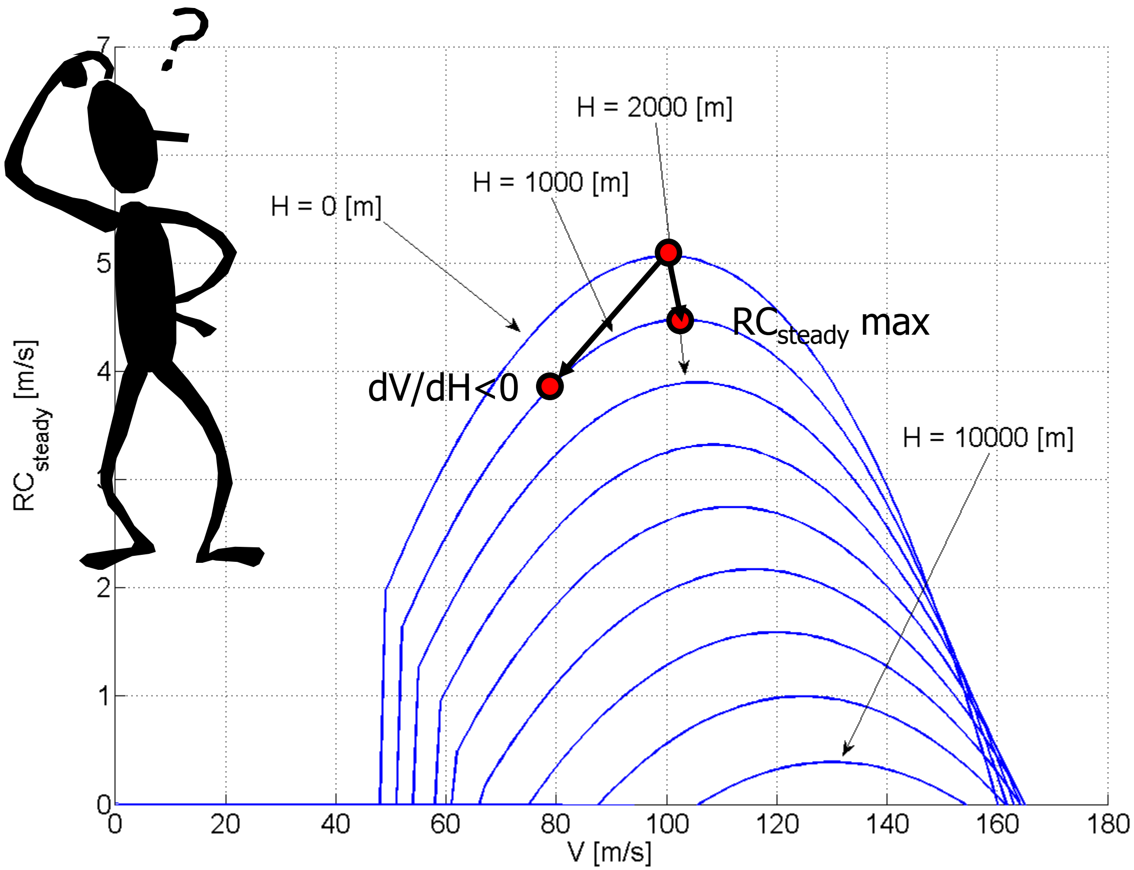
$$\frac{RC}{RC_{steady}} = \frac{1}{1 + \frac{V}{g_0}\frac{dV}{dH}}$$

$$\boxed{t = \int_{H_1}^{H_2} \frac{\left[1 + \frac{V}{g_0}\frac{dV}{dH}\right]}{RC_{steady}} dH}$$

- Time t is not minimal if the integrand is minimal at every altitude H because the term dV/dH is in the integrand
- **Variational calculus** is necessary

Variational calculus?

$$t = \int_{H_1}^{H_2} \frac{\left[1 + \frac{V}{g_0}\frac{dV}{dH}\right]}{RC_{steady}} dH$$



Minimize integrand:

- RC_{steady} maximum
- $dV/dH < 0$

- At H = 0, RC_{st} is maximal at V = 100
- The integrand can be minimized more by choosing $(dV/dH)_1 < 0$
- At altitude H > 0 no optimum V can be chosen
- **Conclusion**: No local optimum but global optimum (consider complete flight path)

Choosing maximum RC_{steady} at sea level seems like a good starting point

Optimal climb

$$t = \int_{H_1}^{H_2} \frac{\left[1 + \dfrac{V}{g_0} \dfrac{dV}{dH}\right]}{RC_{steady}} dH$$

Solution

- Simplify (low speed aircraft)
- Energy method (high subsonic / supersonic aircraft)

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Low speed aircraft

Low speed aircraft

$$t = \int_0^H \frac{dH}{RC} = \int_0^H \frac{\left[1 + \frac{V}{g_0}\frac{dV}{dH}\right]}{RC_{st}} dH$$

Assumption: Low speed, low altitude → dV/dH is small

$$t \approx \int_0^H \frac{dH}{RC_{st}}$$

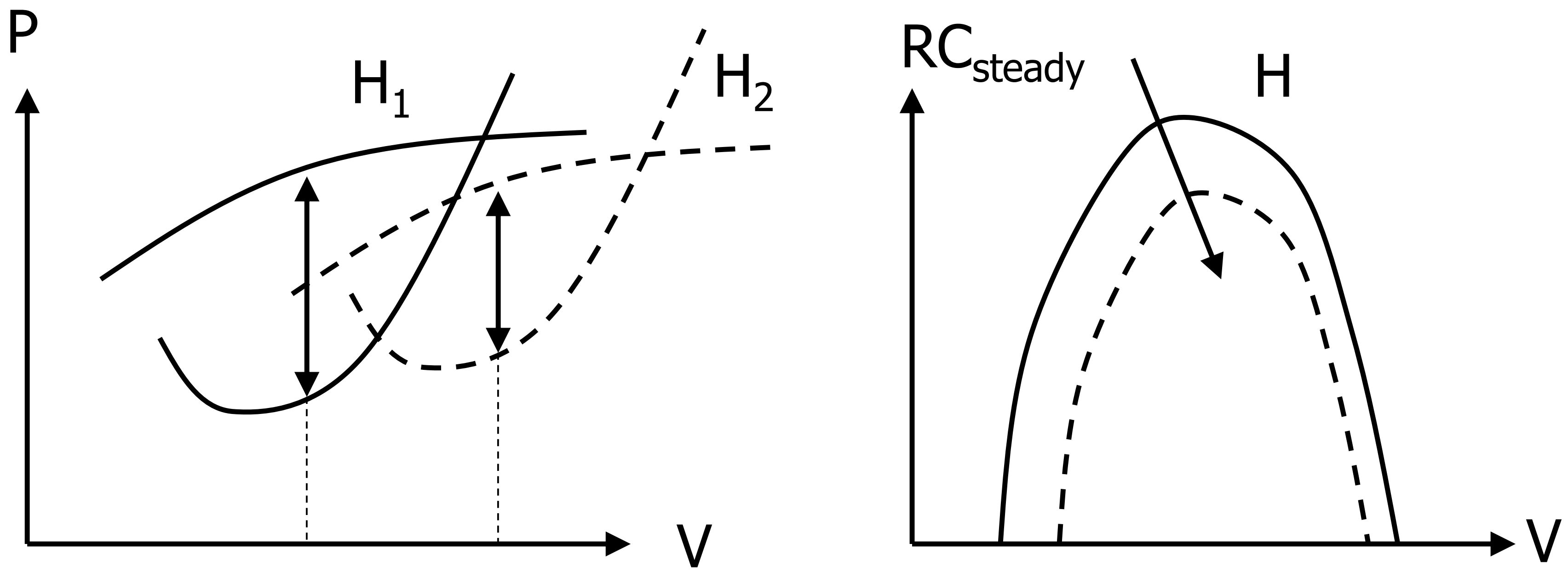
Choose maximum steady rate of climb at each altitude

Low speed aircraft

$$T - D - W\sin\gamma = \frac{W}{g}\frac{dV}{dt}$$

$$\frac{P_a - P_r}{W} = RC + \frac{W}{g}V\frac{dV}{dt}$$

$$\frac{P_a - P_r}{W} = RC_{steady}$$



Low speed aircraft

Solution – propeller aircraft

$$RC_{st} = \frac{P_a - P_r}{W}$$

P_a can be assumed constant as function of airspeed for propeller aircraft

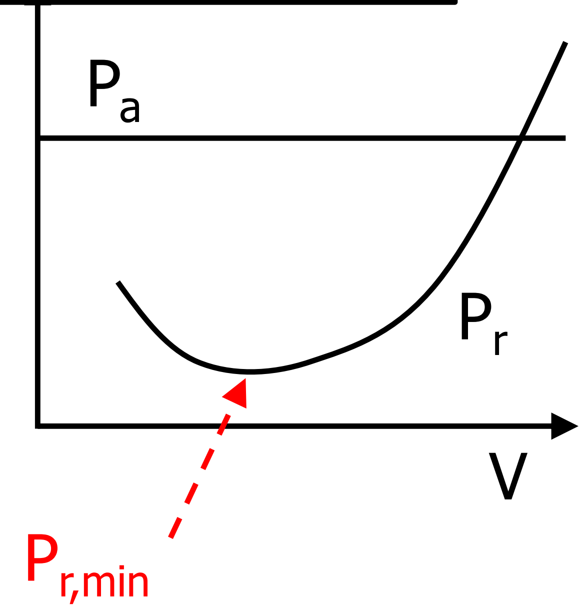
Note: this optimal C_L was determined in the first year lectures!

$$\left(RC_{st}\right)_{\max} \rightarrow P_{r,\min} \rightarrow C_{L,opt} = \sqrt{3C_{D_0}\pi Ae}$$

$$V_{opt} = \sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_{L,opt}}} \propto \frac{1}{\sqrt{\rho}}$$

Corresponding $V_e = V\sqrt{\frac{\rho}{\rho_0}} = \text{constant}$

$\Rightarrow V_i \approx$ constant because $V_i \approx V_e$



Low speed aircraft

Summary

- Conclusion: if the pilot selects the airspeed for the maximum steady rate of climb at the ground and if he / she keeps the indicated airspeed constant, then the climb will be optimal
- Optimal in this case means minimum time to climb (assuming dV/dH is small)

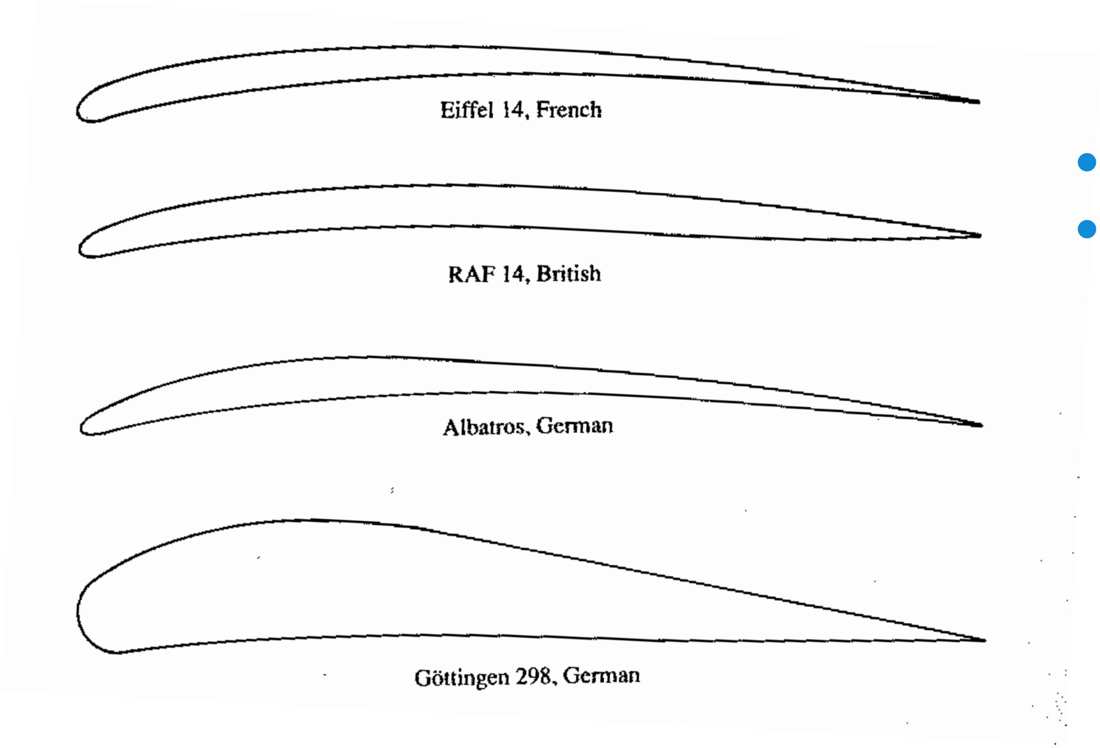
Low speed aircraft

Story – World War I

Low speed aircraft

Story – World War I

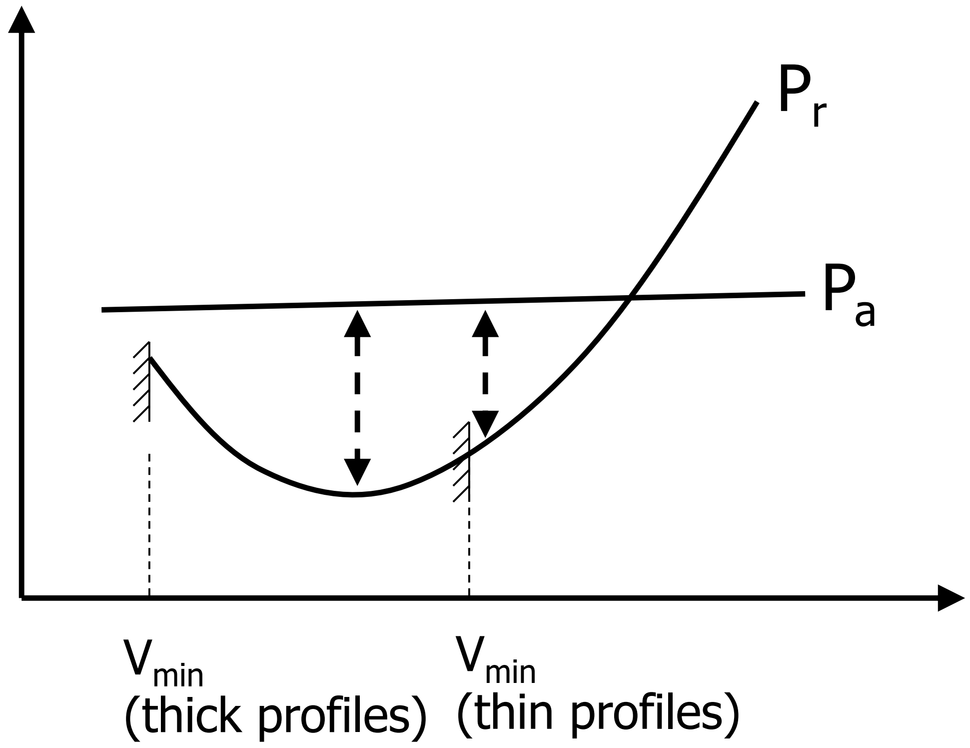
- Allied airfoils (British / French) versus German airfoils



- Thin airfoils have low $C_{L,max}$
- Therefore, German aircraft were able to fly slower British / French) versus German airfoils

Low speed aircraft

Story – World War I



Numerical example

$$C_L = \sqrt{3 C_{D_o} \pi A e}$$

$$C_{D_0} = 0.035$$

$$A = 6$$

$$e = 0.8$$

$$C_{L,opt} = 1.26 \text{ (large !)}$$

$$V_E = V_{ground} = \sqrt{\frac{W}{S} \frac{2}{\rho_0} \frac{1}{C_{L,opt}}}$$

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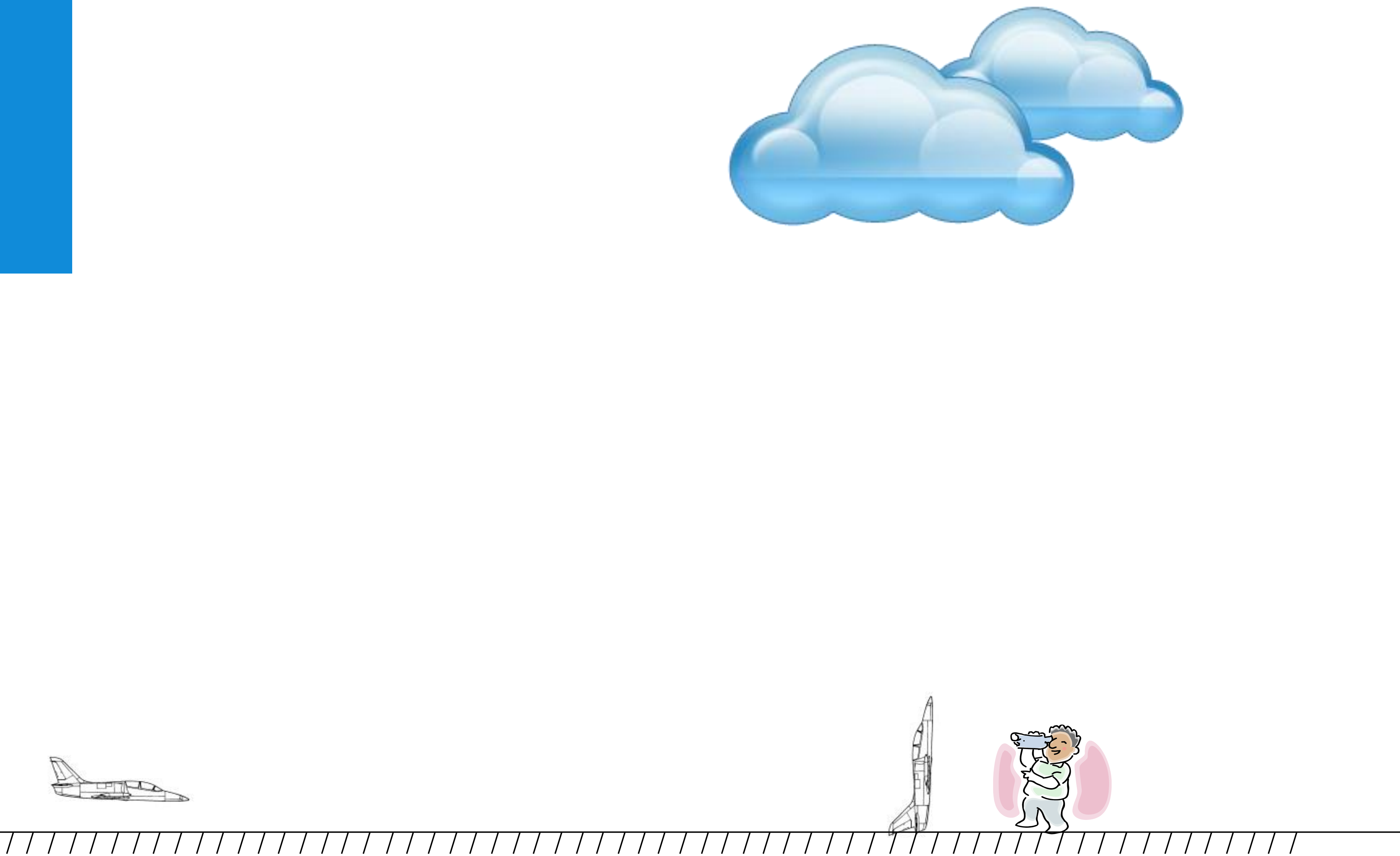


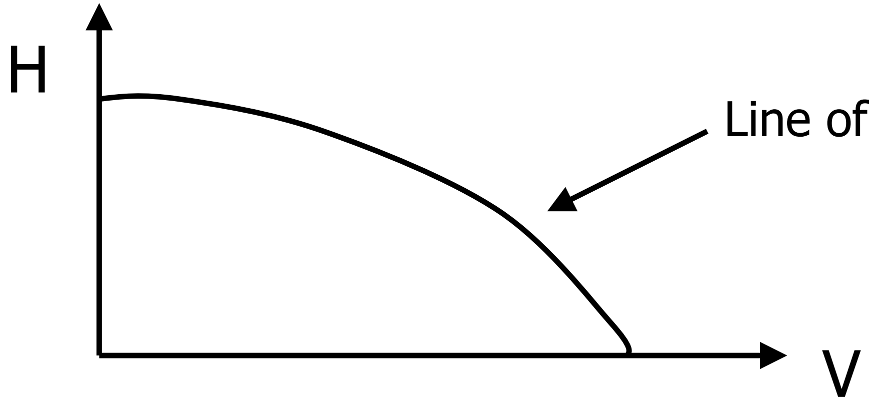
Energy height

Concept to solve the minimum time to climb problem

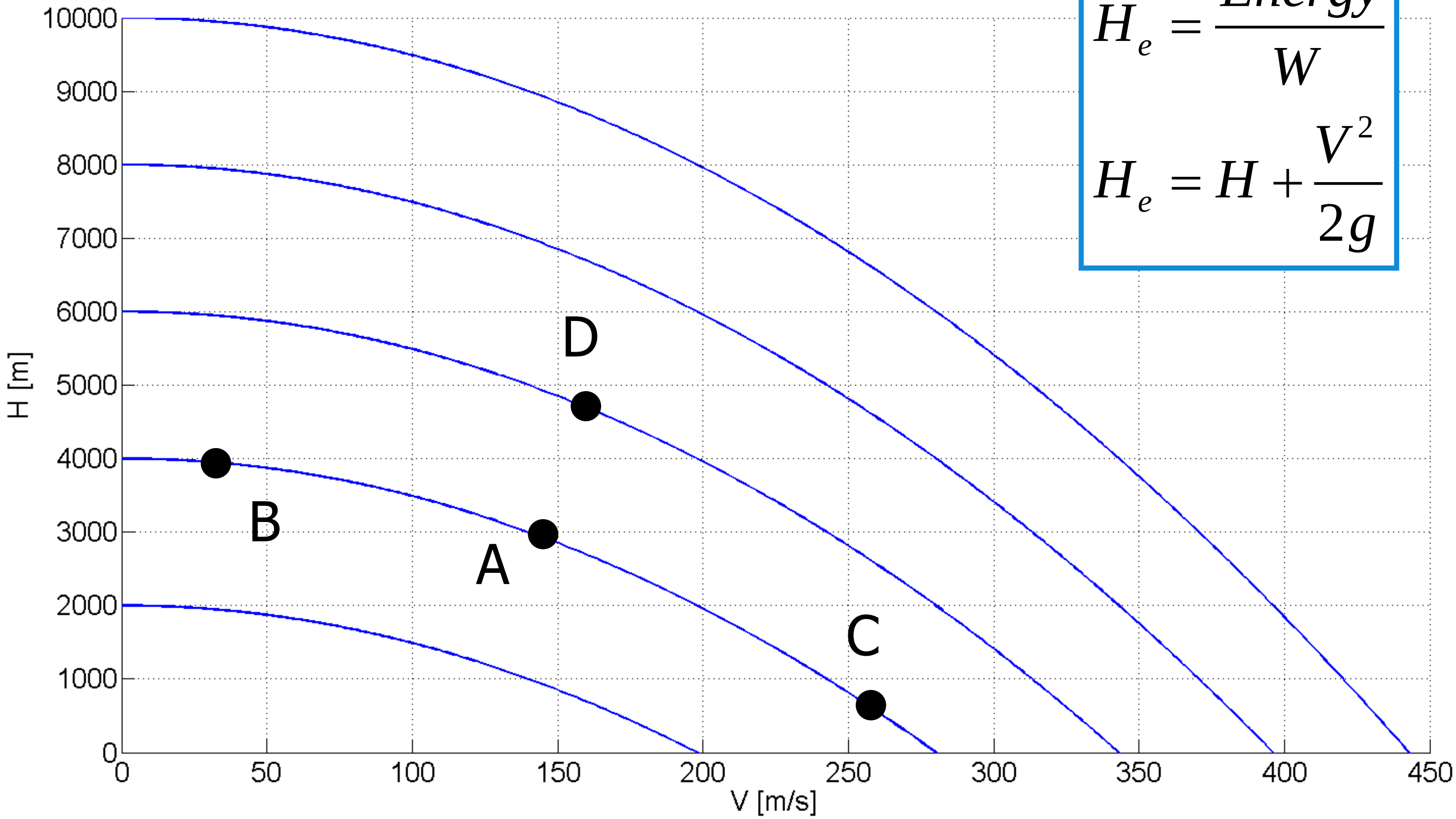
- Altitude and flight speed are rapidly interchangeable (exchange of kinetic energy and potential energy)

- Increasing the total energy is much slower because it depends on the excess power

$$Energy = \tfrac{1}{2} mV^2 + mgH = \tfrac{1}{2}\frac{W}{g}V^2 + WH$$



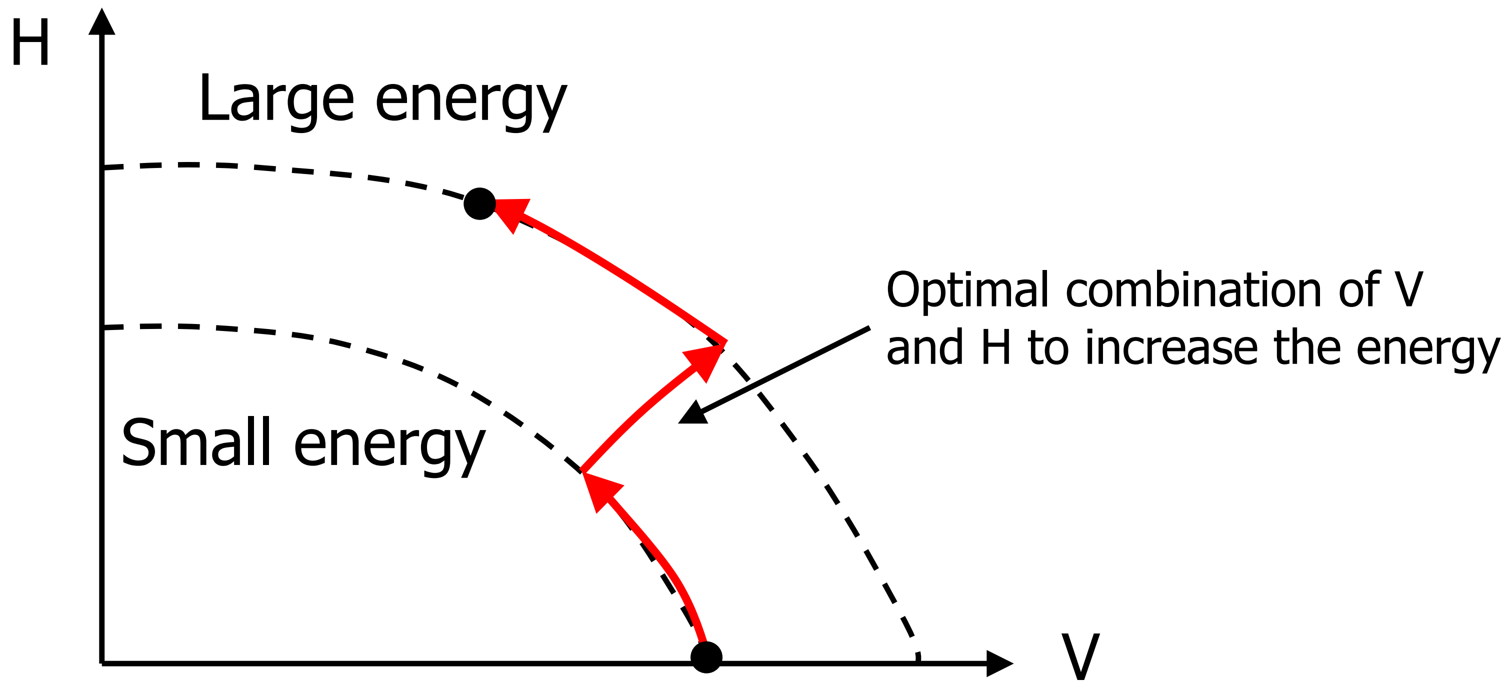
Energy height



$$H_e = \frac{Energy}{W}$$

$$H_e = H + \frac{V^2}{2g}$$

Energy height



So: "Minimum time to climb" problem approximated by another problem: find V_{opt} (H) at which minimum time to total energy

Energy height

Energy method

Energy height

$$E = mg_0 H + \tfrac{1}{2} mV^2$$

$$E = WH + \tfrac{1}{2}\frac{W}{g_0}V^2$$

$$H_e \equiv \frac{E}{W} = H + \frac{V^2}{2g_0}$$

$$\frac{dH_e}{dt} = \boxed{\frac{dH}{dt} + \frac{1}{2g_0}\frac{dV^2}{dt}}$$

Equation of Motion

$$\frac{W}{g_0}\frac{dV}{dt} = T - D - W\sin\gamma$$

$$\frac{1}{g_0}\frac{dV}{dt} + \sin\gamma = \frac{T-D}{W}$$

$$\frac{V}{g_0}\frac{dV}{dt} + V\sin\gamma = \frac{P_a - P_r}{W}$$

$$\boxed{\frac{1}{2g_0}\frac{dV^2}{dt} + \frac{dH}{dt}} = \frac{P_a - P_r}{W}$$

$$\boxed{\frac{dH_e}{dt} = \frac{P_a - P_r}{W} = RC_{steady}}$$

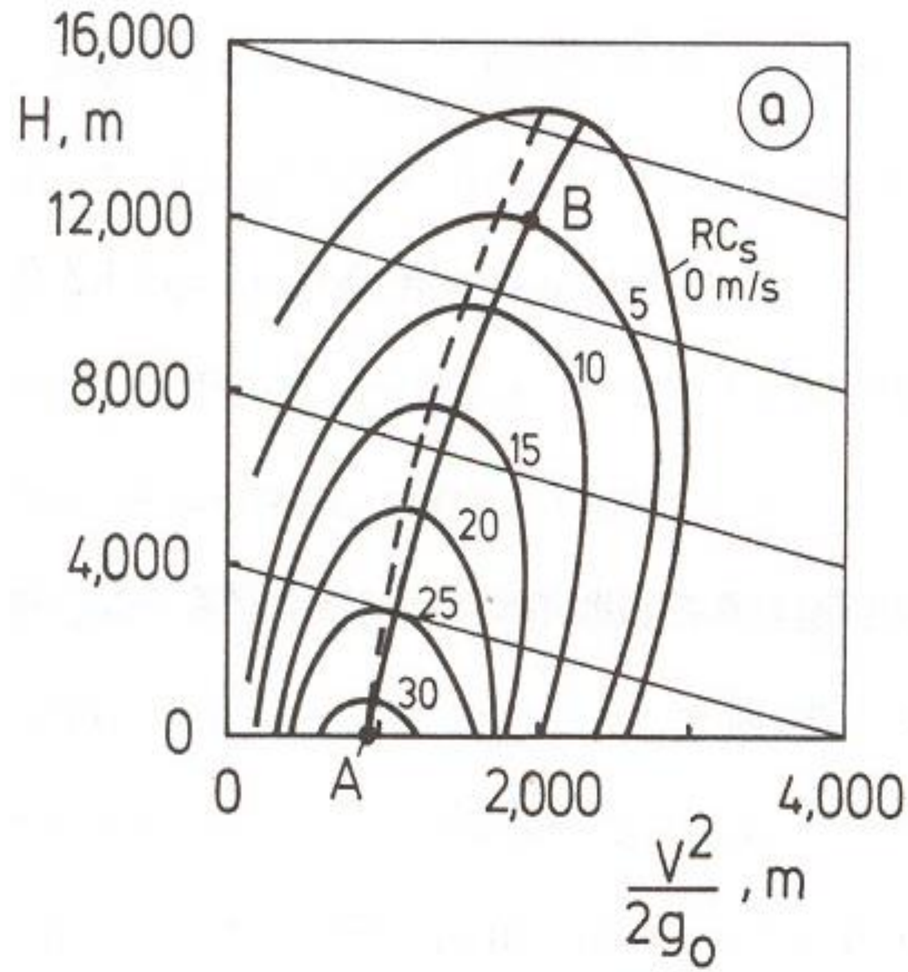
Minimum time to climb

$$\boxed{\frac{dH_e}{dt} = RC_s}$$

$$dt = \frac{dH_e}{RC_{st}}$$

$$t = \int_{H_{e_1}}^{H_{e_2}} \frac{dH_e}{RC_{st}} \quad \text{(time to energy height)}$$

Optimum path subsonic aircraft



(for each constant energy height line, find the maximum steady rate of climb)

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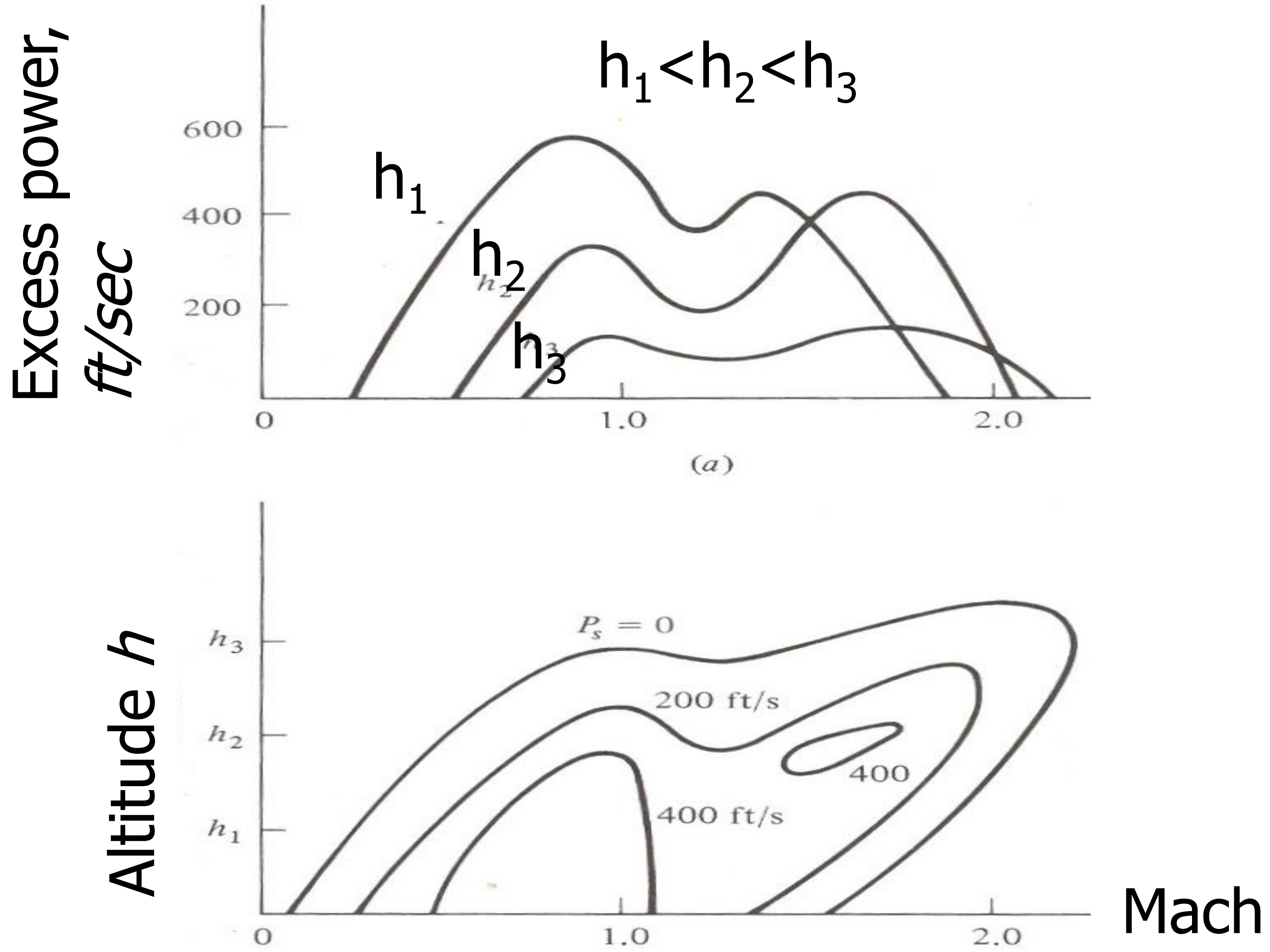
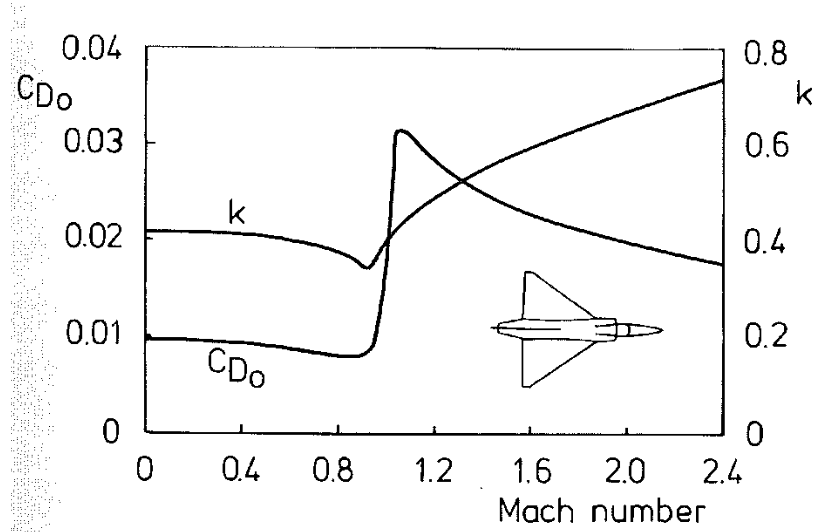
Solution high speed aircraft



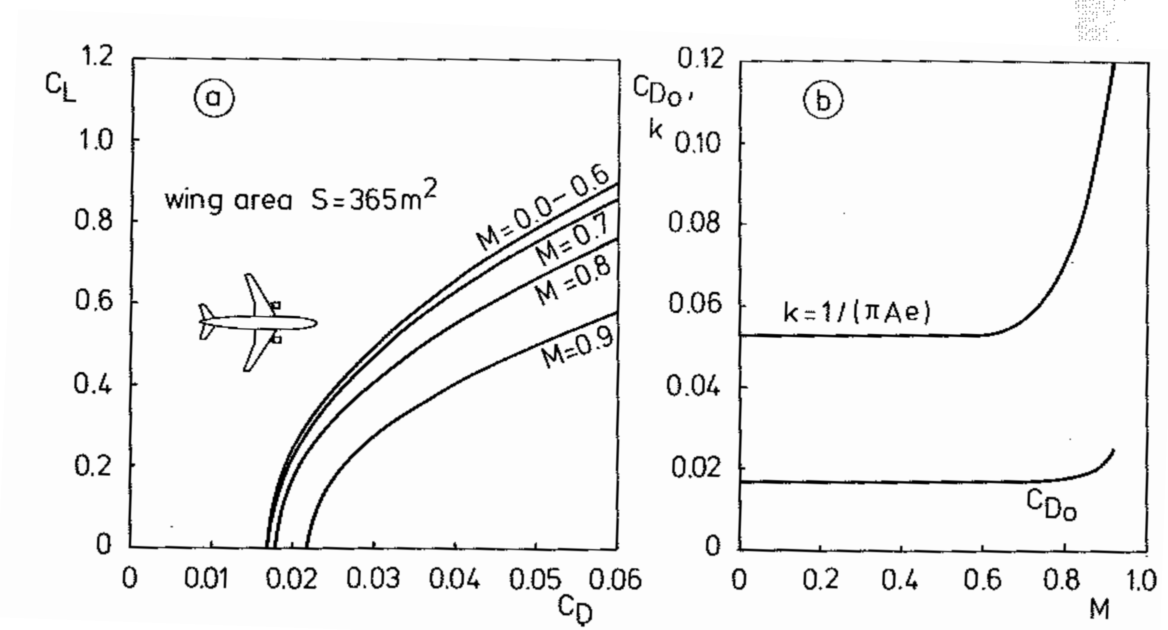
Supersonic aircraft

- Dip in excess power is caused by transonic drag rise:

$$\boxed{C_D = C_{D_0}(M) + k(M)C_L^2}$$

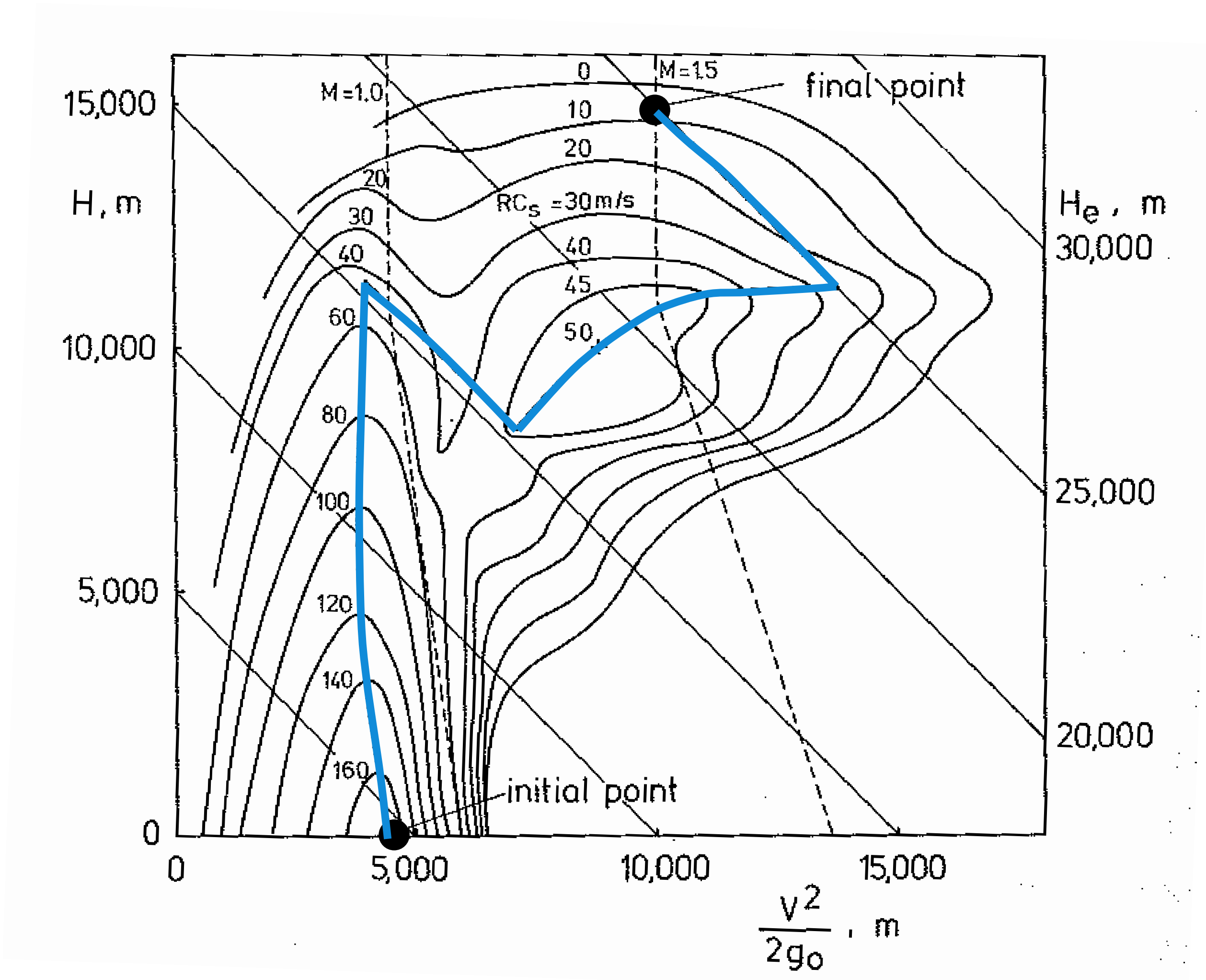


supersonic aircraft



High subsonic aircraft

Solution high speed aircraft



Solution high speed aircraft

Example performance

- Mig-29: Climb from sea level to 6000 m in less than 1 minute

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Example exam question

Question 2 (14 points)
A challenging problem in the field of airplane performance is the so-called **minimum time to climb problem.** The optimum path to climb from an initial condition (sea level, subsonic airspeed) to a final condition (Mach 1.5 at 15000 meter) for a supersonic aircraft is given in the figure below.

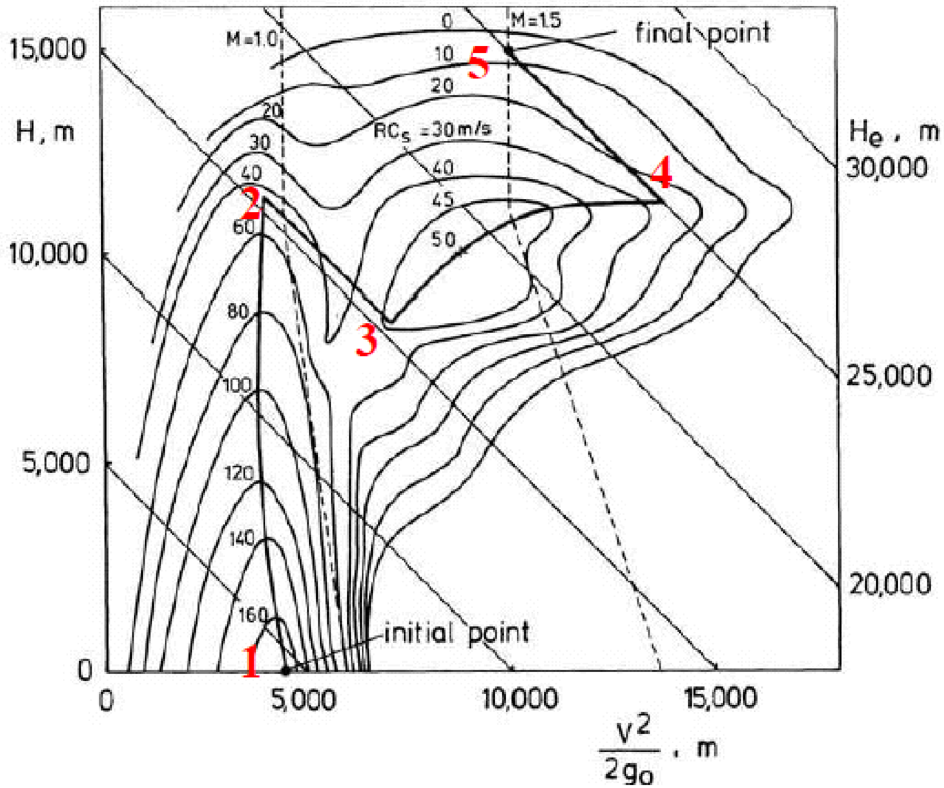

Fig. 1: minimum time to climb trajectory for a supersonic aircraft

Example exam question

a) A useful concept to solve the minimum time to climb problem is the **energy height**. This may be derived from the equation of motion along the path of a climbing and accelerating aircraft. Write down this equation and convert it to an energy equation. Indicate which part of this equation is indicated as energy height, and show how it is related to the "specific excess power" (indicated in the above figure as RC_s) (**3 points**)

b) What is the approximate value and dimension of the energy height at the initial condition? (**2 points**)

c) The optimum path of this aircraft consists of four phases. Clearly explain why this is the optimum path and not for example a straight line from the initial condition to the final condition. (**4 points**)

d) The lines of constant Mach number show a kink at approximately 11000 m. What is the physical cause of this kink? (**2 points**)

e) Determine, by using the figure, (1) the maximum airspeed in steady flight in m/s that can be achieved by this aircraft and (2) the theoretical ceiling of this aircraft. (**3 points**)

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