Flight and Orbital Mechanics

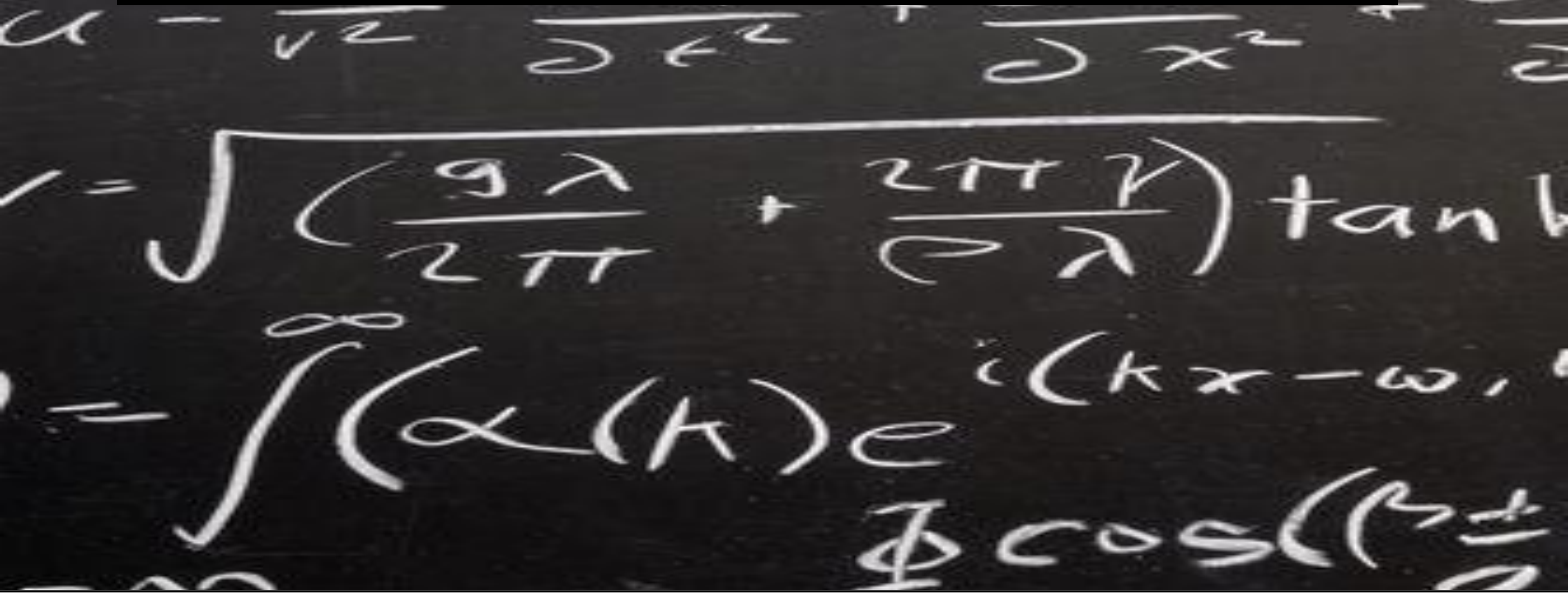
Lecture slides

Flight and Orbital Mechanics

Lecture 7 – Equations of motion

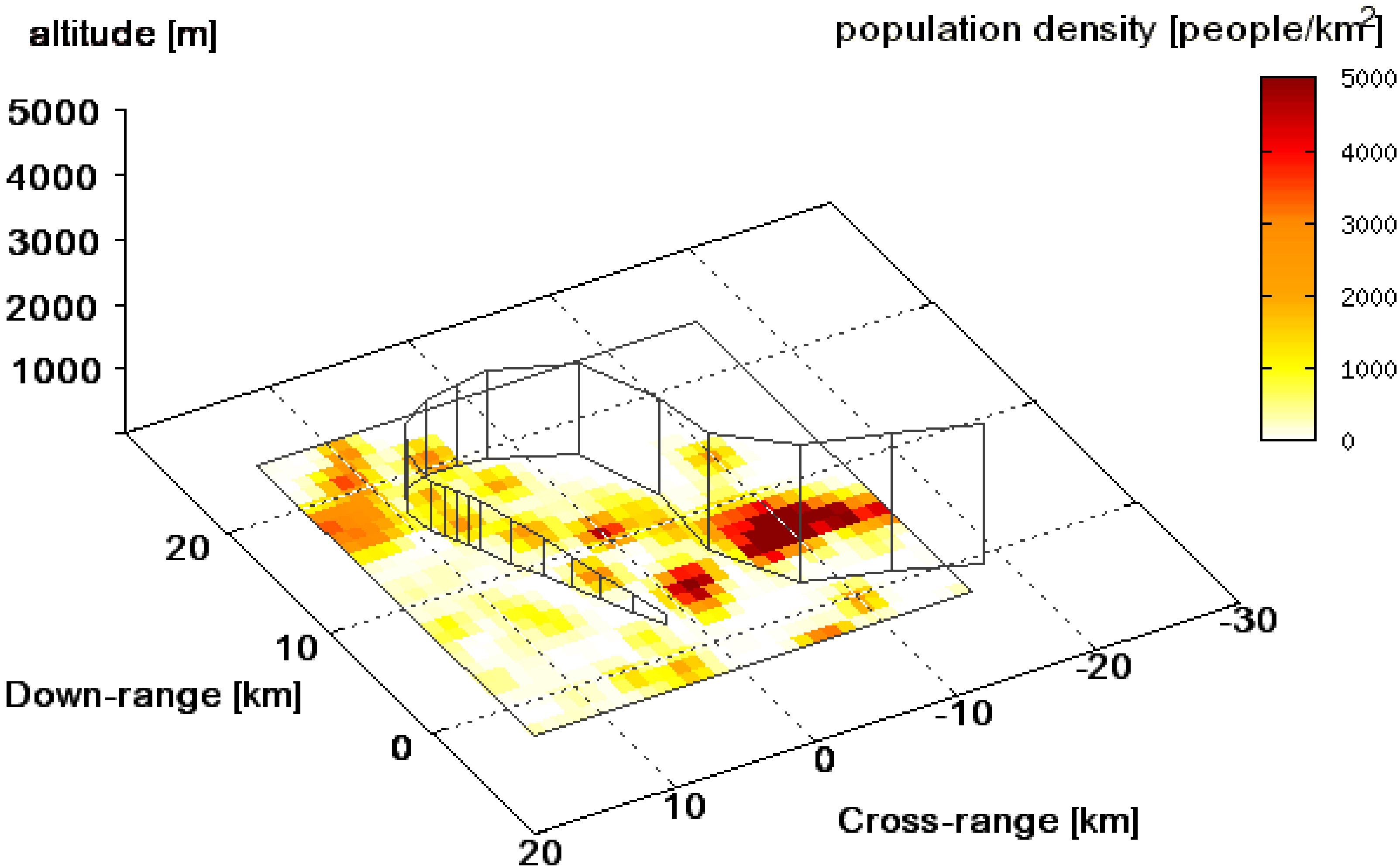
Mark Voskuijl

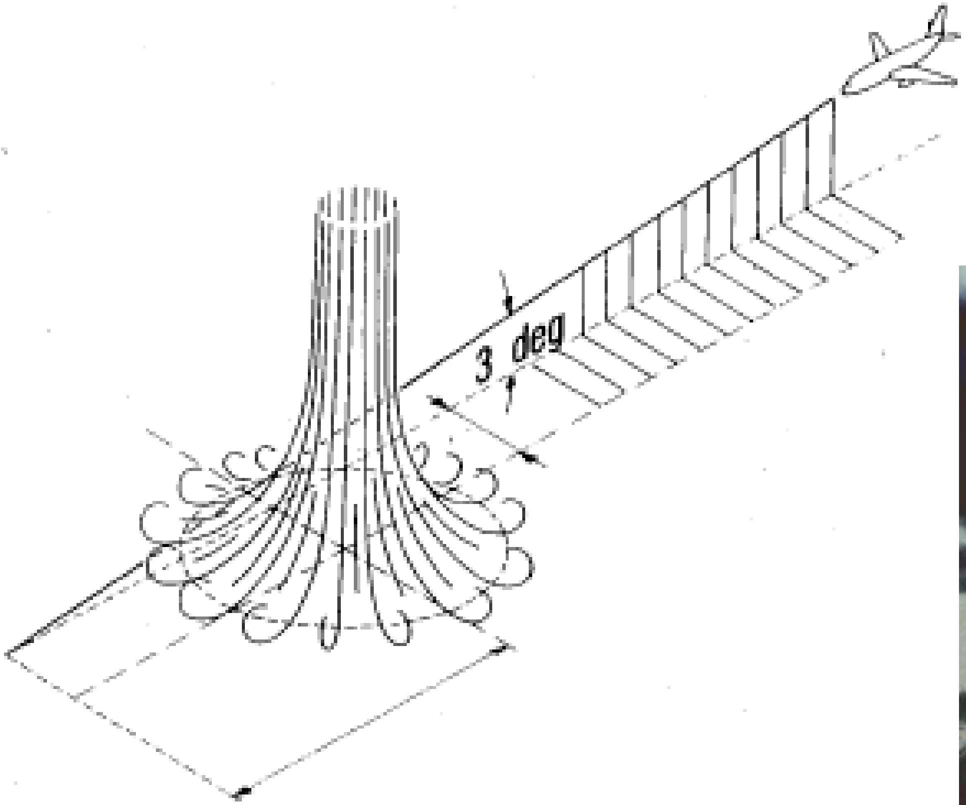
Semester 1 - 2012

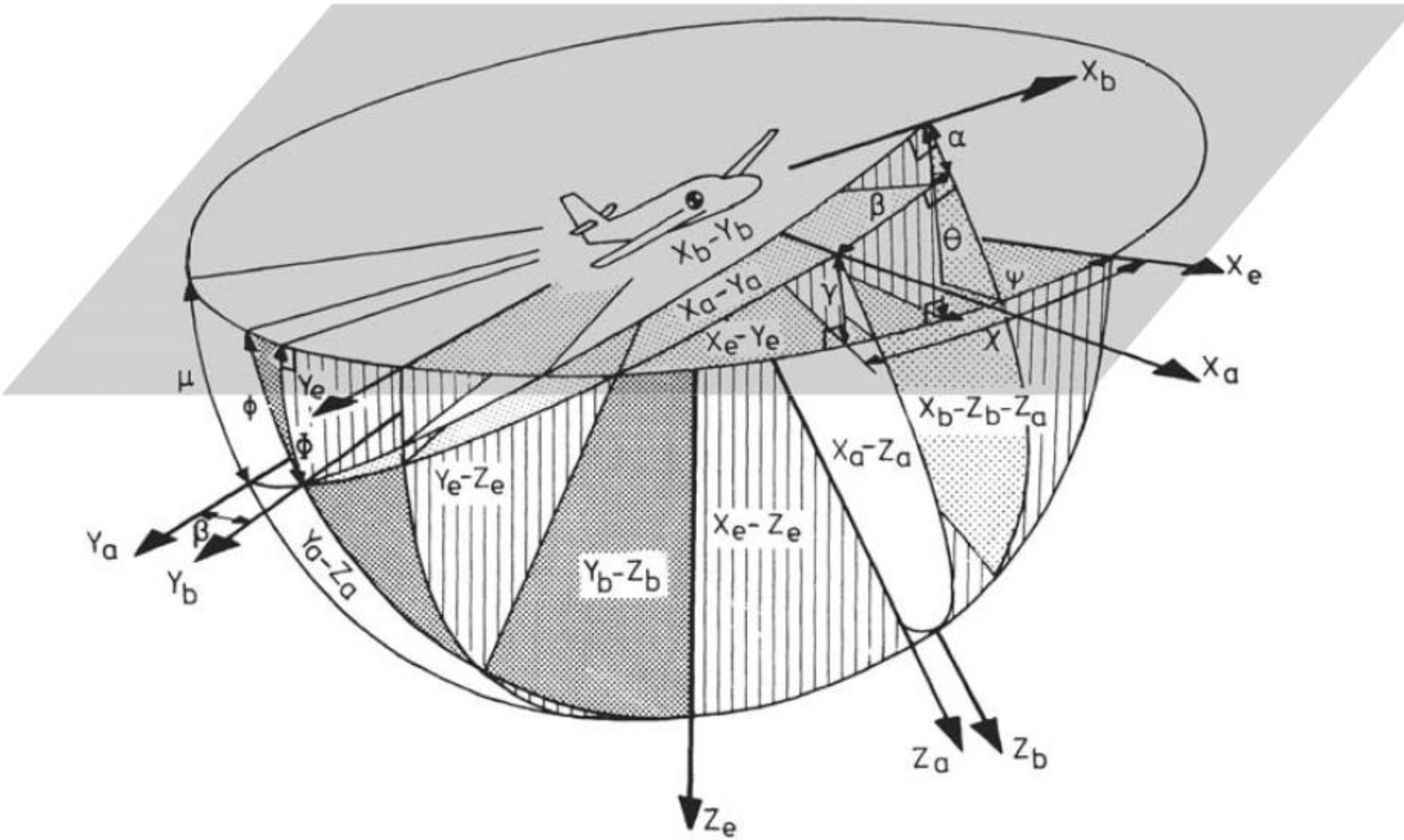


Time schedule

Date	Time	Hours	Topic
4 Sep	10.45 – 12.30	1, 2	Unsteady climb
6 Sep	10.45 – 12.30	3, 4	Minimum time to climb
11 Sep	10.45 – 12.30	5, 6	Turning performance
13 Sep	10.45 – 12.30	7, 8	Take – off
18 Sep	10.45 – 12.30	9, 10	Landing
20 Sep	10.45 – 12.30	11, 12	Cruise
25 Sep	10.45 – 12.30	13, 14	**Equations of motion (wind gradient)**
27 Sep	10.45 – 12.30	15, 16	Kepler orbits, gravity, Earth-repeat orbits, sun-synchronous orbits, geostationary satellites
2 Oct	10.45 – 12.30	17, 18	Third-body perturbation, atmospheric drag, solar radiation, thrust
4 Oct	10.45 – 12.30	19, 20	Eclipse, maneuvers
9 Oct	10.45 – 12.30	21, 22	Interplanetary flight
11 Oct	10.45 – 12.30	23, 24	Interplanetary flight
16 Oct	10.45 – 12.30	25, 26	Launcher, ideal vs. real flight, staging, design
18 Oct	10.45 – 12.30	27, 28	Exam practice


altitude [m]
population density [people/km^2]
5000
4000
3000
2000
1000
5000
4000
3000
2000
1000
0
20
10
Down-range [km]
0
20
10
0
-10
-20
-30
Cross-range [km]


3 deg



X_b
α
β
Θ
Ψ
X_e
X_b-Y_b
X_a-Y_a
X_e-Y_e
γ
χ
X_a
μ
ϕ
Y_e
Φ
$X_b-Z_b-Z_a$
X_a-Z_a
Y_e-Z_e
X_e-Z_e
Y_a
β
Y_b
Y_a-Z_a
Y_b-Z_b
Z_a
Z_b
Z_e

Content

- **Introduction**
- Axis systems and Euler angles
- Vector / matrix notation
- Accelerations
- Forces
- General equations of motion 3D flight
- Effect of a wind gradient

Introduction

Newton's laws

Newton's laws only hold with respect to a frame of reference which is in **absolute rest**. This is called an **inertial frame of reference**

Coordinate systems **translating uniformly** to the frame of reference in absolute rest are also **inertial frames of reference**

A **rotating** frame of reference is **not** an inertial frame of reference

Introduction

Objective

- Derivation of equations of motion
  - General 3 dimensional flight
  - 2 dimensional flight with a wind gradient
- General approach!

Content

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- **Axis systems and Euler angles**
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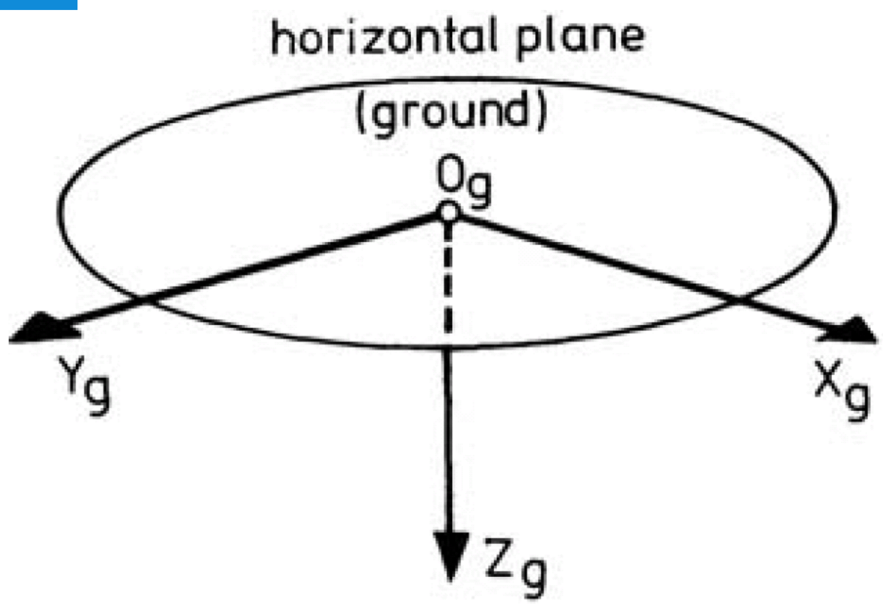
Axis systems and Euler angles

Earth axis system



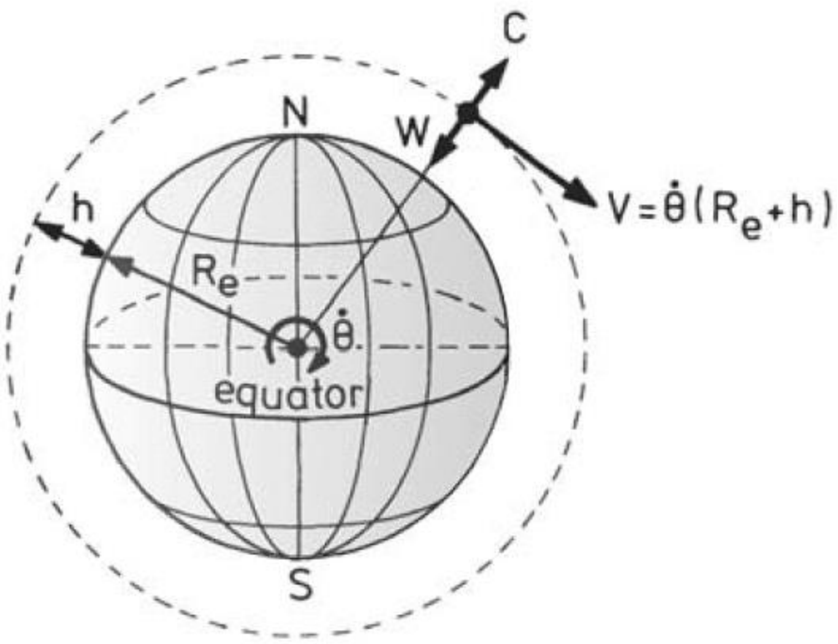
Earth axis system: {E_g}

1. X_g axis in the horizontal plane, orientation is arbitrary
2. Y_g axis in the horizontal plane, orientation: perpendicular to X_g
3. Z_g axis points downwards

Axis systems and Euler angles

Assumption

Assumption 1: the earth is flat



'Centrifugal force'

$$C = \frac{W}{g}\frac{V^2}{R_e + h}$$

$$\frac{C}{W} = \frac{V^2}{\left(R_e + h\right)g}$$

Example

$V = 100$ [m/s]

$R_e = 6371$ [km]

$g = 9.80665$ [m/s^2]

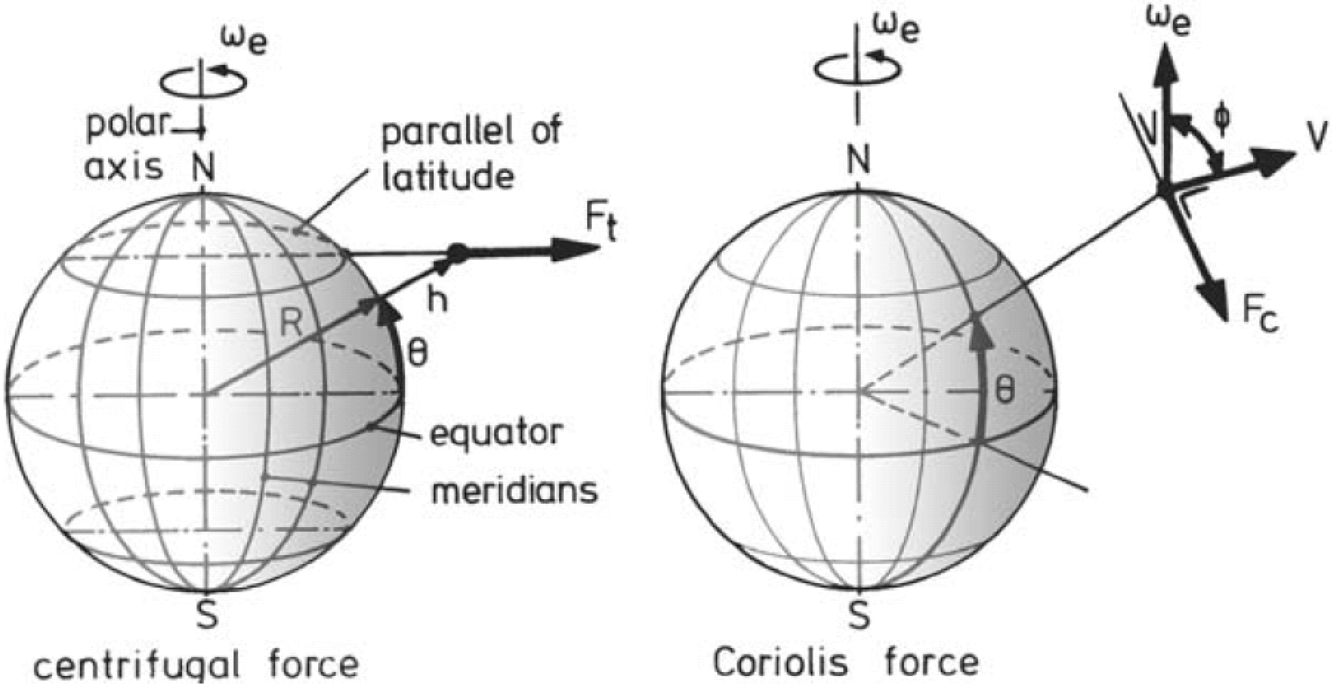
$h = 0$ [m]

√ Valid assumption

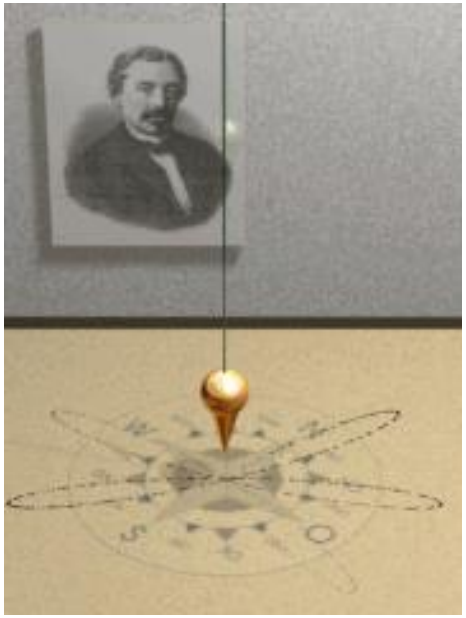
Axis systems and Euler angles

Assumptions

Assumption 2: the earth is non-rotating

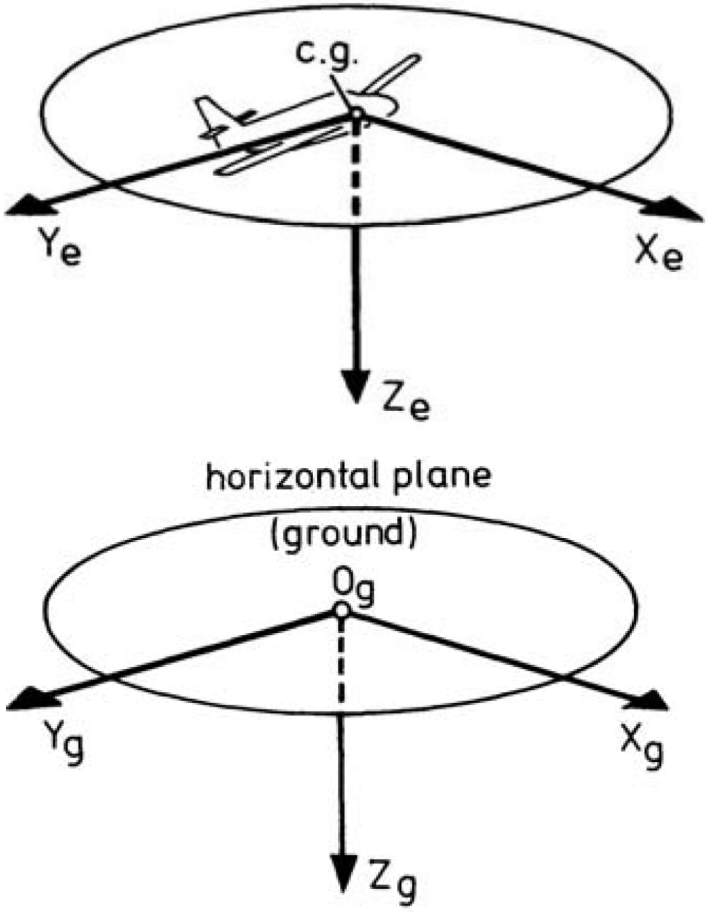



18 17 16 15 14 13 12 11



Axis systems and Euler angles

Moving earth axis system

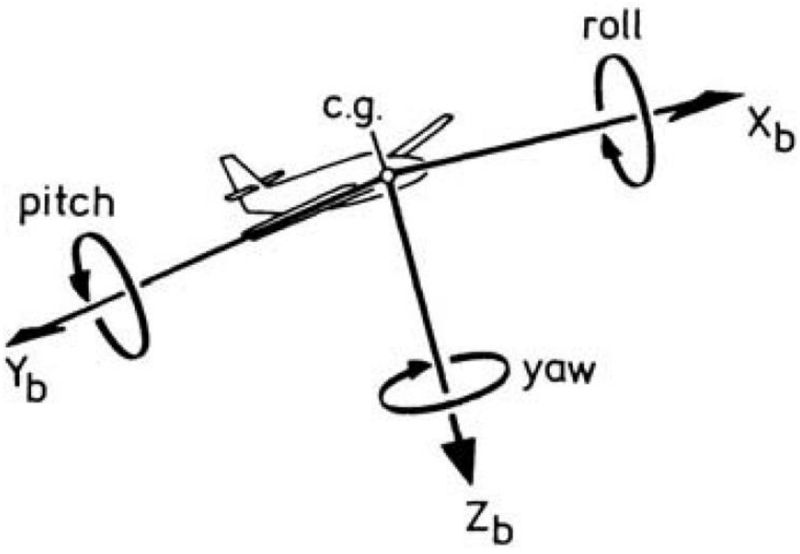


Moving earth axis system: {E$_e$}

1. X_e parallel to X_g axis but attached to c.g. of aircraft
2. Y_e parallel to Y_g axis but attached to c.g. of aircraft
3. Z_e axis points downwards

Axis systems and Euler angles

Body axis system

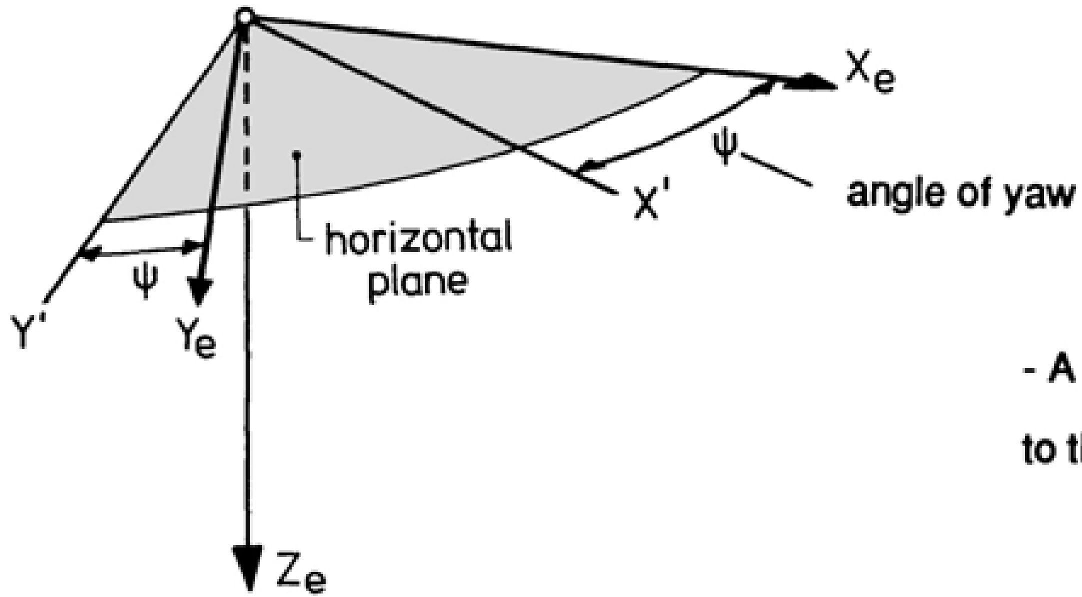


Body axis system: {E_b}

1. Origin is fixed to the aircraft c.g.
2. X_b lies in plane of symmetry and points towards the nose
3. Y_b is perpendicular to the plane of symmetry and is directed to the right wing
4. Z_b is perpendicular to X_b and Y_b

Axis systems and Euler angles

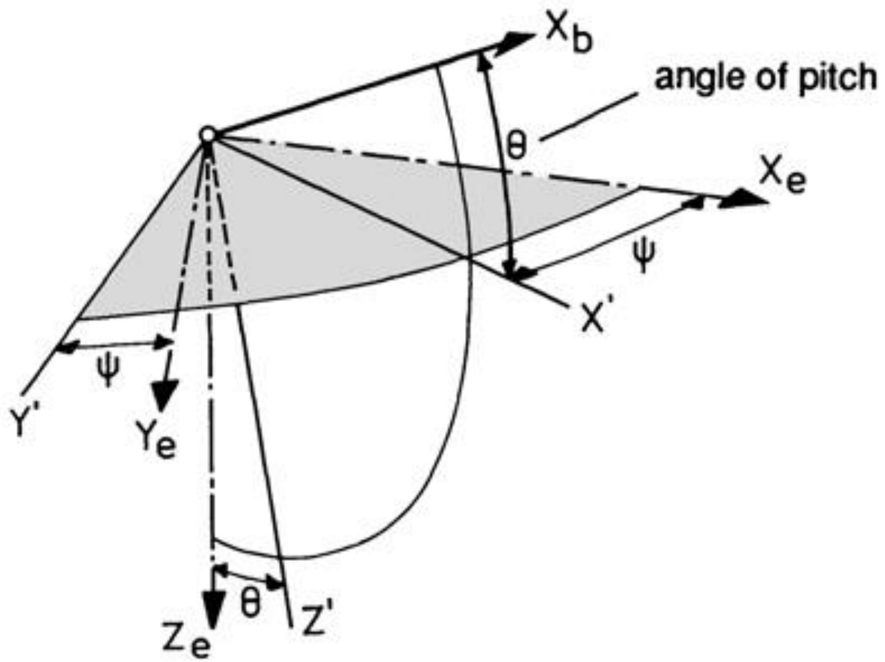
Yaw angle (body axis)



- A rotation by ψ about the Z_e axis to the intermediate position $X'Y'Z_e$.

Axis systems and Euler angles

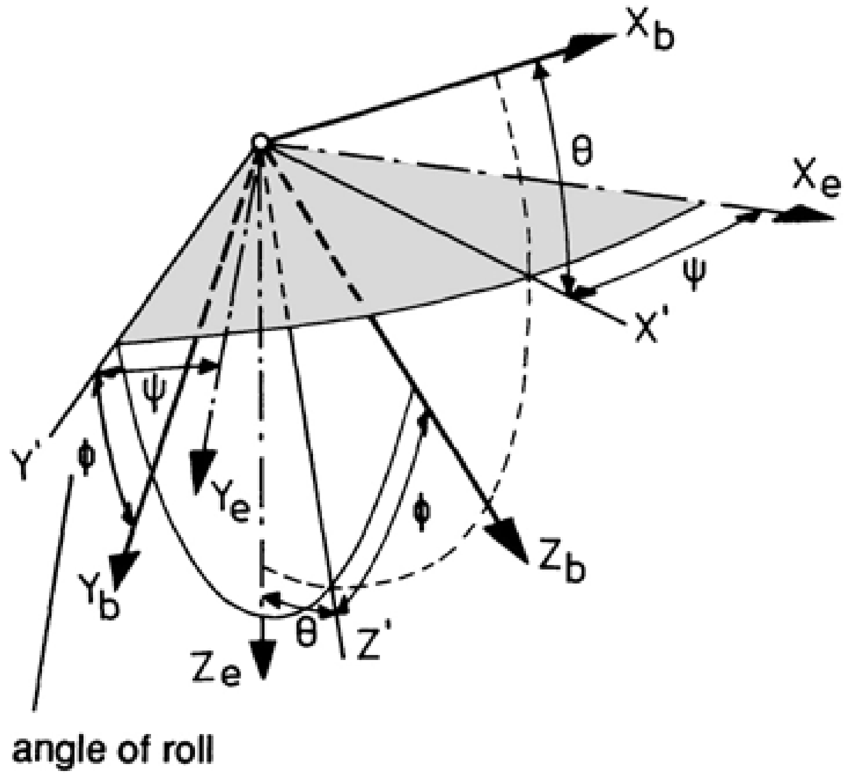
Pitch angle (body axis)



- A rotation by θ about the Y'-axis to the intermediate position $X_b Y'Z'$.

Axis systems and Euler angles

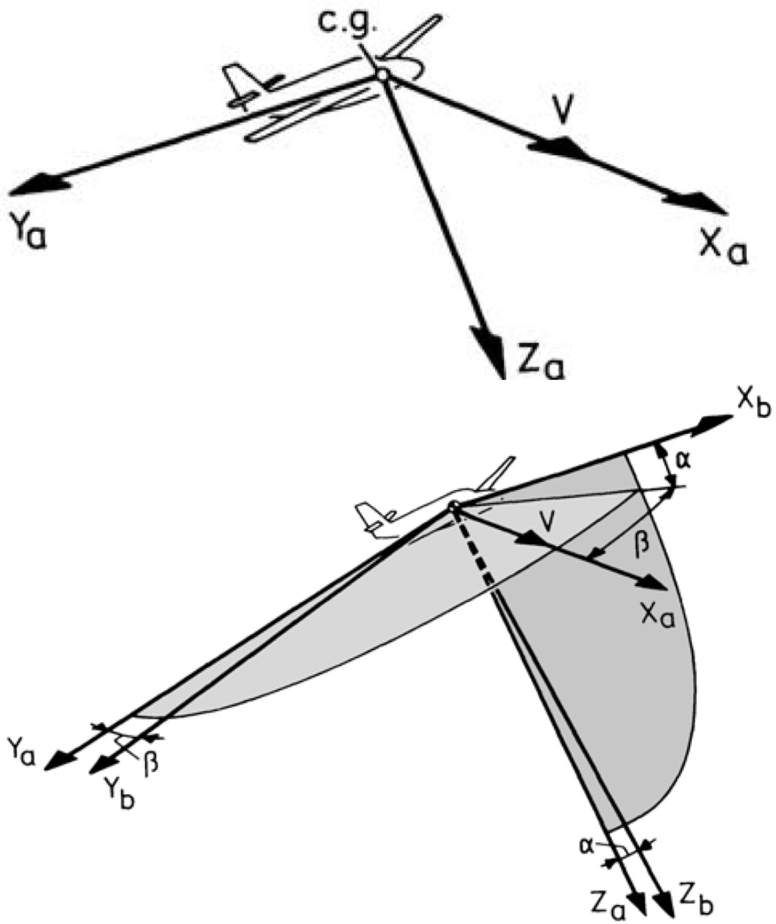
Roll angle (body axis)



- A rotation by ϕ about the X_b-axis to the final position $X_bY_bZ_b$.

Axis systems and Euler angles

Air path axis system

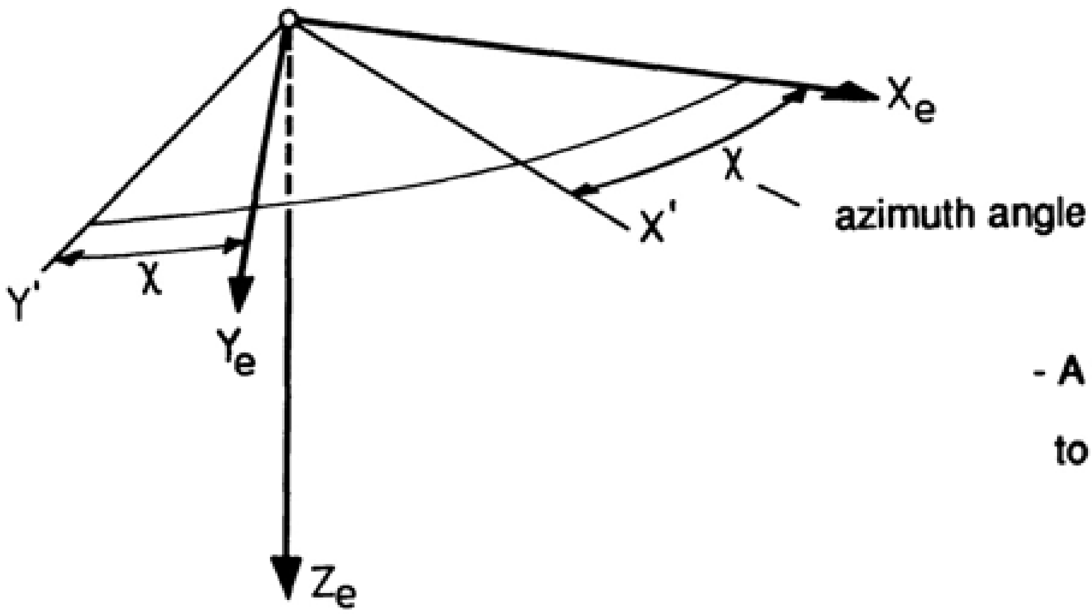


<u>Air path axis system: {E_a}</u>

1. Origin is fixed to the aircraft c.g.
2. X_a lies along the velocity vector
3. Z_a taken in the plane of symmetry of the airplane
4. Y_a is positive starboard

Axis systems and Euler angles

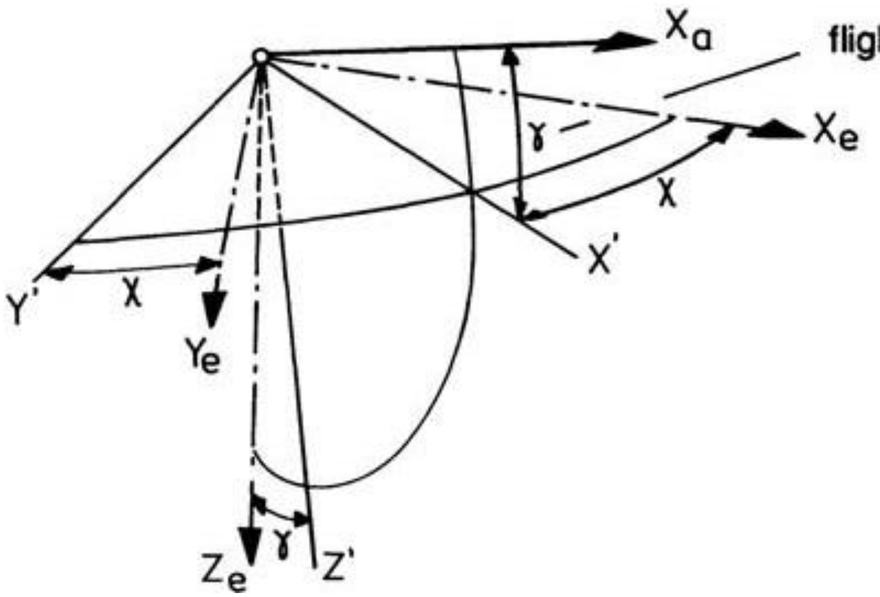
Azimuth angle (air path axis)



- A rotation by χ about the Z_e-axis to the intermediate position $X'Y'Z_e$.

Axis systems and Euler angles

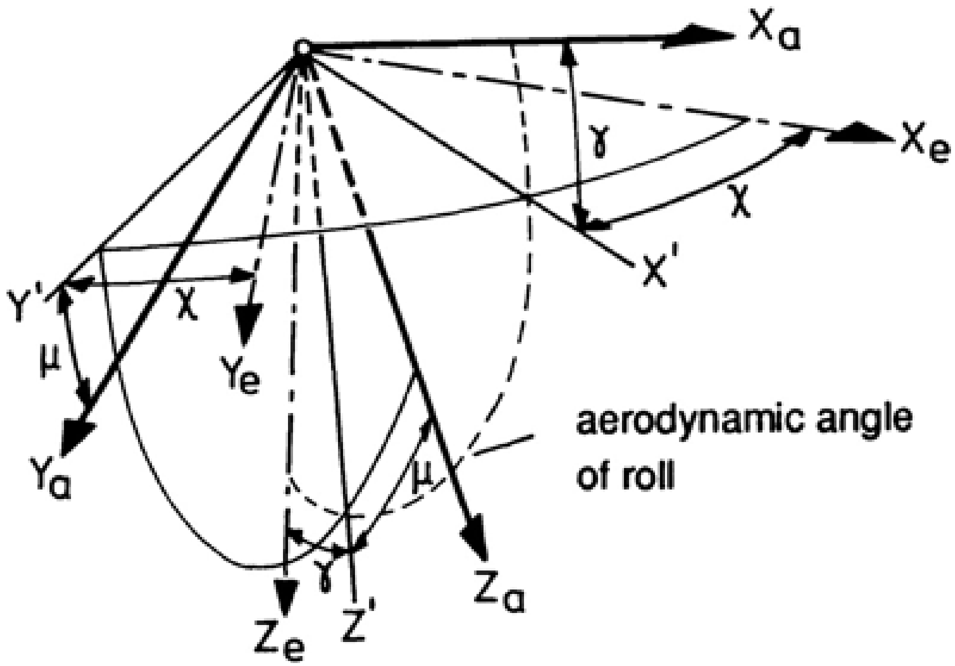
Flight path angle (air path axis system)



- A rotation by γ about the Y'-axis to the intermediate position $X_a Y'Z'$.

Axis systems and Euler angles

Aerodynamic angle of roll (air path axis system)



- A rotation by μ about the X_a-axis to the final position $X_aY_aZ_a$.

Axis systems and Euler angles

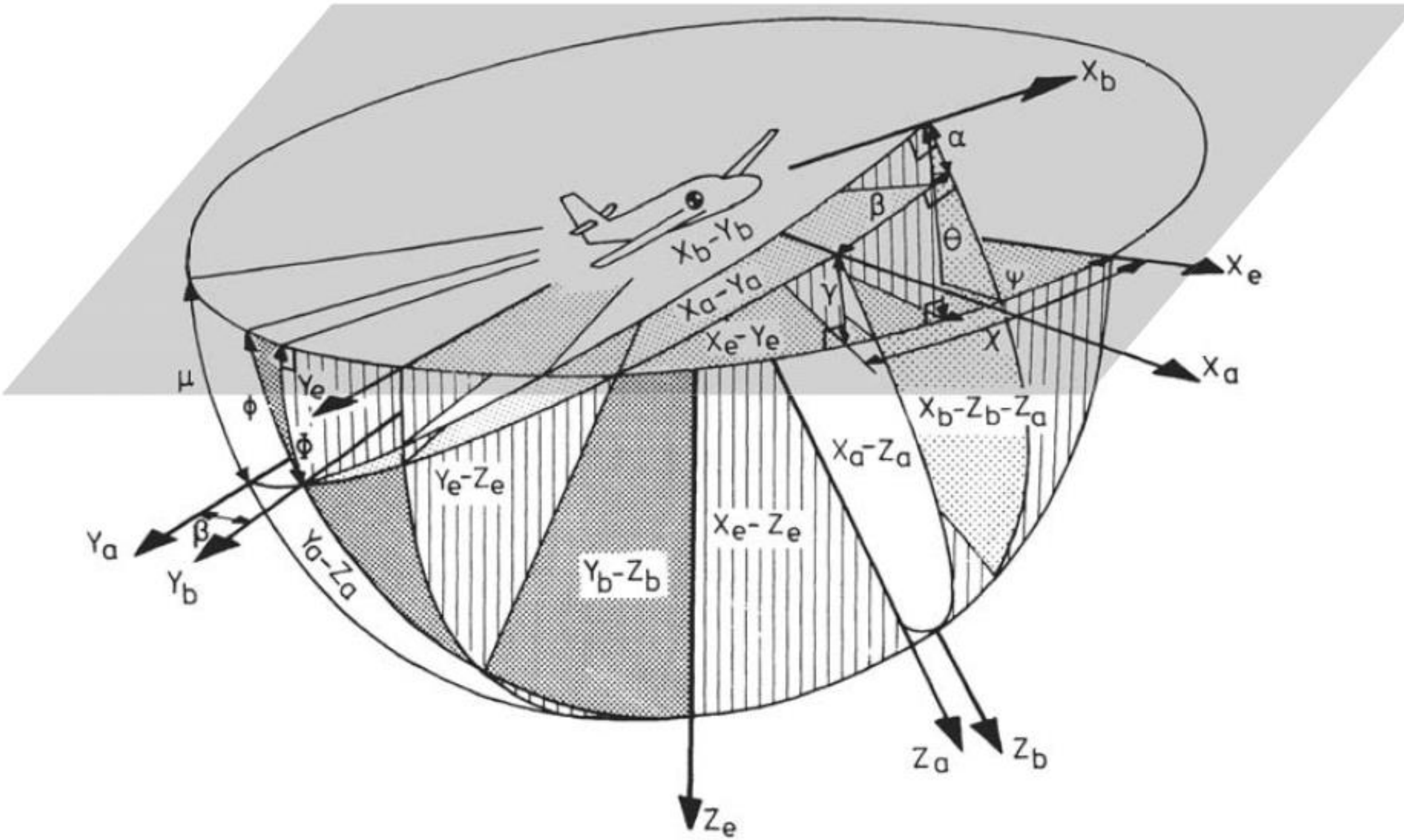
Summary

- Four axes systems can be defined
  - Earth
  - Moving Earth
  - Body axes
  - Air path axes

- Three **Euler angles** define the orientation of the aircraft (body axes) **Yaw** ψ → **Pitch** θ → **Roll** ϕ

- Three Euler angles define the orientation of the aircraft (body axes) **Azimuth** χ → **Flight path** γ → **Aerodynamic roll** μ

- **The sequence of the Euler angles is very important!!!**


Xb
α
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θ
Ψ
Xe
Xb-Yb
Xa-Ya
Xe-Ye
γ
χ
Xa
μ
Ye
φ
Φ
Xb-Zb-Za
Xa-Za
Ye-Ze
Xe- Ze
Ya
β
Yb
Ya-Za
Yb-Zb
Za
Zb
Ze

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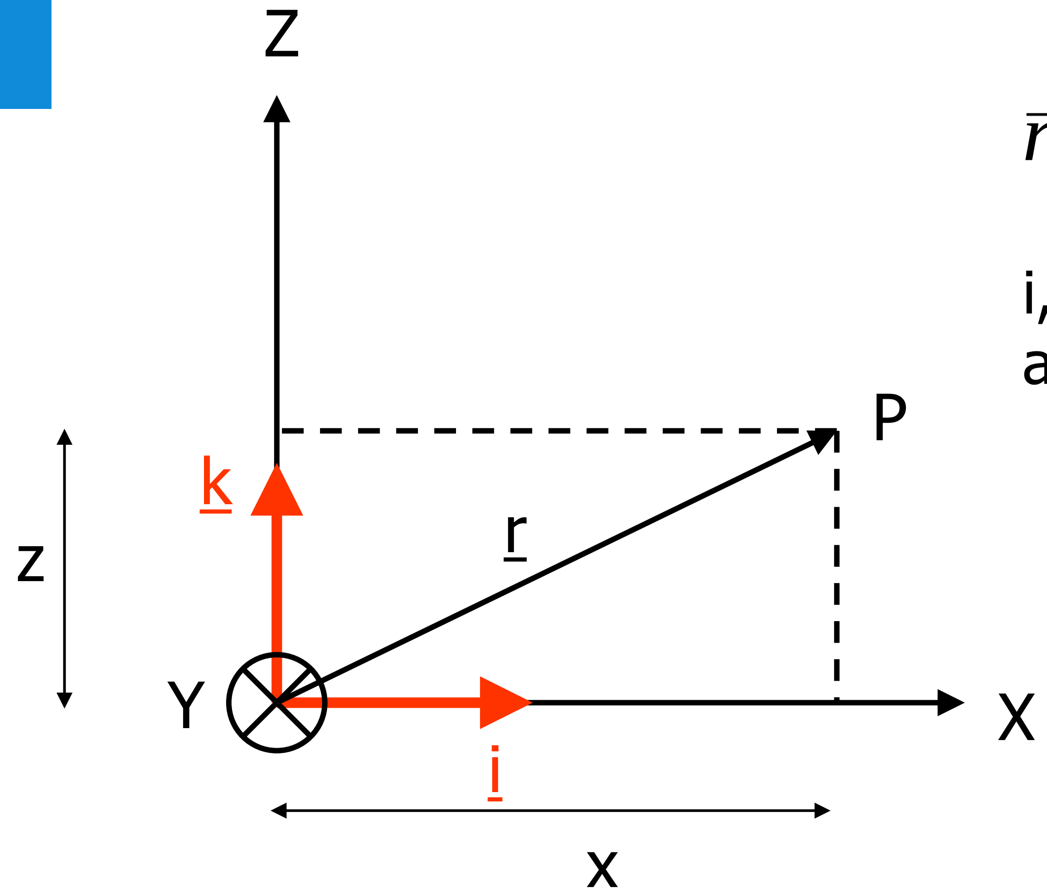
- Introduction
- Axis systems and Euler angles
- **Vector / matrix notation**
- Accelerations
- Forces
- General equations of motion 3D flight
- Effect of a wind gradient

Vector / matrix notation



$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

i, j and k are unit vectors along the axes

Vector / matrix notation

$$\vec{r} = x \cdot \vec{i} + y \cdot \vec{j} + z \cdot \vec{k} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$$

$$= \begin{pmatrix} x & y & z \end{pmatrix} \left\{ \underline{E} \right\}$$

$(\quad)$: Row

$\{\quad\}$: Column

$[\quad]$: Square matrix

Content

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Accelerations

$$\vec{a} = \frac{d\vec{V}}{dt}$$

$$\vec{V} = \begin{bmatrix} V & 0 & 0 \end{bmatrix} \{E_a\}$$

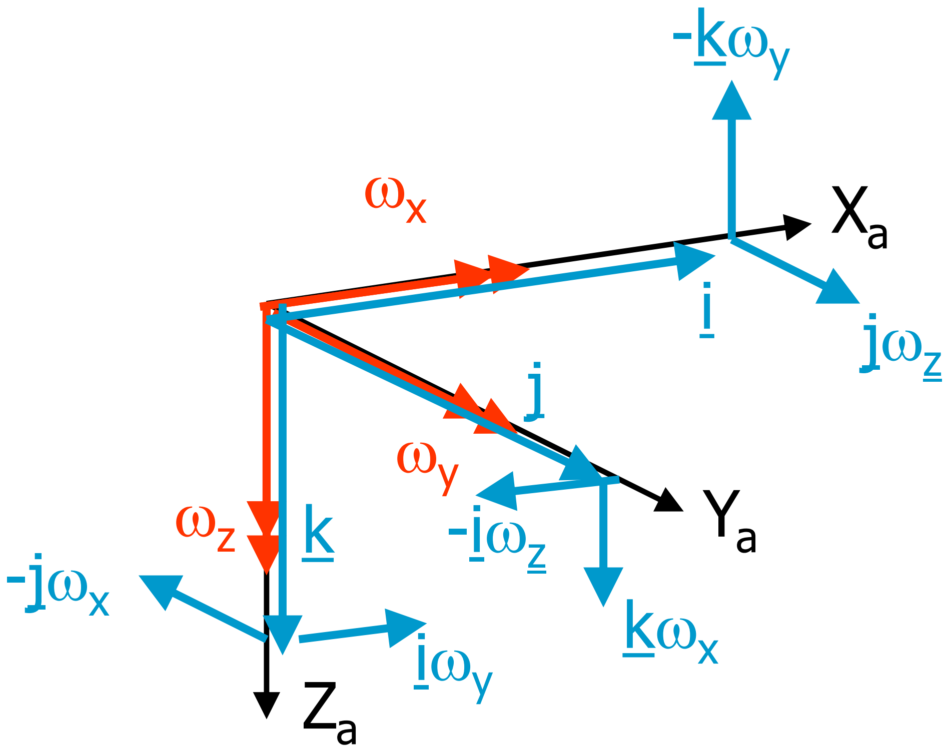
?

$$\frac{d\vec{V}}{dt} = \begin{bmatrix} \dot{V} & 0 & 0 \end{bmatrix} \{E_a\} + \begin{bmatrix} V & 0 & 0 \end{bmatrix} \{\dot{E}_a\}$$

What is the time derivative of the air path axis system?

Accelerations

Time derivative of the air path axis system



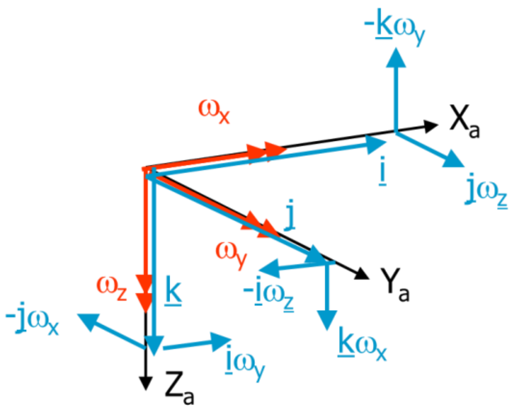
$$\frac{d\vec{i}}{dt} = 0 \cdot \vec{i} + \omega_z \cdot \vec{j} - \omega_y \cdot \vec{k}$$

$$\frac{d\vec{j}}{dt} = -\omega_z \cdot \vec{i} + 0 \cdot \vec{j} + \omega_x \cdot \vec{k}$$

$$\frac{d\vec{k}}{dt} = \omega_y \cdot \vec{i} - \omega_x \cdot \vec{j} + 0 \cdot \vec{k}$$

$$\left\{ \underline{\dot{E}}_a \right\} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \left\{ \underline{E}_a \right\}$$

Accelerations



$$\frac{d\vec{V}}{dt} = \begin{bmatrix} \dot{V} & 0 & 0 \end{bmatrix} \left\{ E_a \right\} + \begin{bmatrix} V & 0 & 0 \end{bmatrix} \left\{ \dot{E}_a \right\}$$

$$\frac{d\vec{V}}{dt} = \begin{bmatrix} \dot{V} & 0 & 0 \end{bmatrix} \left\{ E_a \right\} + \begin{bmatrix} V & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \left\{ \underline{E}_a \right\}$$

$$\boxed{\frac{d\vec{V}}{dt} = \begin{bmatrix} \dot{V} & V\omega_z & -V\omega_y \end{bmatrix} \left\{ E_a \right\}}$$

Content

- Introduction
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- **Forces**
- General equations of motion 3D flight
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Forces

$$\vec{F} = \vec{L} + \vec{D} + \vec{T} + \vec{W}$$

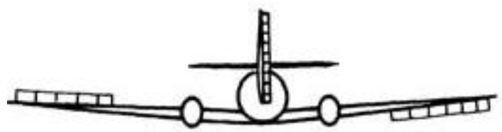
$$\vec{F} = \begin{pmatrix} T - D & 0 & -L \end{pmatrix} \left\{ \underline{E}_a \right\} + W \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \left\{ \underline{E}_e \right\}$$

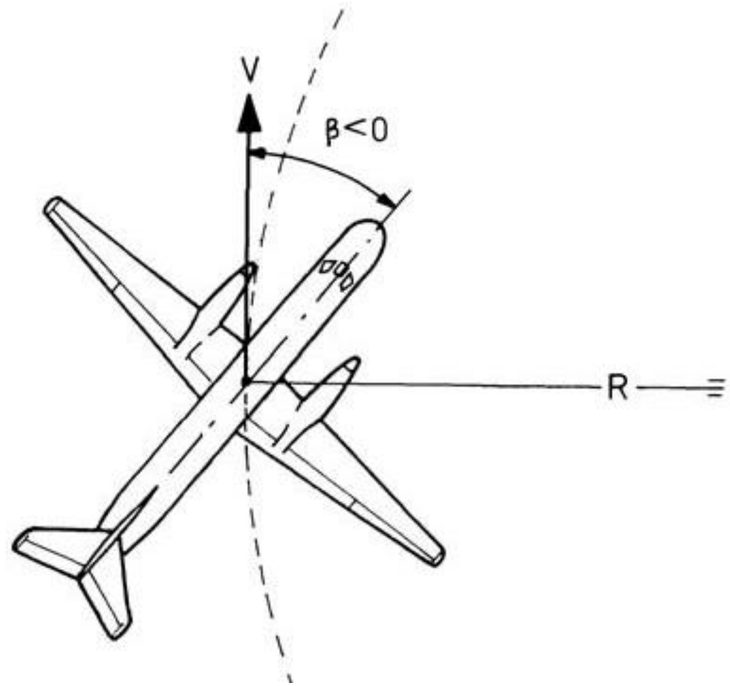
No sideslip

Assume thrust in direction of airspeed vector

Forces

Sideslip angle

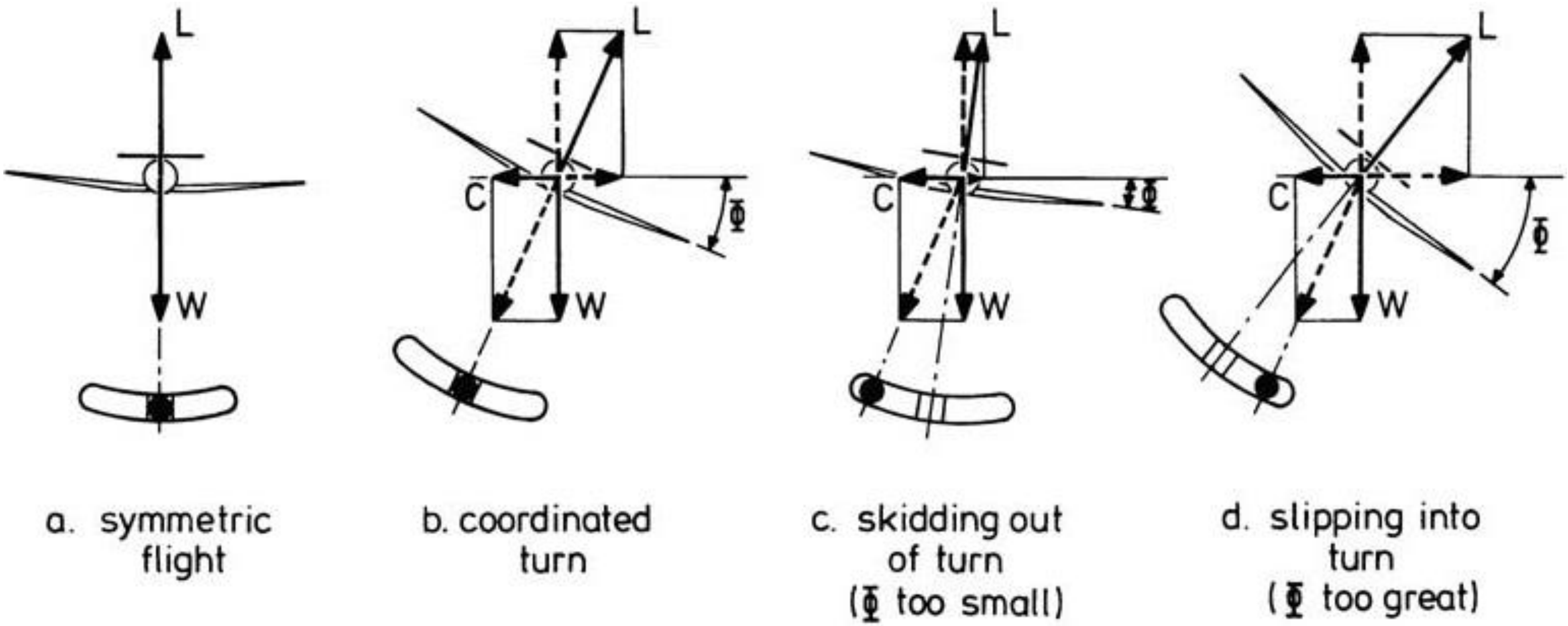




Coordinated Turn

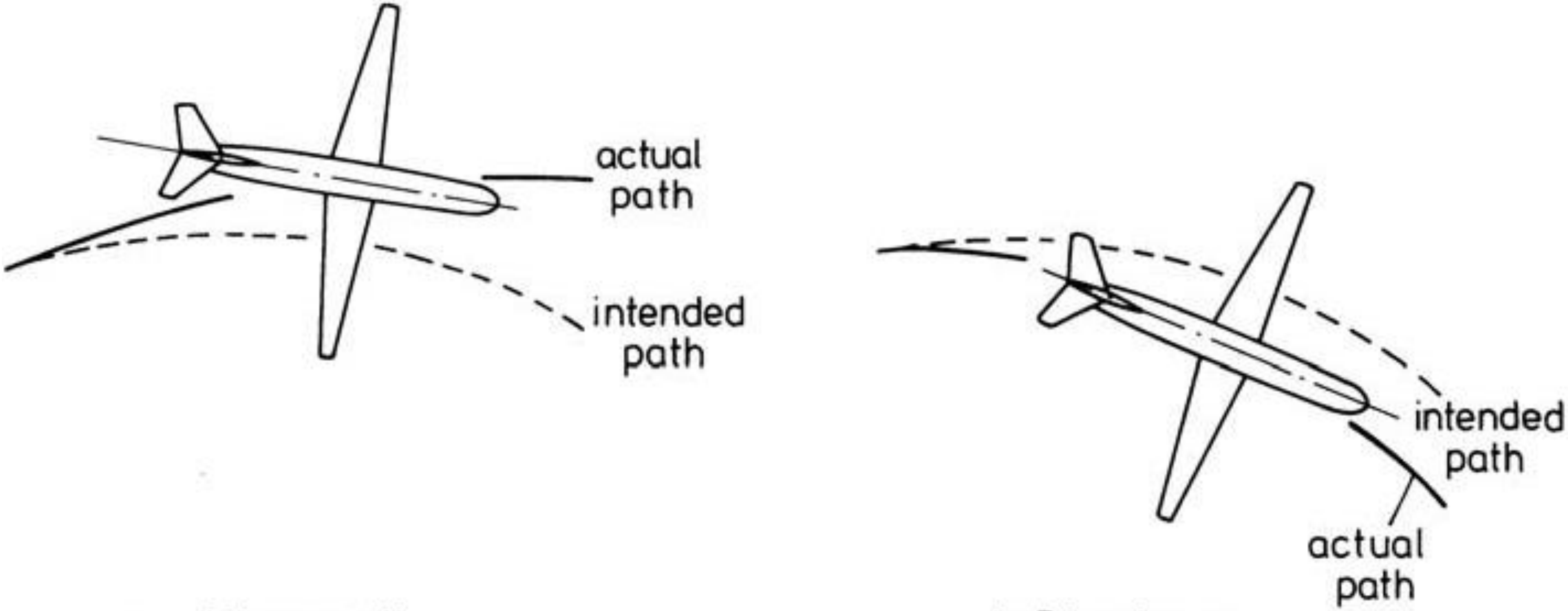
Forces

Sideslip angle



Forces

Sideslip angle



a. Φ too small:
skidding out of turn

b. Φ too large:
slipping into turn

Forces

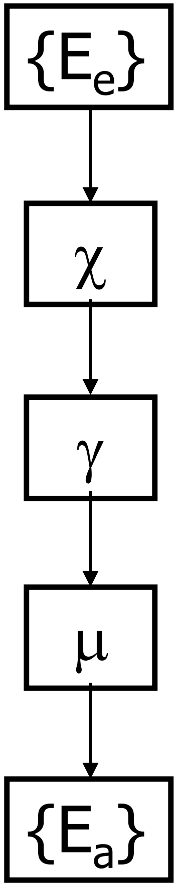
$$\vec{F} = \vec{L} + \vec{D} + \vec{T} + \vec{W}$$

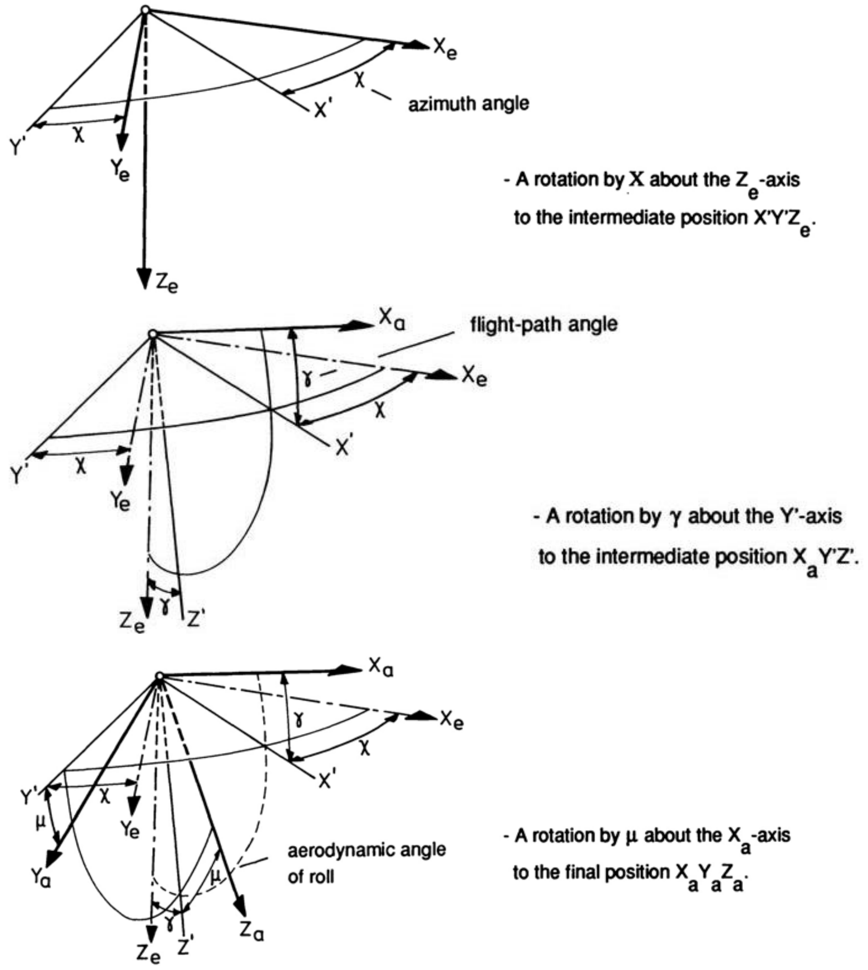
$$\vec{F} = \begin{pmatrix} T - D & 0 & -L \end{pmatrix} \left\{ \underline{E}_a \right\} + W \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \left\{ \underline{E}_e \right\}$$

Problem: different axis systems
→ Express all forces in 1 axis system

Forces

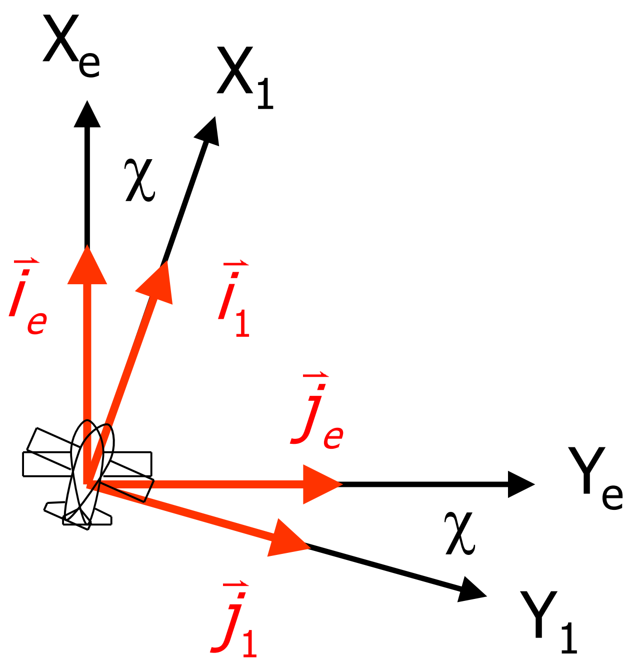
Transformation matrices





Forces

Rotation over azimuth angle (χ)



$$\vec{i}_1 = \vec{i}_e \cos\chi + \vec{j}_e \sin\chi$$

$$\vec{j}_1 = -\vec{i}_e \sin\chi + \vec{j}_e \cos\chi$$

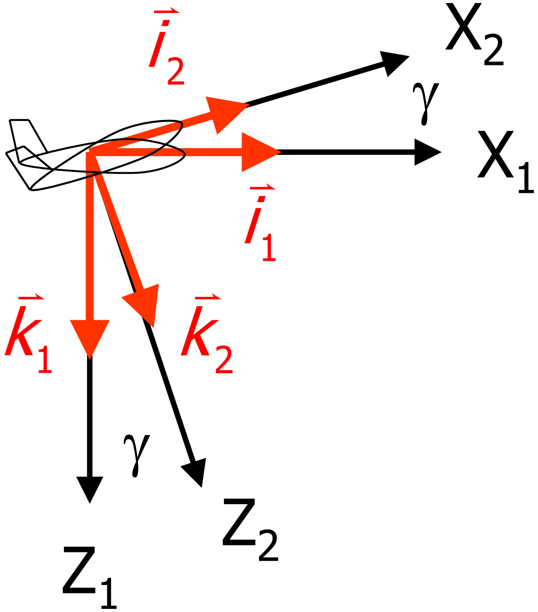
$$\vec{k}_1 = \vec{k}_e$$

$$\begin{Bmatrix} \vec{i}_1 \\ \vec{j}_1 \\ \vec{k}_1 \end{Bmatrix} = \begin{bmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i}_e \\ \vec{j}_e \\ \vec{k}_e \end{Bmatrix}$$

$$\{\underline{E}_1\} = [T_\chi]\{\underline{E}_e\}$$

Forces

Rotation over flight path angle (γ)



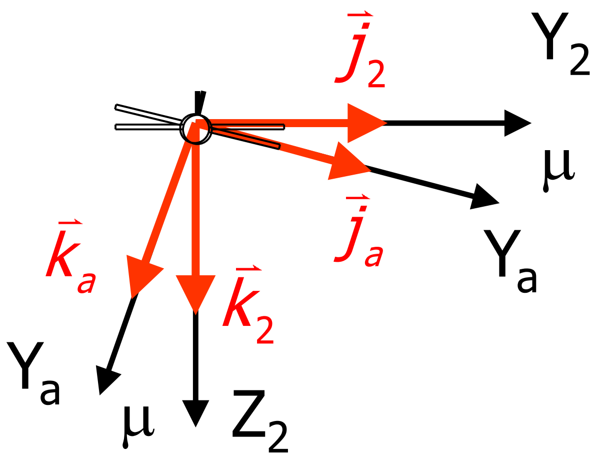
$$\begin{cases} \vec{i}_2 = \vec{i}_1 \cos\gamma - \vec{k}_1 \sin\gamma \\ \vec{j}_2 = \vec{j}_1 \\ \vec{k}_2 = \vec{i}_1 \sin\gamma + \vec{k}_1 \cos\gamma \end{cases}$$

$$\{\underline{E}_2\} = [T_\gamma]\{\underline{E}_1\}$$

$$[T_\gamma] = \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix}$$

Forces

Rotation over aerodynamic angle of roll (μ)



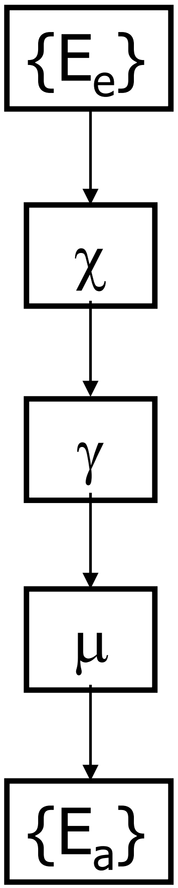
$$\begin{cases} \vec{i}_a = \vec{i}_2 \\ \vec{j}_a = \vec{j}_2 \cos\mu + \vec{k}_2 \sin\mu \\ \vec{k}_a = -\vec{j}_2 \sin\mu + \vec{k}_2 \cos\mu \end{cases}$$

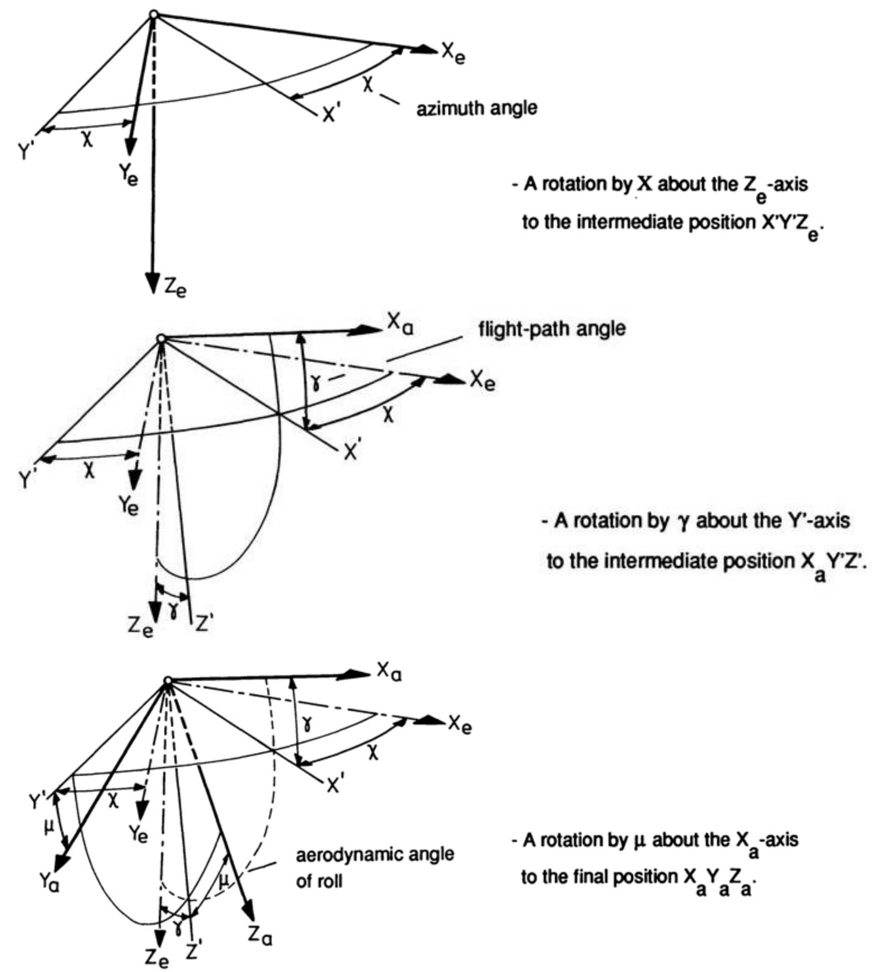
$$\left\{\underline{E}_a\right\} = \left[T_\mu\right]\left\{\underline{E}_2\right\}$$

$$\left[T_\mu\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{bmatrix}$$

Forces

Transformation matrices





Forces

Transformation matrices

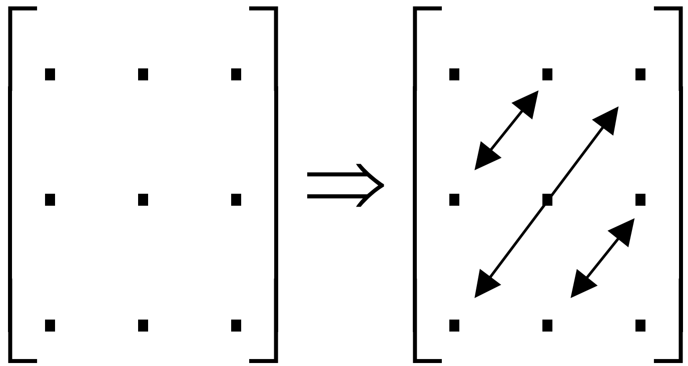
$$\left\{\underline{E}_1\right\} = \left[T_\chi\right]\left\{\underline{E}_e\right\}$$

Forces

Properties of transformation matrices

$$\left[\quad\right]^{-1} = \left[\quad\right]^{T}$$

$$\left[\quad\right]\cdot\left[\quad\right]^{-1} = \left[\quad\right]^{-1}\cdot\left[\quad\right] = \left[I\right]$$



Forces

All results combined

$$\vec{F} = \vec{L} + \vec{D} + \vec{T} + \vec{W}$$

$$= \begin{pmatrix} T-D & 0 & -L \end{pmatrix}\{\underline{E}_a\} + W\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\{\underline{E}_e\}$$

$$\vec{F} = \begin{pmatrix} T-D & 0 & -L \end{pmatrix}\{\underline{E}_a\} + W\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\left[T_\chi\right]^T\left[T_\gamma\right]^T\left[T_\mu\right]^T\{\underline{E}_a\}$$

$$W\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\{E_e\} = W\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix}^T\begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix}^T\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{bmatrix}^T\{E_a\}$$

$$W\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\{E_e\} = \begin{pmatrix} -W\sin\gamma & W\cos\gamma\sin\mu & W\cos\gamma\cos\mu \end{pmatrix}\{E_a\}$$

$$\boxed{\vec{F} = \begin{pmatrix} T-D-W\sin\gamma & W\cos\gamma\sin\mu & -L+W\cos\gamma\cos\mu \end{pmatrix}\{\underline{E}_a\}}$$

Equations of motion

$$\vec{F} = m \cdot \vec{a}$$

$$\vec{F} = \begin{pmatrix} T - D - W\sin\gamma & W\cos\gamma\sin\mu & -L + W\cos\gamma\cos\mu \end{pmatrix} \{\underline{E}_a\}$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \begin{bmatrix} \dot{V} & V\omega_z & -V\omega_y \end{bmatrix} \{E_a\}$$

$$T - D - W\sin\gamma = m\dot{V}$$

$$W\cos\gamma\sin\mu = mV\omega_z$$

$$-L + W\cos\gamma\cos\mu = -mV\omega_y$$

3 equations of motion!

Equations of motion

Rewrite in traditional form

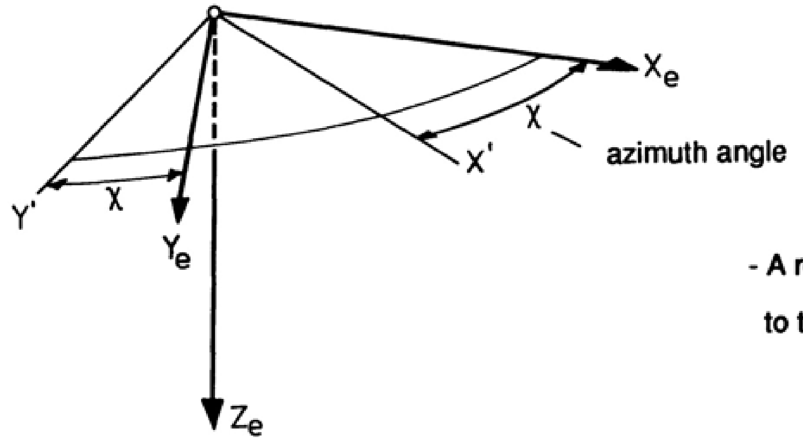
$$\begin{cases} T - D - W\sin\gamma = m\dot{V} \\ W\cos\gamma\sin\mu = mV\omega_z \\ -L + W\cos\gamma\cos\mu = -mV\omega_y \end{cases}$$

⇓ Using transformation matrices to convert ω_x, ω_y and ω_z

$$\begin{cases} m\dot{V} = T - D - W\sin\gamma \\ mV\dot{\chi}\cos\gamma = L\sin\mu \\ mV\dot{\gamma} = L\cos\mu - W\cos\gamma \end{cases}$$

Equations of motion

Conversion ω_x, ω_y, ω_z to $d\chi/dt$, $d\gamma/dt$, $d\mu/dt$



- A rotation by χ about the Z_e-axis to the intermediate position $X'Y'Z_e$.

$$\vec{\dot{\chi}} = \begin{bmatrix} 0 & 0 & \dot{\chi} \end{bmatrix} \{E_1\}$$

$$\{E_1\} = \left[T_\gamma\right]^{-1} \left[T_\mu\right]^{-1} \{E_a\}$$

$$\vec{\dot{\chi}} = \begin{bmatrix} 0 & 0 & \dot{\chi} \end{bmatrix} \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & -\sin\mu \\ 0 & \sin\mu & \cos\mu \end{bmatrix} \{E_a\}$$

$$\vec{\dot{\chi}} = \begin{bmatrix} -\dot{\chi}\sin\gamma & \dot{\chi}\cos\gamma\sin\mu & \dot{\chi}\cos\gamma\cos\mu \end{bmatrix} \{E_a\}$$

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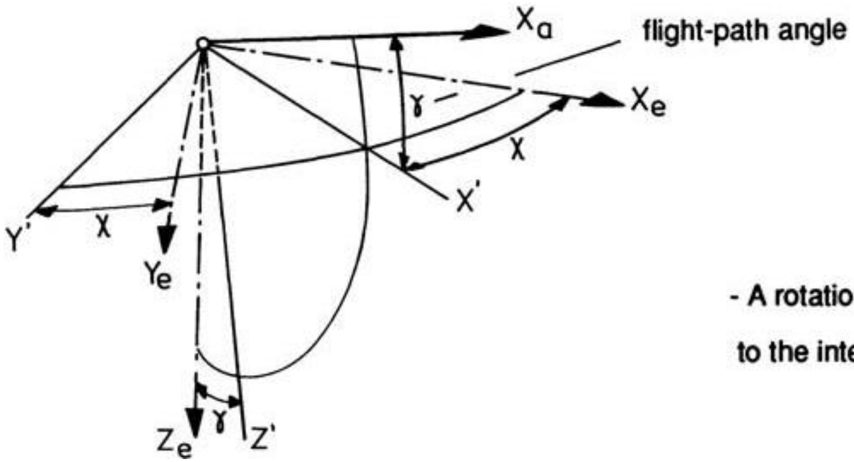


- A rotation by γ about the Y'-axis to the intermediate position X_aY'Z'.

$$\vec{\dot{\gamma}} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \end{bmatrix} \{E_2\}$$

$$\{E_2\} = \left[T_\mu\right]^{-1} \{E_a\}$$

$$\vec{\dot{\gamma}} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & -\sin\mu \\ 0 & \sin\mu & \cos\mu \end{bmatrix} \{E_a\}$$

$$\vec{\dot{\gamma}} = \begin{bmatrix} 0 & \dot{\gamma}\cos\mu & -\dot{\gamma}\sin\mu \end{bmatrix} \{E_a\}$$

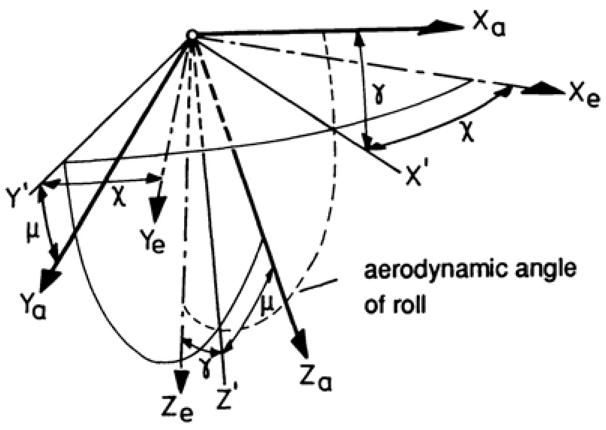
Equations of motion

Conversion ω_x, ω_y, ω_z to $d\chi/dt$, $d\gamma/dt$, $d\mu/dt$

$$\vec{\mu} = [\dot{\mu} \quad 0 \quad 0]\{E_3\}$$

$$\{E_3\} = \{E_a\}$$

$$\vec{\mu} = [\dot{\mu} \quad 0 \quad 0]\{E_a\}$$



- A rotation by μ about the X_a-axis to the final position $X_aY_aZ_a$.

Equations of motion

Conversion ω_x, ω_y, ω_z to $d\chi/dt$, $d\gamma/dt$, $d\mu/dt$

$$\begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}\left\{E_a\right\} =$$

$$\begin{bmatrix}-\dot{\chi}\sin\gamma + \dot{\mu} & \dot{\chi}\cos\gamma\sin\mu + \dot{\gamma}\cos\mu & \dot{\chi}\cos\gamma\cos\mu - \dot{\gamma}\sin\mu\end{bmatrix}\left\{E_a\right\}$$

Content

- Introduction
- Axes systems and Euler angles
- Vector / matrix notation
- Accelerations
- Forces
- **General equations of motion 3D flight**
- Effect of a wind gradient

Equations of motion

Final result

$$m\dot{V} = T - D - W\sin\gamma$$

$$mV\dot{\chi}\cos\gamma = L\sin\mu$$

$$mV\dot{\gamma} = L\cos\mu - W\cos\gamma$$

Content

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- **Effect of a wind gradient**

Effect of a wind gradient

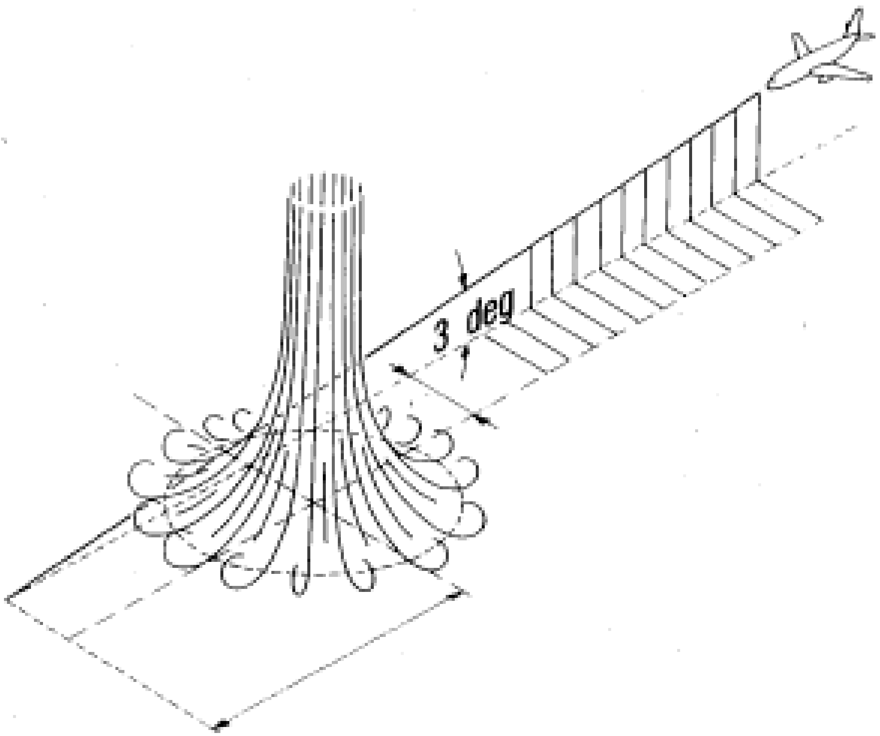
Question

An aircraft is flying from A to B over a distance of 1000 nautical miles. The True Airspeed of this aircraft is 120 knots (1kt = 1 nautical mile per hour). The aircraft is experiencing a constant headwind of 20 kts.

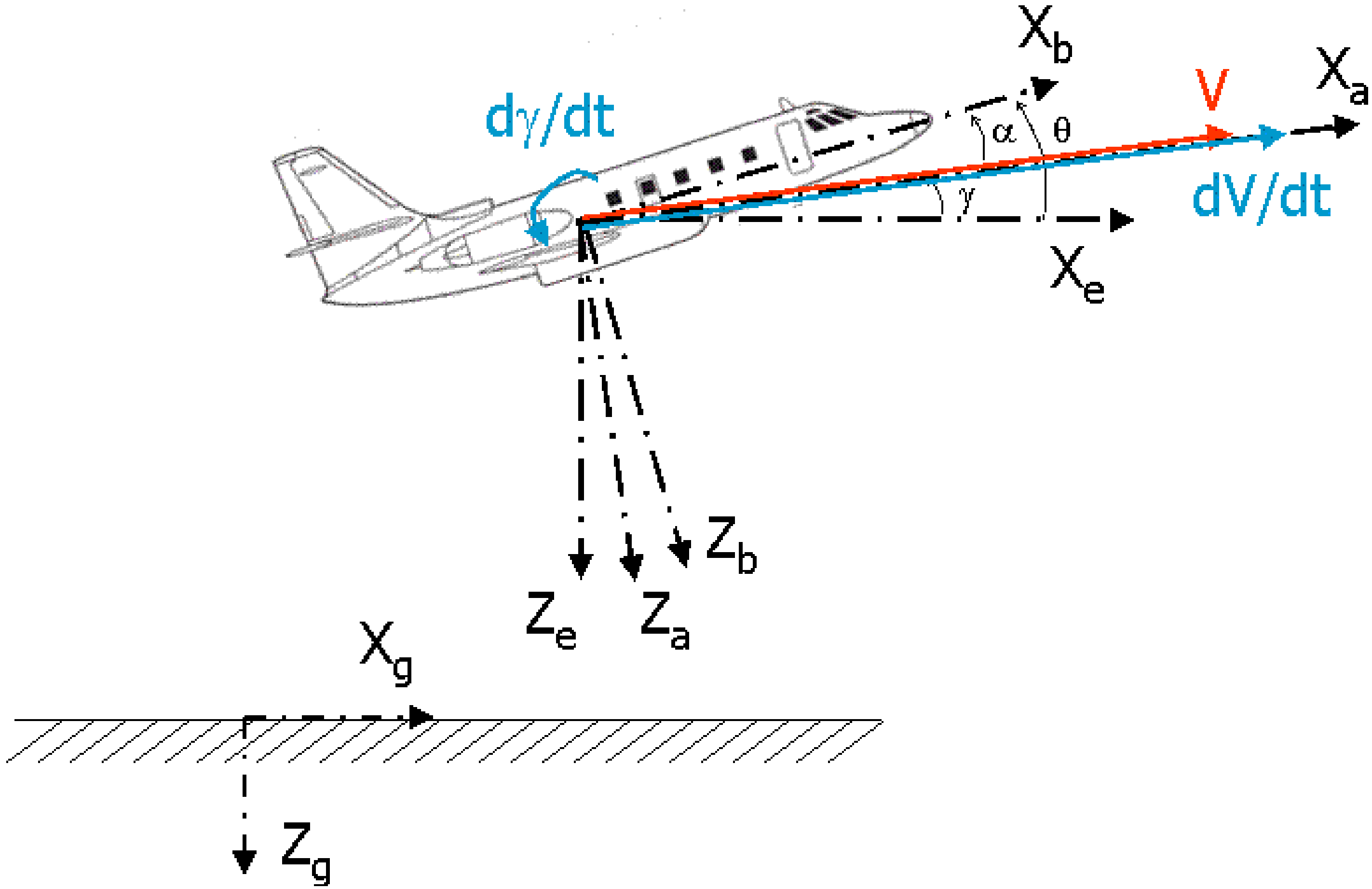
How long does it take to fly from A to B?

A. Less than 10 hours
B. 10 hours
C. More than 10 hours

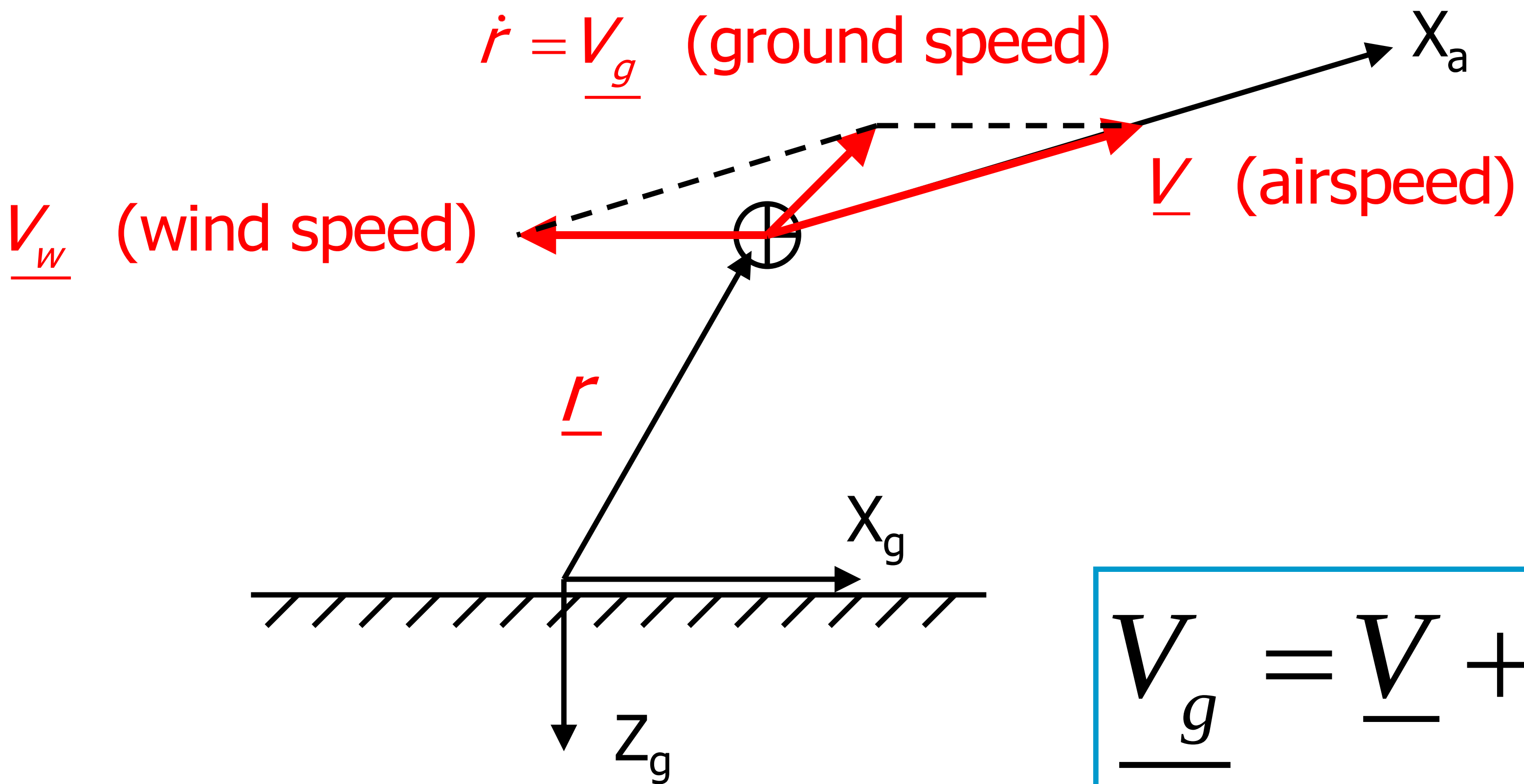
Effect of a wind gradient



Effect of a wind gradient

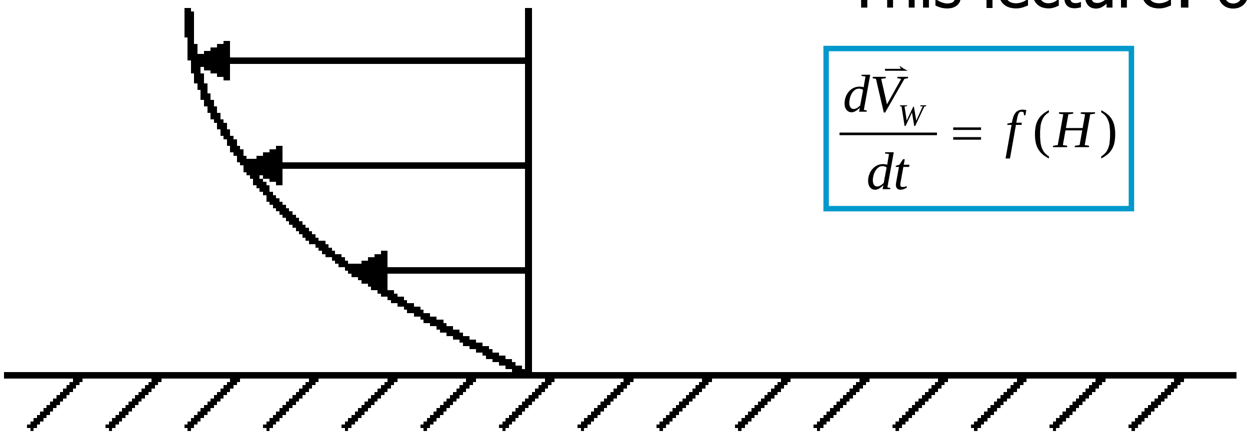


Effect of a wind gradient



$$\underline{V}_g = \underline{V} + \underline{V}_w$$

Effect of a wind gradient



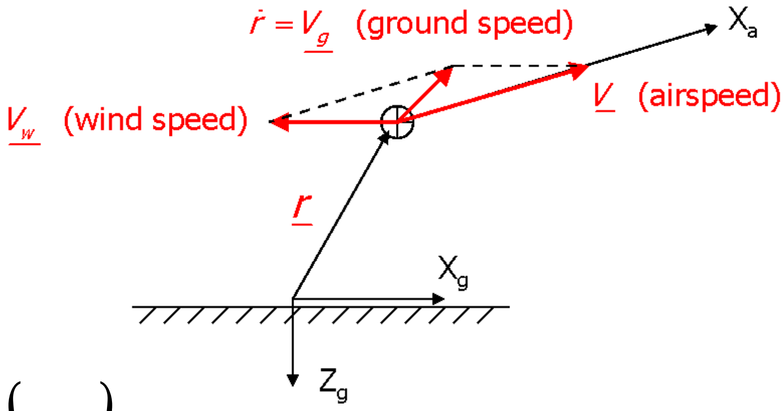
Wind is not constant:

$$\frac{d\vec{V}_W}{dt} \neq 0$$

This lecture: only horizontal wind

$$\frac{d\vec{V}_W}{dt} = f(H)$$

Effect of a wind gradient



Absolute acceleration

$$\underline{a} = \underline{\dot{V}}_g$$

So we need to define the velocity first

$$\underline{V}_g = \underline{V} + \underline{V}_w$$

$$\underline{V}_g = \begin{pmatrix} V & 0 & 0 \end{pmatrix} \left\{ \underline{E}_a \right\} + \begin{pmatrix} -V_w & 0 & 0 \end{pmatrix} \left\{ \underline{E}_g \right\}$$

The acceleration can be determined by taking the time derivative

$$\underline{a} = \underline{\dot{V}}_g = \underline{\dot{V}} + \underline{\dot{V}}_w$$

$$\underline{a} = \frac{d}{dt}\left(\begin{pmatrix} V & 0 & 0 \end{pmatrix} \left\{ \underline{E}_a \right\} \right) + \frac{d}{dt}\left(\begin{pmatrix} -V_w & 0 & 0 \end{pmatrix} \left\{ \underline{E}_g \right\} \right)$$

$$\boxed{\underline{a} = \begin{pmatrix} \dot{V} & 0 & 0 \end{pmatrix} \left\{ \underline{E}_a \right\} + \begin{pmatrix} V & 0 & 0 \end{pmatrix} \left\{ \underline{\dot{E}}_a \right\} + \begin{pmatrix} -\underline{\dot{V}}_w & 0 & 0 \end{pmatrix} \left\{ \underline{E}_g \right\} + \begin{pmatrix} -\underline{V}_w & 0 & 0 \end{pmatrix} \left\{ \underline{\dot{E}}_g \right\}}$$

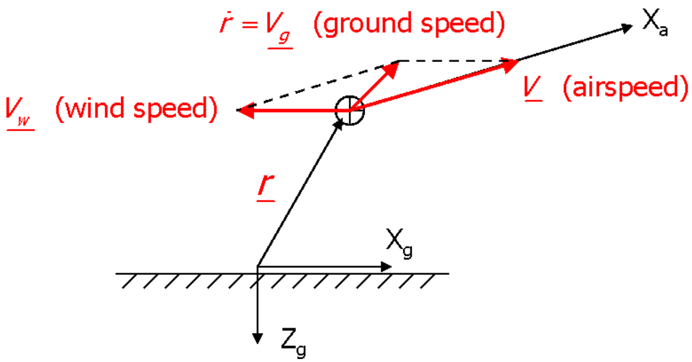
$$\underline{a} = \begin{pmatrix}\dot{V} & 0 & 0\end{pmatrix}\left\{\underline{E}_a\right\} + \begin{pmatrix}V & 0 & 0\end{pmatrix}\left\{\underline{\dot{E}}_a\right\} + \begin{pmatrix}-\dot{V}_w & 0 & 0\end{pmatrix}\left\{\underline{E}_g\right\} + \begin{pmatrix}-V_w & 0 & 0\end{pmatrix}\left\{\underline{\dot{E}}_g\right\}$$

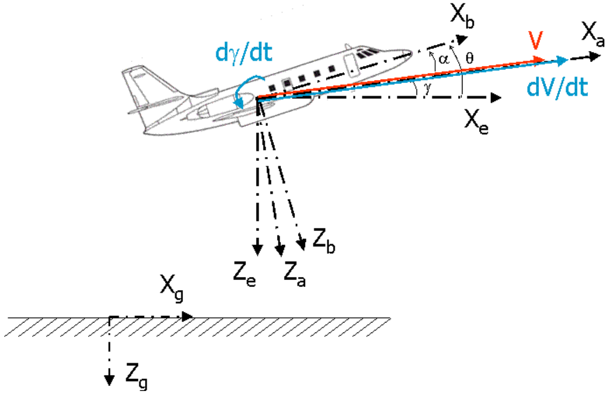
What are $\left\{\dot{E}_a\right\}$ and $\left\{\dot{E}_g\right\}$???

The ground axis system is at rest

$$\left\{\dot{E}_g\right\} = \vec{0}$$

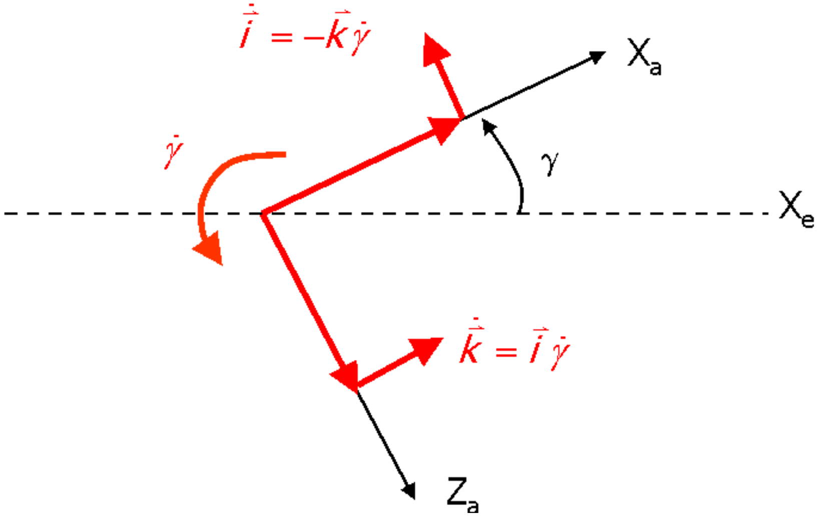
However, the air path axis system is rotating and translating





Effect of a wind gradient

$$\{\dot{E}_a\} = \begin{pmatrix} \frac{d}{dt}(\vec{i}) \\ \frac{d}{dt}(\vec{j}) \\ \frac{d}{dt}(\vec{k}) \end{pmatrix}$$



$$\frac{d}{dt}(\vec{i}) = 0\cdot\vec{i} + 0\cdot\vec{j} - \dot{\gamma}\vec{k}$$

$$\frac{d}{dt}(\vec{j}) = 0\cdot\vec{i} + 0\cdot\vec{j} + 0\cdot\vec{k}$$

$$\frac{d}{dt}(\vec{k}) = \dot{\gamma}\cdot\vec{i} + 0\cdot\vec{j} + 0\cdot\vec{k}$$

$$\{\dot{E}_a\} = \begin{pmatrix} \frac{d}{dt}(\vec{i}) \\ \frac{d}{dt}(\vec{j}) \\ \frac{d}{dt}(\vec{k}) \end{pmatrix} = \begin{bmatrix} 0 & 0 & -\dot{\gamma} \\ 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0 \end{bmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} = \begin{bmatrix} 0 & 0 & -\dot{\gamma} \\ 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0 \end{bmatrix} \{E_a\}$$

Effect of a wind gradient

$$\underline{a} = \left(\dot{V} \quad 0 \quad 0\right)\left\{\underline{E}_a\right\} + \left(V \quad 0 \quad 0\right)\left\{\underline{\dot{E}}_a\right\} + \left(-\dot{V}_w \quad 0 \quad 0\right)\left\{\underline{E}_g\right\} + \left(-\underline{V}_w \quad 0 \quad 0\right)\left\{\underline{\dot{E}}_g\right\}$$

Fill in the results

$$\underline{a} = \left(\dot{V} \quad 0 \quad 0\right)\left\{\underline{E}_a\right\} + \left(V \quad 0 \quad 0\right)\begin{bmatrix} 0 & 0 & -\dot{\gamma} \\ 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0 \end{bmatrix}\left\{\underline{E}_a\right\} + \left(-\dot{V}_w \quad 0 \quad 0\right)\left\{\underline{E}_g\right\}$$

Write out

$$\underline{a} = \left(\dot{V} \quad 0 \quad 0\right)\left\{\underline{E}_a\right\} + \left(0 \quad 0 \quad -V\dot{\gamma}\right)\left\{\underline{E}_a\right\} + \left(-\dot{V}_w \quad 0 \quad 0\right)\left\{\underline{E}_g\right\}$$

Simplify

$$\underline{a} = \left(\dot{V} \quad 0 \quad -V\dot{\gamma}\right)\left\{\underline{E}_a\right\} + \left(-\dot{V}_w \quad 0 \quad 0\right)\left\{\underline{E}_g\right\}$$

Two axis systems…

Effect of a wind gradient

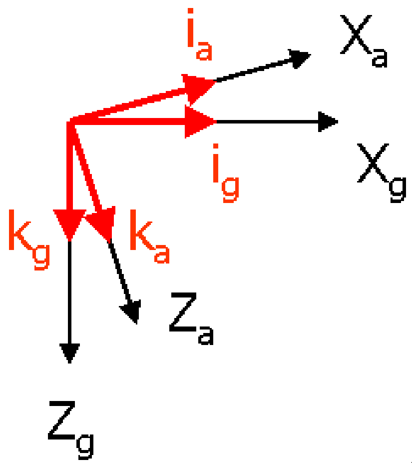
$$\underline{a} = \begin{pmatrix} \dot{V} & 0 & -V\dot{\gamma} \end{pmatrix} \left\{ \underline{E}_a \right\} + \begin{pmatrix} -\underline{\dot{V}}_w & 0 & 0 \end{pmatrix} \left\{ \underline{E}_g \right\}$$

$$\underline{i}_g = \cos\gamma \cdot \underline{i}_a + 0 \cdot \underline{j}_a + \sin\gamma \cdot \underline{k}_a$$

$$\underline{j}_g = 0 \cdot \underline{i}_a + 1 \cdot \underline{j}_a + 0 \cdot \underline{k}_a$$

$$\underline{k}_g = -\sin\gamma \cdot \underline{i}_a + 0 \cdot \underline{j}_a + \cos\gamma \cdot \underline{k}_a$$

Rotation over angle γ



$$\left\{ \underline{E}_g \right\} = \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \left\{ \underline{E}_a \right\}$$

$$\underline{a} = \begin{pmatrix} \dot{V} & 0 & -V\dot{\gamma} \end{pmatrix} \left\{ \underline{E}_a \right\} + \begin{pmatrix} -\underline{\dot{V}}_w & 0 & 0 \end{pmatrix} \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \left\{ \underline{E}_a \right\}$$

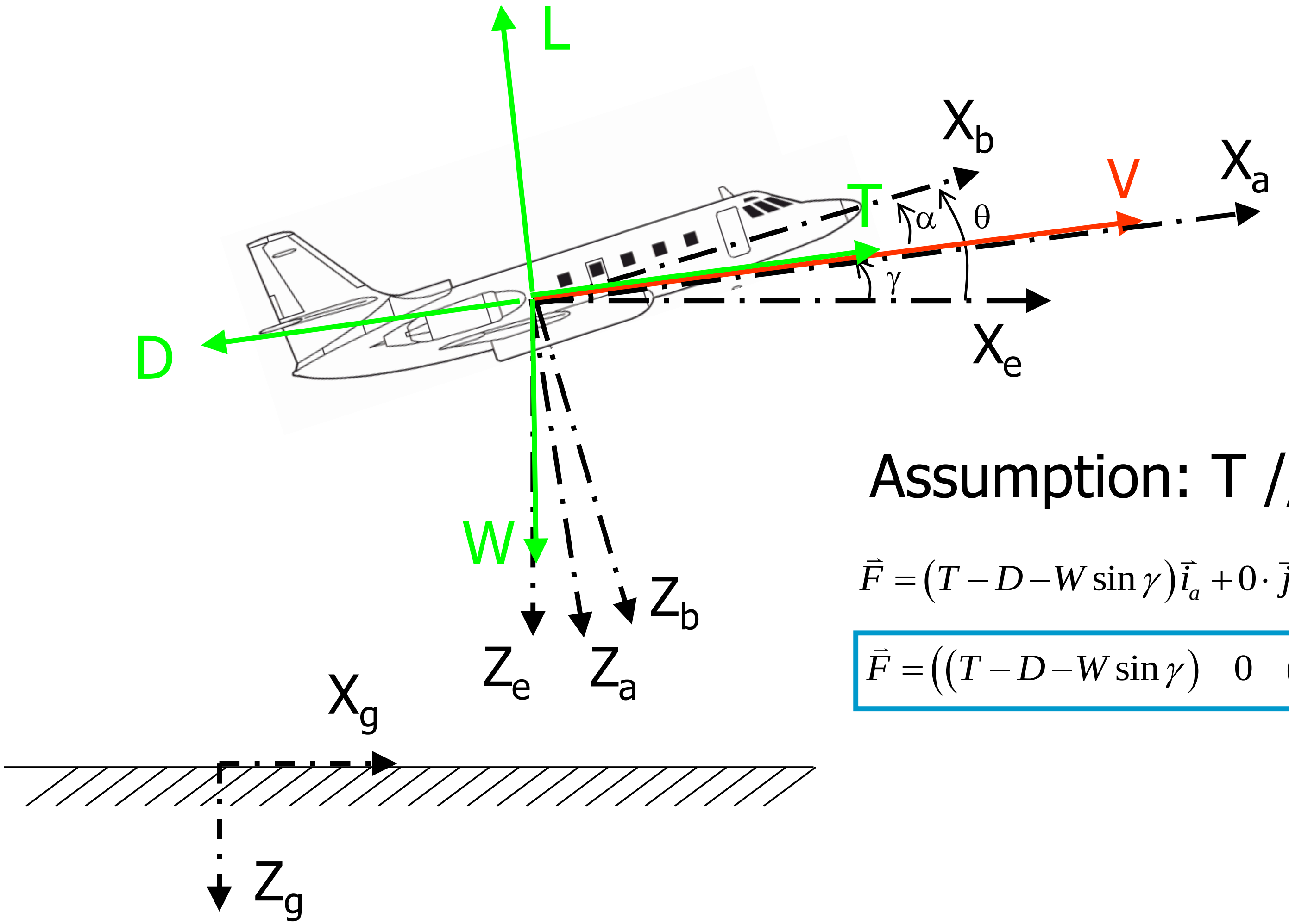
Effect of a wind gradient

$$\underline{a} = \begin{pmatrix} \dot{V} & 0 & -V\dot{\gamma} \end{pmatrix} \left\{ \underline{E_a} \right\} + \begin{pmatrix} -\dot{V}_w & 0 & 0 \end{pmatrix} \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \left\{ \underline{E_g} \right\}$$

$$\underline{a} = \begin{pmatrix} \dot{V} & 0 & -V\dot{\gamma} \end{pmatrix} \left\{ \underline{E_a} \right\} + \begin{pmatrix} -\underline{\dot{V}_w} \cos\gamma & 0 & -\underline{\dot{V}_w} \sin\gamma \end{pmatrix} \left\{ \underline{E_a} \right\}$$

$$\boxed{\underline{a} = \begin{pmatrix} \dot{V} - \dot{V}_w \cos\gamma & 0 & -V\dot{\gamma} - \dot{V}_w \sin\gamma \end{pmatrix} \left\{ \underline{E_a} \right\}}$$

Effect of a wind gradient



Assumption: T // V

$$\vec{F} = \left(T - D - W\sin\gamma\right)\vec{i}_a + 0\cdot\vec{j}_a + \left(L - W\cos\gamma\right)\cdot\vec{k}_a$$

$$\boxed{\vec{F} = \left(\left(T - D - W\sin\gamma\right) \quad 0 \quad \left(L - W\cos\gamma\right)\right)\left\{E_a\right\}}$$

Effect of a wind gradient

$$\underline{F} = m \cdot \dot{V}_g$$

$$\underline{F} = \begin{pmatrix} T - D - W\sin\gamma & 0 & W\cos\gamma - L \end{pmatrix} \left\{ \underline{E}_a \right\}$$

$$\underline{a} = \begin{pmatrix} \dot{V} - \underline{\dot{V}_w}\cos\gamma & 0 & -V\dot{\gamma} - \underline{\dot{V}_w}\sin\gamma \end{pmatrix} \left\{ \underline{E}_a \right\}$$

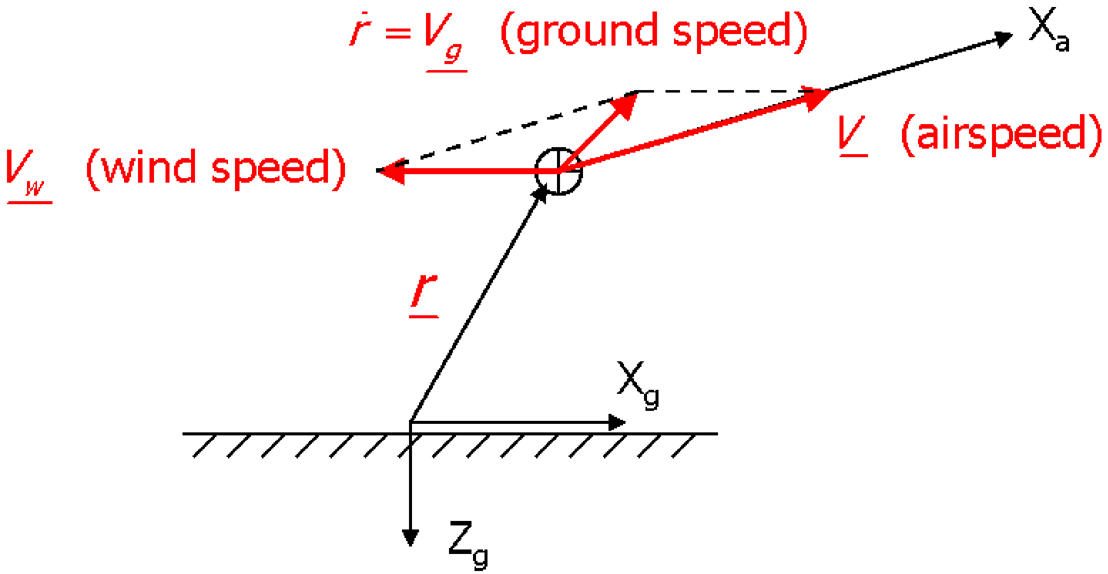
$$T - D - W\sin\gamma = \frac{W}{g}\left(\dot{V} - \dot{V}_w\cos\gamma\right)$$

$$0 = 0$$

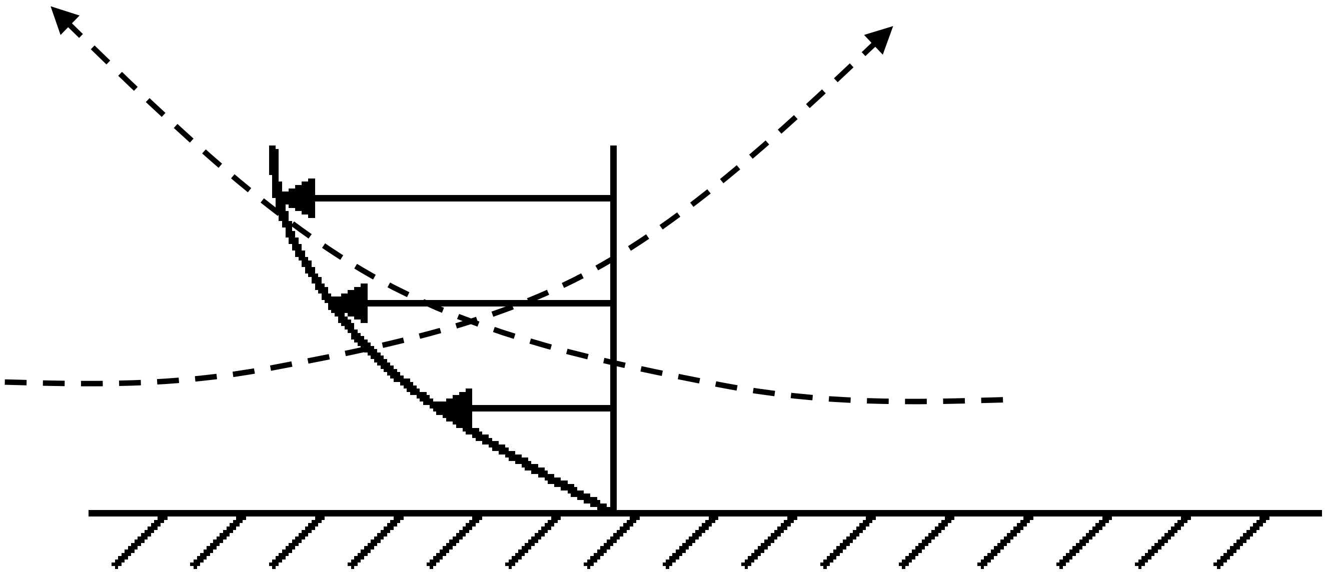
$$L - W\cos\gamma = \frac{W}{g}\left(V\dot{\gamma} + \dot{V}_w\sin\gamma\right)$$

Effect of a wind gradient

$$\dot{H} = RC = V \sin\gamma$$
$$\dot{s} = V\cos\gamma - V_w$$



Effect of a wind gradient



Questions?