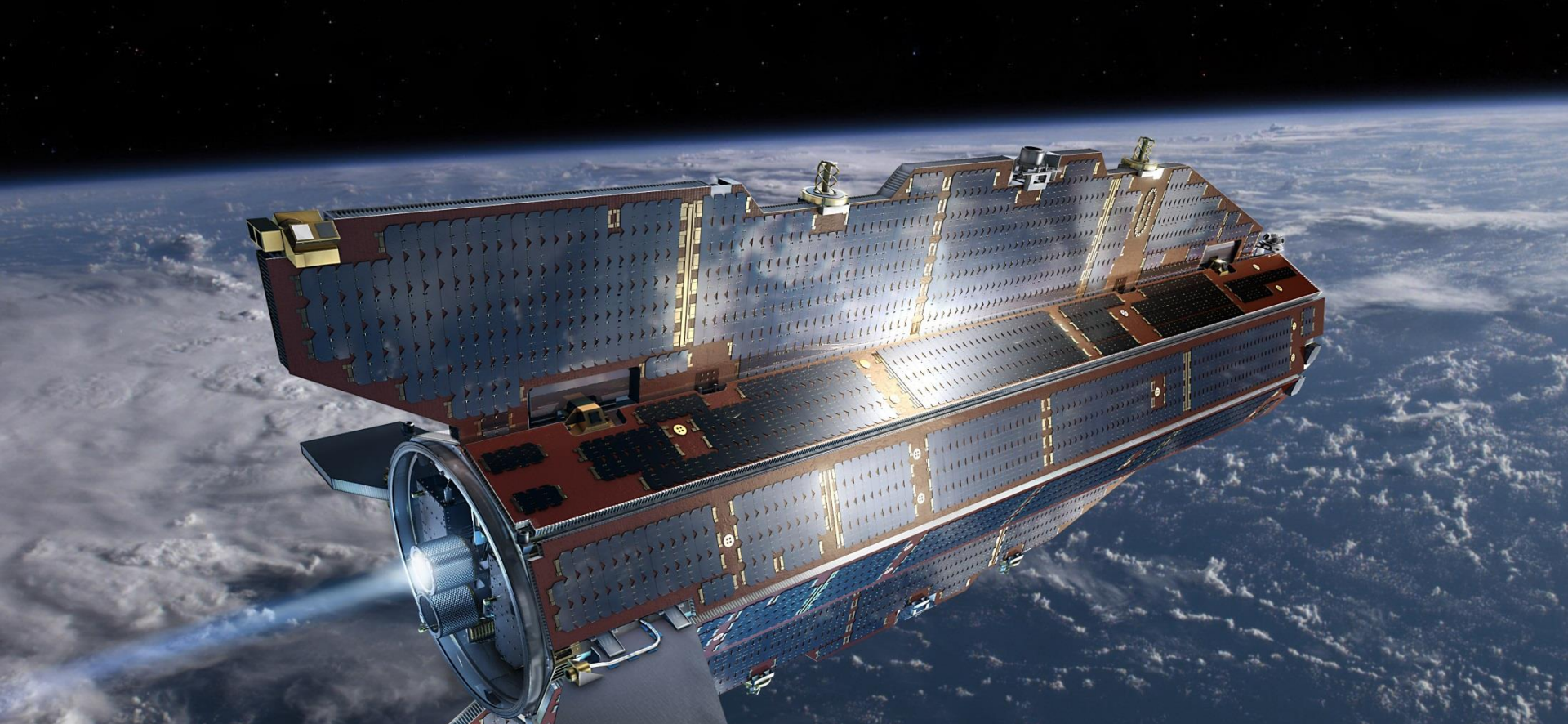


Flight and Orbital Mechanics

Lecture slides



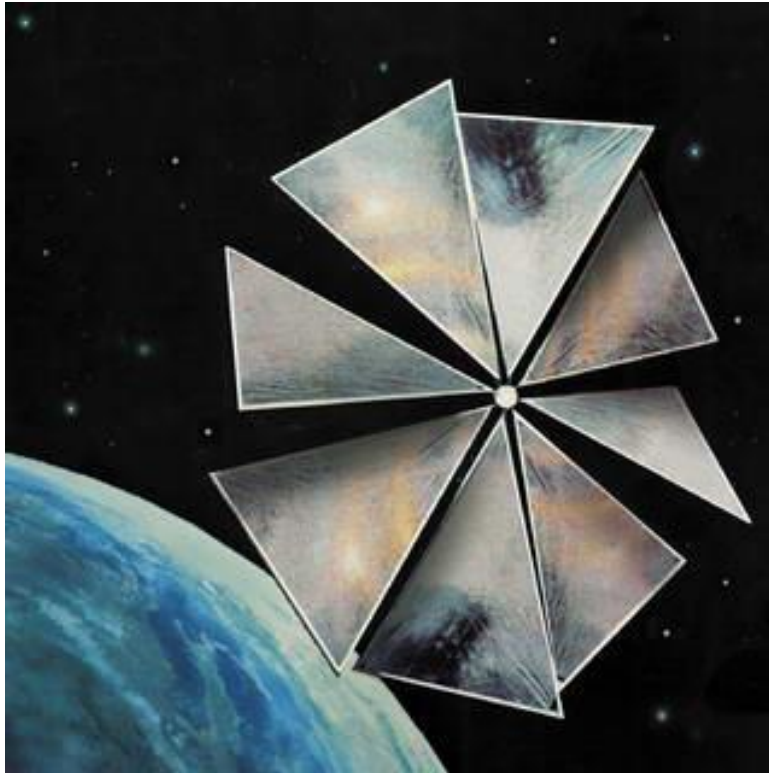
Flight and Orbital Mechanics

AE2-104, lecture hours 17+18: Perturbations

Ron Noomen

October 25, 2012

Example: solar sail spacecraft



Questions:

- what is the purpose of this mission?
- where is the satellite located?
- why does it use a solar sail?
-

[Wikipedia, 2010]

Overview

- Orbital mechanics (recap)
 - Irregularities gravity field
 - Third-body perturbations
 - Atmospheric drag
 - Solar radiation pressure
 - Thrust
 - Relativistic effects
 - Tidal forces
 - Thermal forces
 - Impacts debris/micrometeoroids
- ← Special missions only; not treated here

Learning goals

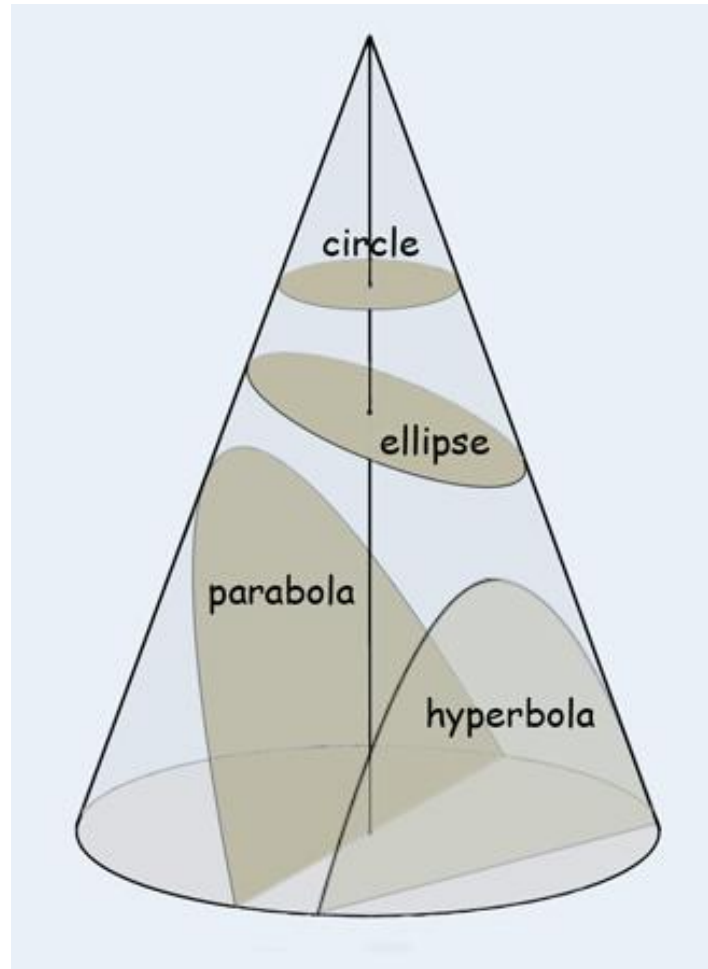
The student should be able to:

- mention and describe the various perturbing forces that may act on an arbitrary spacecraft;
- quantify the resulting accelerations;
- make an assessment of the importance of the different perturbing forces, depending on the specifics of the mission (phase) at hand.

Lecture material:

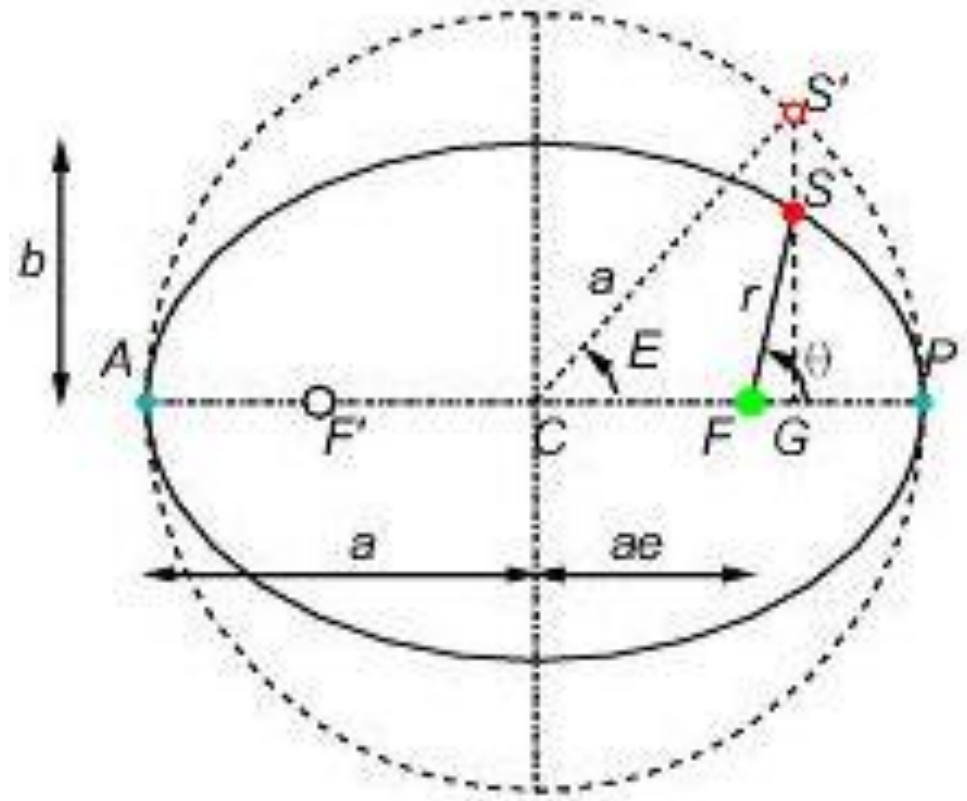
- these slides (incl. footnotes)

2-dimensional Kepler orbits



[Seligman, 2010]

2-dimensional Kepler orbits



[Cornellise, Schöyer and Wakker, 1979]

a : semi-major axis [m]

e : eccentricity [-]

θ : true anomaly [deg]

E : eccentric anomaly [deg]

2-dimensional Kepler orbits: general equations

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} = \frac{p}{1 + e \cos \theta} ; r_p = a(1 - e) ; r_a = a(1 + e)$$

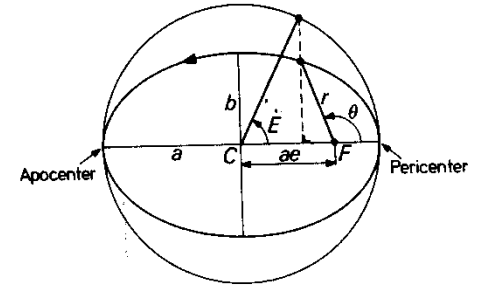
$$E_{tot} = E_{kin} + E_{pot} = \frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$V^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) ; V_{circ} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{a}} ; V_{esc} = \sqrt{\frac{2\mu}{r}}$$

$$T = 2\pi \sqrt{\frac{a^3}{\mu}}$$

2-dimensional Kepler orbits: equations (cnt'd)

ellips: $0 \leq e < 1$ $a > 0$ $E_{tot} < 0$



$$n = \sqrt{\frac{\mu}{a^3}}$$

$$\tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$$

→ $M = E - e \sin E$

$$M = n(t - t_0)$$

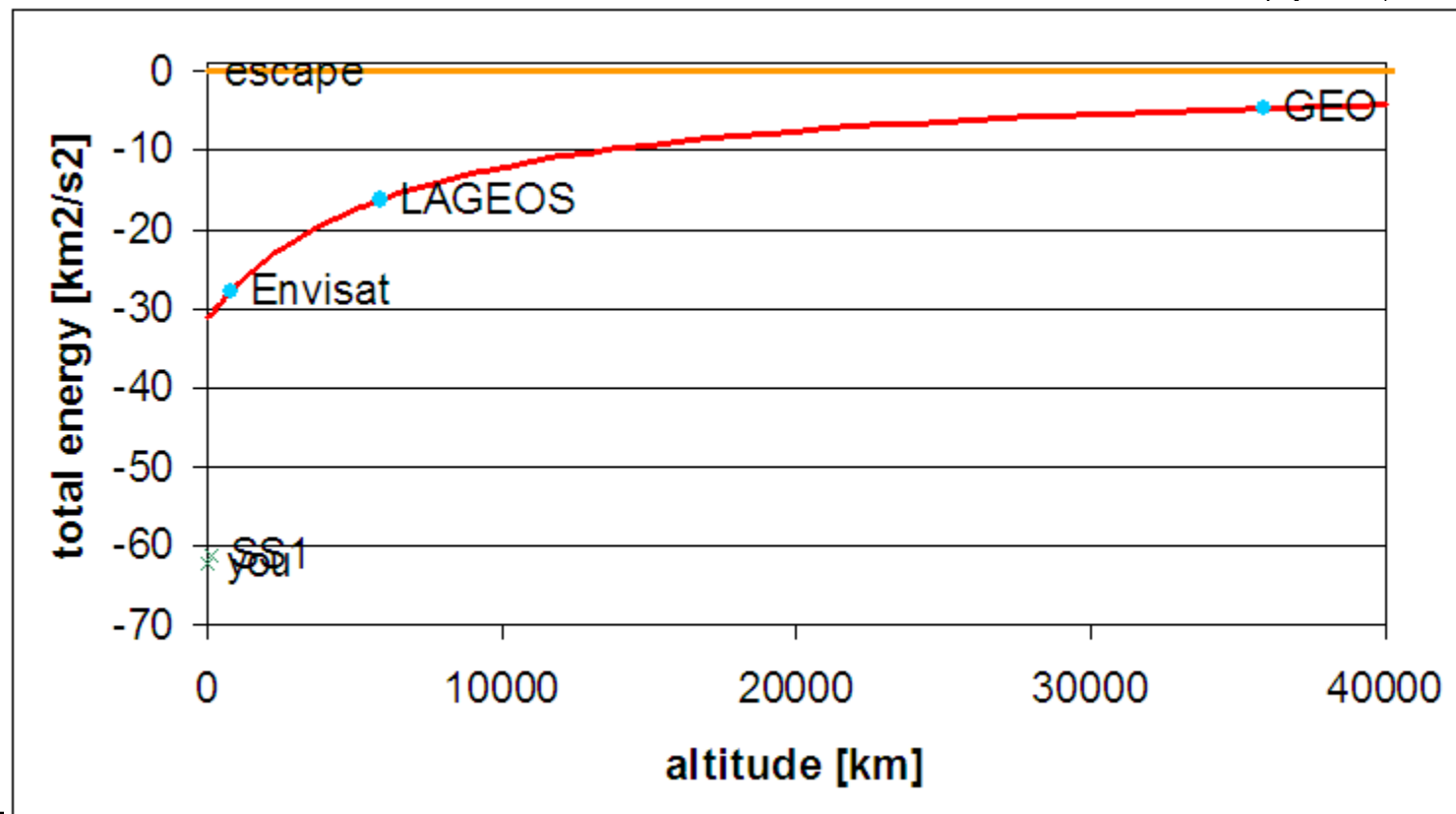
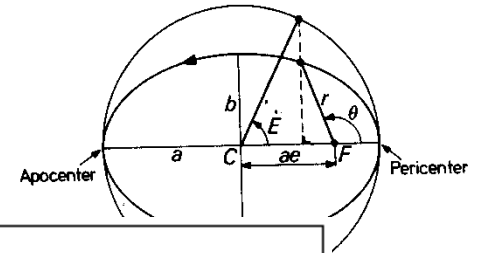
$$E_{i+1} = E_i + \frac{M - E_i + e \sin E_i}{1 - e \cos E_i}$$

$$r = a(1 - e \cos E)$$

satellite	altitude [km]	specific energy [km ² /s ² /kg]
launch platform	0	-62.4
SpaceShipOne	100+ (culmination)	-61.4
imaginary sat	100	-30.8
Envisat	800	-27.8
LAGEOS	5900	-16.2
GEO	35900	-4.7

2-dimensional Kepler orbits: equations (cnt'd)

ellips: $0 \leq e < 1$ $a > 0$ $E_{tot} < 0$



2-dimensional Kepler orbits: equations (cnt'd)

parabola: $e = 1$ $a = \infty$ $E_{tot} = 0$

$$r = \frac{p}{1 + \cos \theta}$$

$$M = \frac{1}{2} \tan \frac{\theta}{2} + \frac{1}{6} \tan^3 \frac{\theta}{2}$$

$$M = n (t - t_0)$$

$$n = \sqrt{\frac{\mu}{p^3}}$$

$$V^2 = V_{esc}^2 = \frac{2\mu}{r}$$

2-dimensional Kepler orbits: equations (cnt'd)

hyperbola: $e > 1$ $a < 0$ $E_{tot} > 0$

$$\tan \frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}} \tanh \frac{F}{2}$$

$$M = e \sinh F - F$$

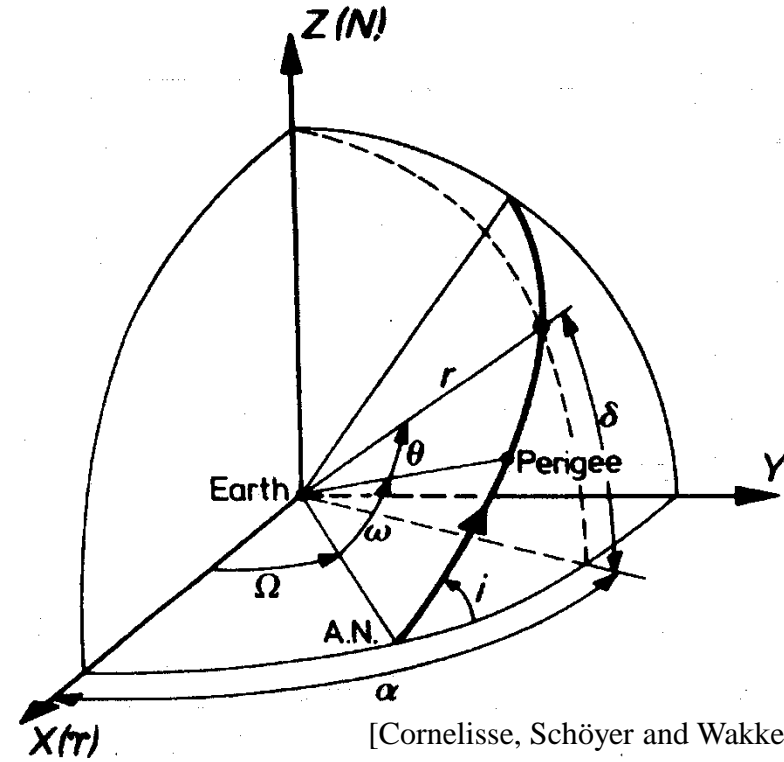
$$M = n (t - t_0)$$

$$n = \sqrt{\frac{\mu}{(-a)^3}}$$

$$r = a (1 - e \cosh F)$$

$$V^2 = V_{esc}^2 + V_{\infty}^2 = \frac{2\mu}{r} + V_{\infty}^2$$

3-dimensional Kepler orbits



i : inclination [deg]

Ω : right ascension of the ascending node,
or longitude of the ascending node [deg]

ω : argument of pericenter [deg]

$u = \omega + \theta$: argument of latitude [deg]

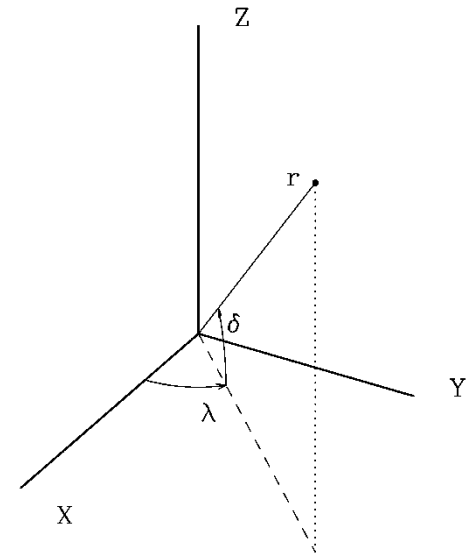
coordinates transformations

1) from spherical (r, λ, δ) (λ in X-Y plane; δ w.r.t. X-Y plane) to cartesian (x, y, z) :

$$x = r \cos \delta \cos \lambda$$

$$y = r \cos \delta \sin \lambda$$

$$z = r \sin \delta$$



2) from cartesian (x, y, z) to spherical (r, λ, δ) :

$$r = \text{sqrt} (x^2 + y^2 + z^2)$$

$$r_{xy} = \text{sqrt} (x^2 + y^2)$$

$$\lambda = \text{atan2} (y / r_{xy} , x / r_{xy})$$

$$\delta = \text{asin} (z / r)$$

Orbital perturbations: introduction

Questions:

- **what forces?**
- **magnitude of forces?**
- **how to model/compute in satellite orbit?**
- **analytically? numerically?**
- **accuracy?**
- **efficiency?**



Inclusion in orbit modeling

Option 1: include directly in equation of motion

$$\frac{d^2 \mathbf{x}}{dt^2} = \mathbf{a}_{main} + \mathbf{a}_{\Delta grav} + \mathbf{a}_{drag} + \mathbf{a}_{solrad} + \mathbf{a}_{3rdbody} + \text{etcetera}$$

Option 2: express as variation of orbital elements

$$\text{e.g. } \frac{da}{dt} = \frac{2a^2}{\sqrt{\mu p}} \left[S e \sin \theta + N \frac{p}{r} \right]$$

(not further treated here; cf. ae4-874 and ae4-878)

Perturbations:

- **Irregularities gravity field**
- **Third body**
- **Atmospheric drag**
- **Solar radiation pressure**
- **Thrust**

Perturbations:

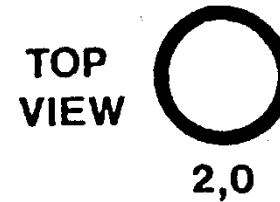
- **Irregularities gravity field**
- **Third body**
- **Atmospheric drag**
- **Solar radiation pressure**
- **Thrust**

Irregularities gravity field

Already treated in lecture hours 15+16

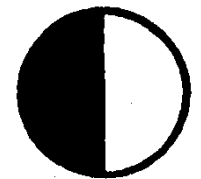


e.g., North-South acceleration due to J_2 :

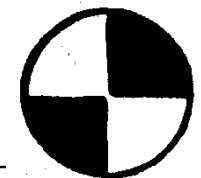


$$acc = -3\mu J_2 R_e^2 r^{-4} \sin(\delta) \cos(\delta)$$

e.g., East-West acceleration due to $J_{2,2}$:



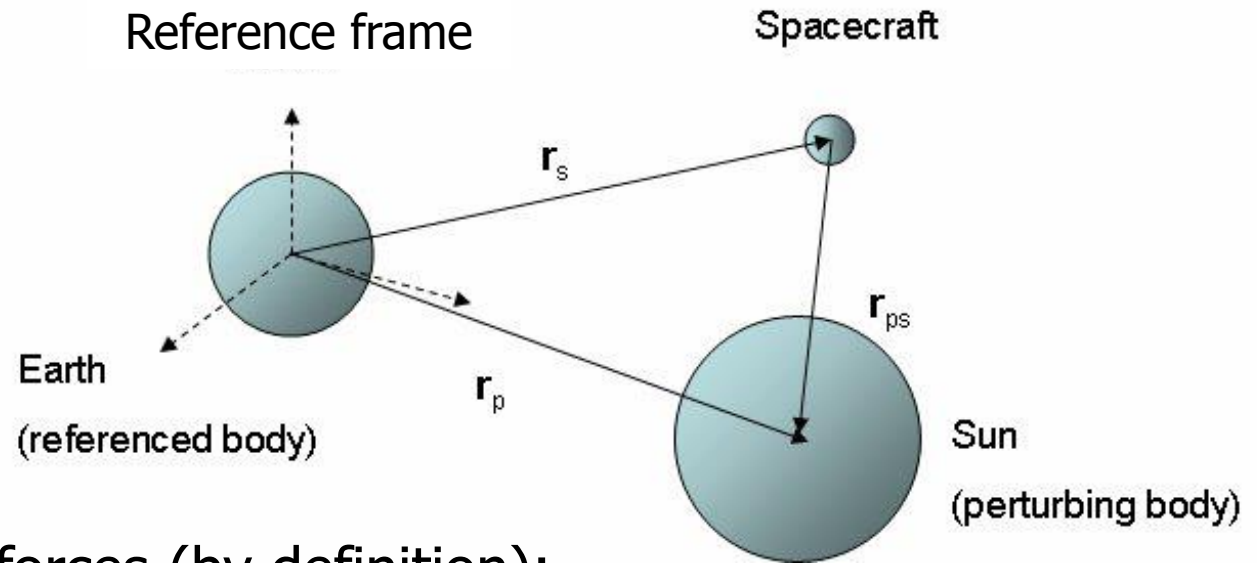
$$acc = -6\mu J_{2,2} R_e^2 r^{-4} 3\cos(\delta) \sin(2(\lambda - \lambda_{2,2}))$$



Perturbations:

- Irregularities gravity field
- **Third body**
- Atmospheric drag
- Solar radiation pressure
- Thrust

Third-body perturbations



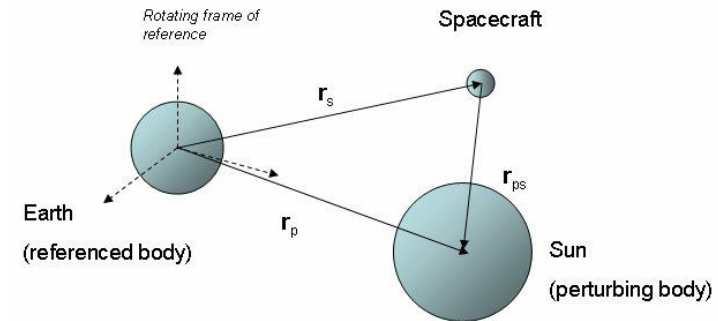
Attractational forces (by definition):

- satellite \leftrightarrow Earth
- satellite \leftrightarrow perturbing body
- Earth \leftrightarrow perturbing body

Third-body perturbations (cnt'd)

Attractational forces (practice):

- Earth attracts satellite
- perturbing body attracts satellite
- perturbing body attracts Earth
- net effect counts



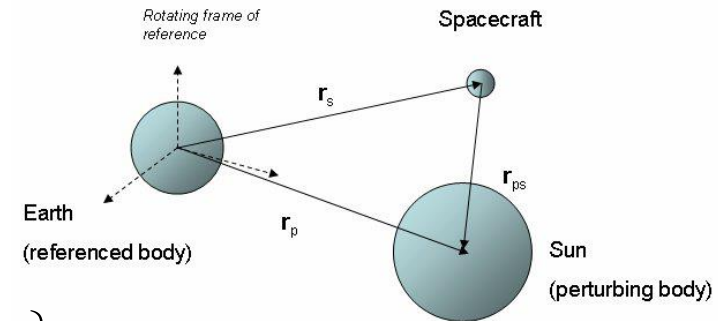
$$M_{\text{sat}} \ll M_E, M_S$$

$$\rightarrow \ddot{\mathbf{r}}_s = -G \frac{M_{\text{main}}}{r_s^3} \mathbf{r}_s + G M_p \left(\frac{\mathbf{r}_{ps}}{r_{ps}^3} - \frac{\mathbf{r}_p}{r_p^3} \right)$$

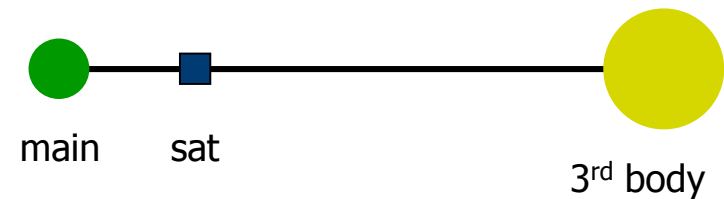
Third-body perturbations (cnt'd)

perturbing acceleration \mathbf{a}_p :

$$\mathbf{a}_p = \mu_p \left\{ \frac{\mathbf{r}_{ps}}{\|\mathbf{r}_{ps}\|^3} - \frac{\mathbf{r}_p}{\|\mathbf{r}_p\|^3} \right\}$$

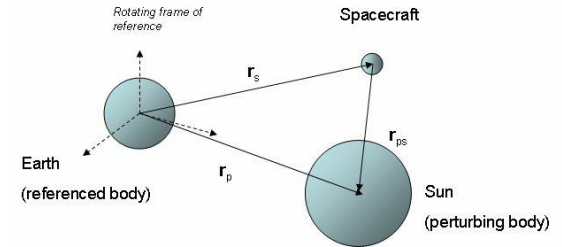


maximum when on straight line:



$$\left(\frac{a_p}{a_{main}} \right)_{\max} = 2 \frac{m_p}{m_{main}} \left(\frac{r_s}{r_p} \right)^3$$

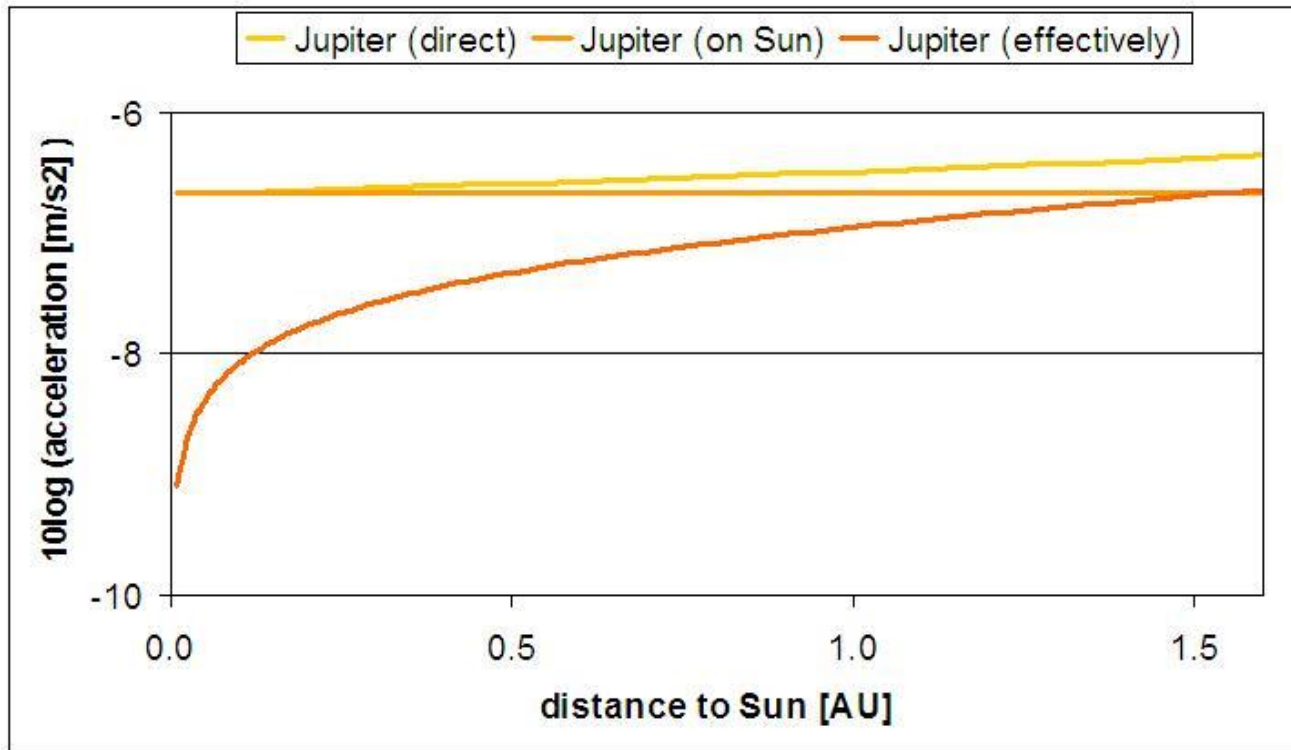
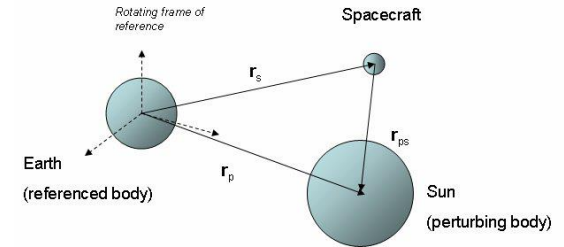
Third-body perturbations (cnt'd)



2 situations:

1. heliocentric (*i.e.*, orbits around Sun)
2. planetocentric (*i.e.*, orbits around Earth)

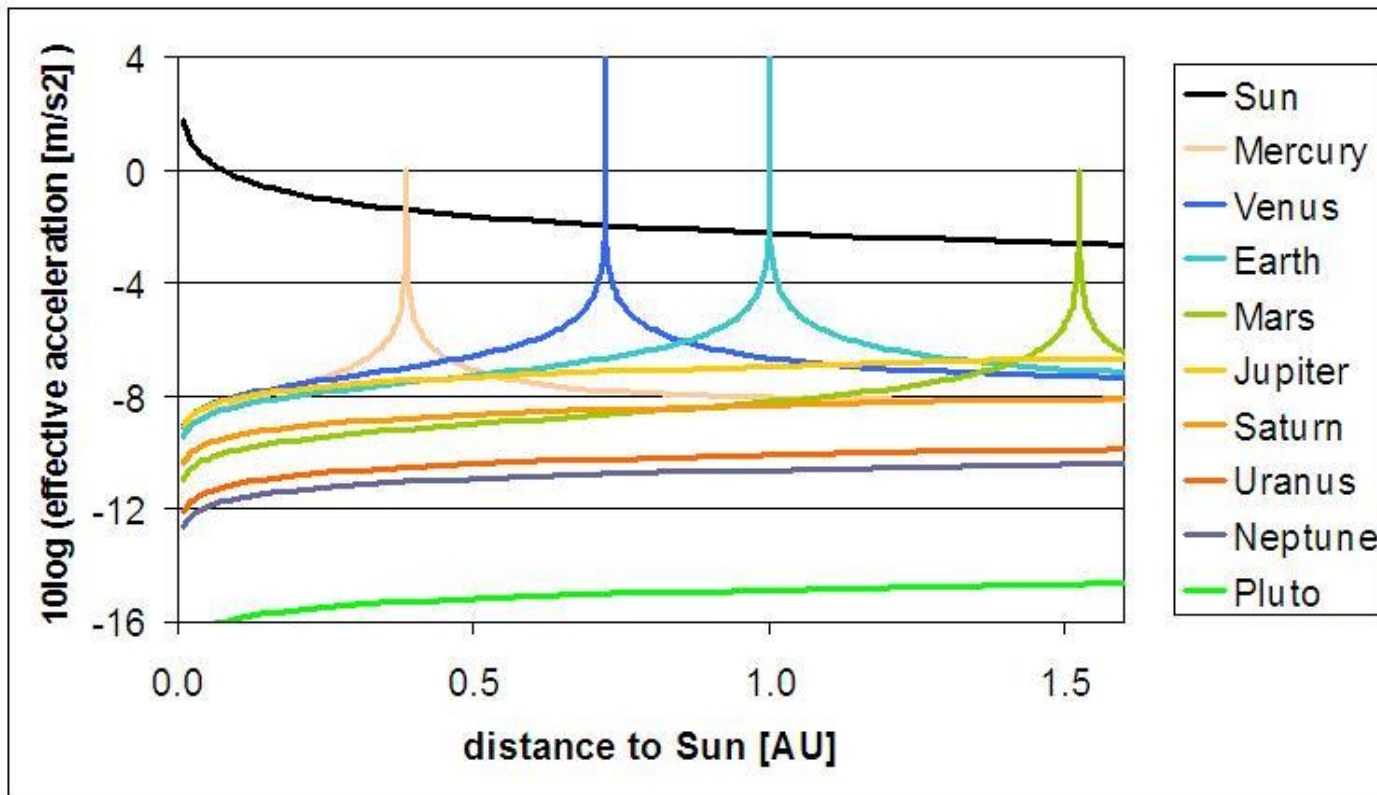
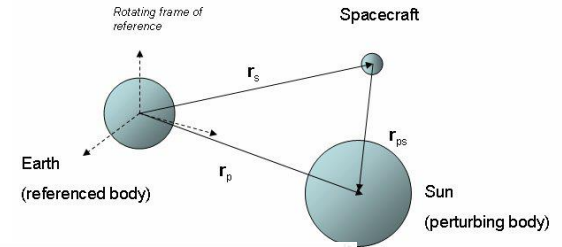
Third-body perturbations (cnt'd)



Conclusions:

- influence increases with distance to Sun
- perturbation $O(10^{-7}) \text{ m/s}^2$

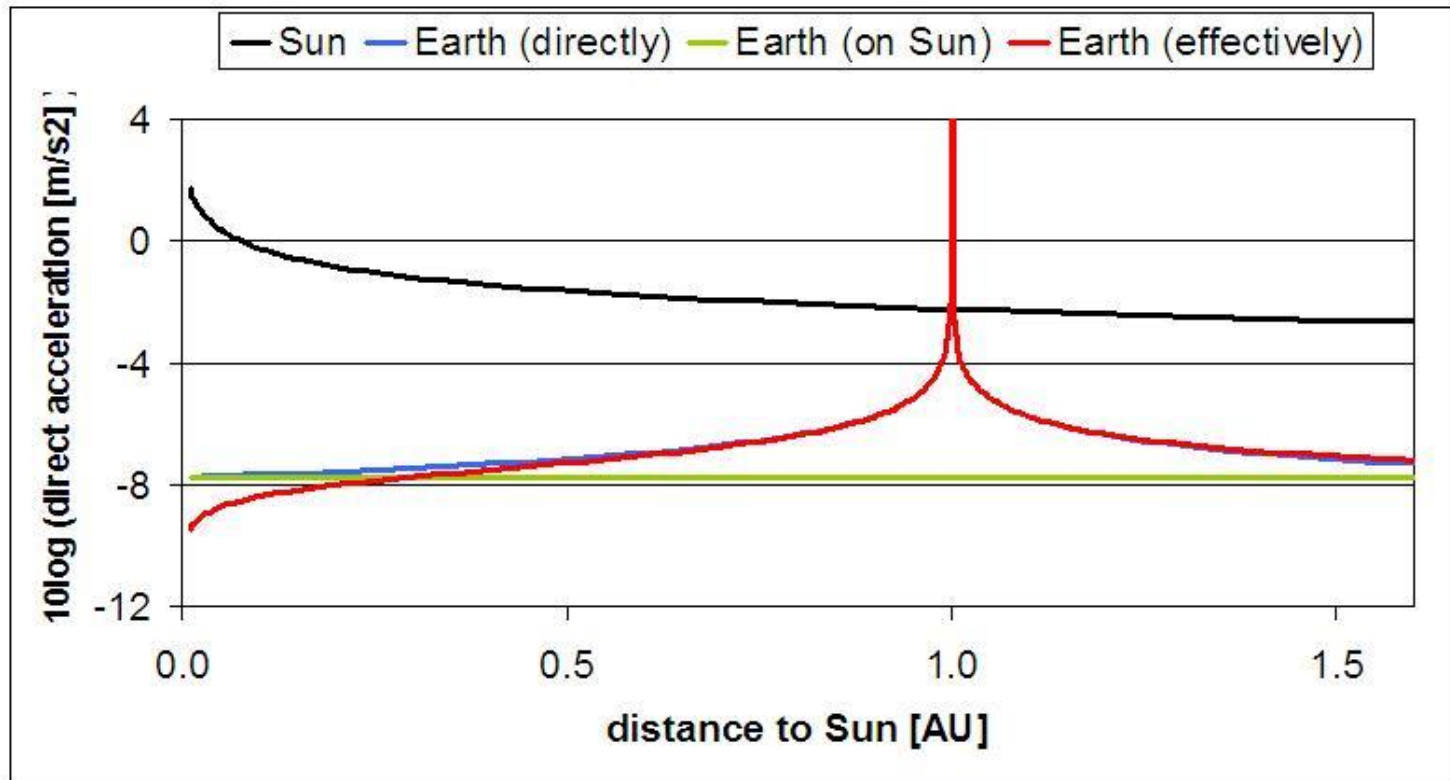
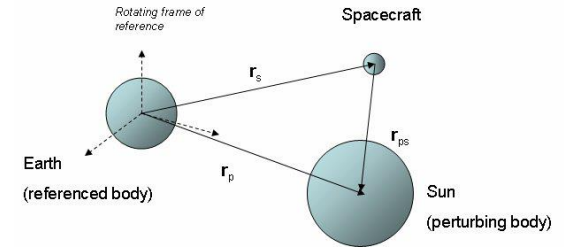
Third-body perturbations (cnt'd)



Conclusions:

- influence Sun decreases with distance to Sun
- influence planets increases with distance to Sun
- acceleration from Sun $O(10^{-2})$ m/s²; dominant (central body!!)
- near planet: 3rd body becomes dominant

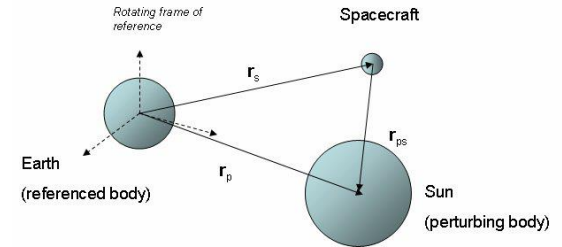
Third-body perturbations (cnt'd)



Conclusions:

- in Solar System: Sun dominant
- near Earth: Earth itself dominant
- central body vs. 3rd body perturbation ??

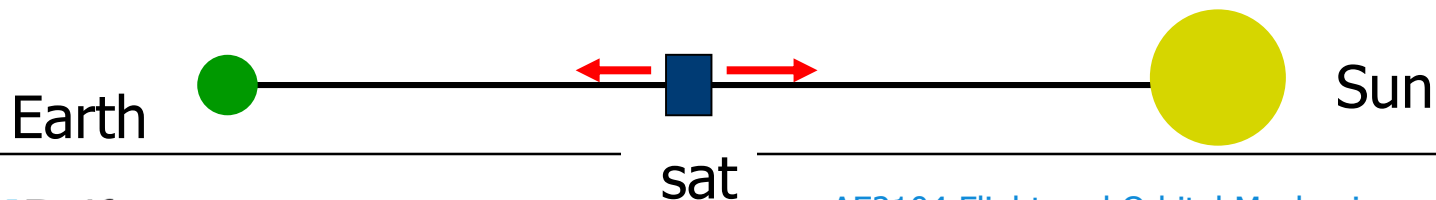
Third-body perturbations (cnt'd)



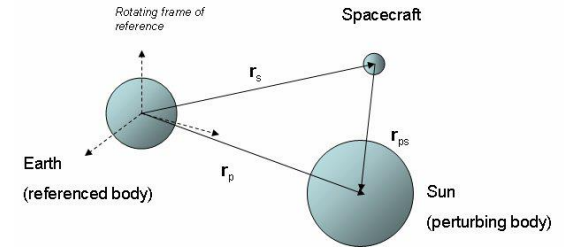
Sphere of Influence:

- area around planet where gravity from planet is dominant (compared with gravity of other celestial bodies)
- 3-dimensional shape
- boundary
- 1st-order approximation: sphere with constant radius
- definition for determination location:

relative acceleration w.r.t. system 1 = relative acceleration w.r.t. system 2



Third-body perturbations (cnt'd)



$$\frac{acc_{Sun,3rd}}{acc_{Earth,main}} = \frac{acc_{Earth,3rd}}{acc_{Sun,main}}$$

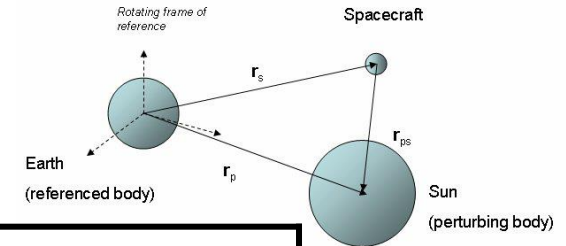
without derivation :

$$r_{SoI} = r_{3rd} \left(\frac{M_{main}}{M_{3rd}} \right)^{0.4}$$

example : Earth main body, Sun 3rd body :

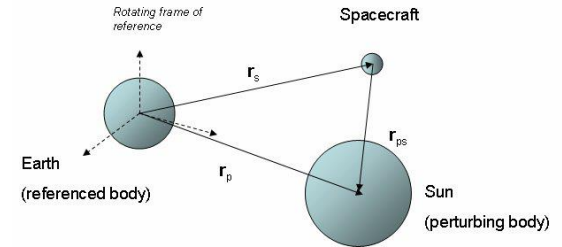
$$r_{SoI,Earth} = dist_{Earth-Sun} \left(\frac{M_{Earth}}{M_{Sun}} \right)^{0.4} \approx 930,000 \text{ km}$$

Third-body perturbations (cnt'd)



planet	Sphere of Influence		
	[km]	[% AU]	[% distance planet-Sun]
Mercury	1.1×10^5	0.08	0.2
Venus	6.2×10^5	0.4	0.6
Earth	9.3×10^5	0.6	0.6
Mars	5.8×10^5	0.4	0.3
Jupiter	4.8×10^7	32.2	6.2
Saturn	5.5×10^7	36.5	3.8
Uranus	5.2×10^7	34.6	1.8
Neptune	8.7×10^7	57.9	1.9
Pluto	3.2×10^6	2.1	0.1

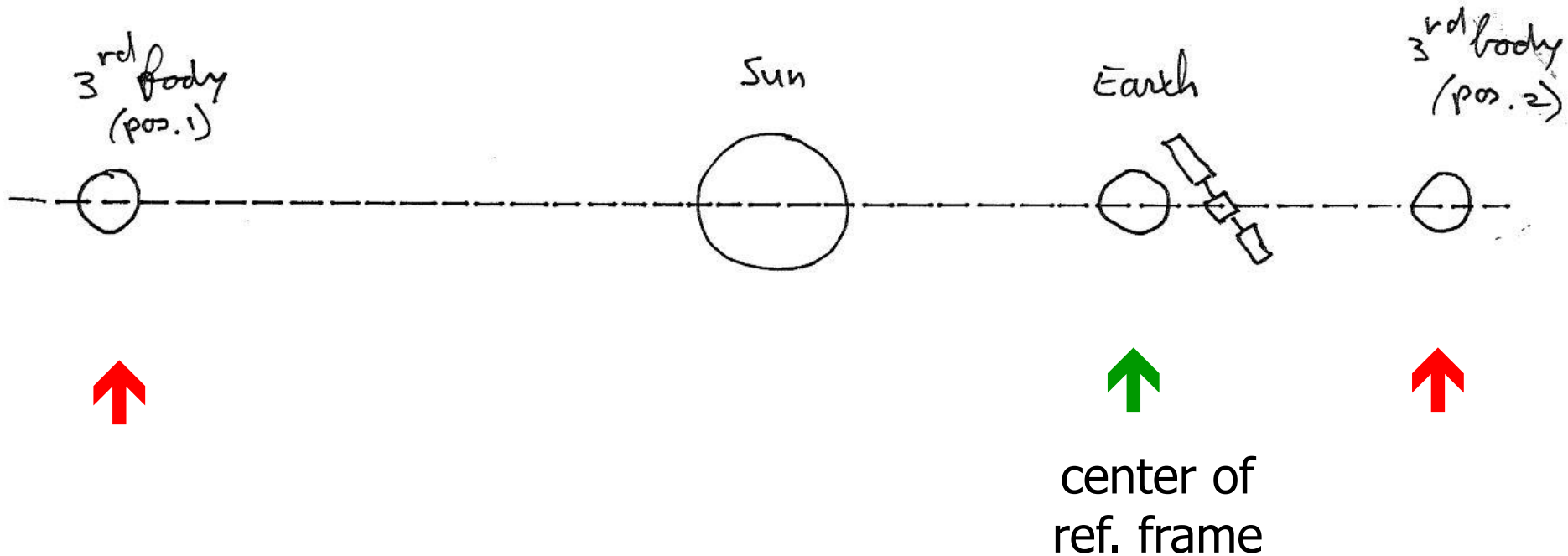
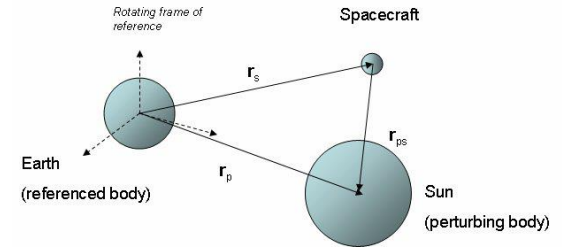
Third-body perturbations (cnt'd)



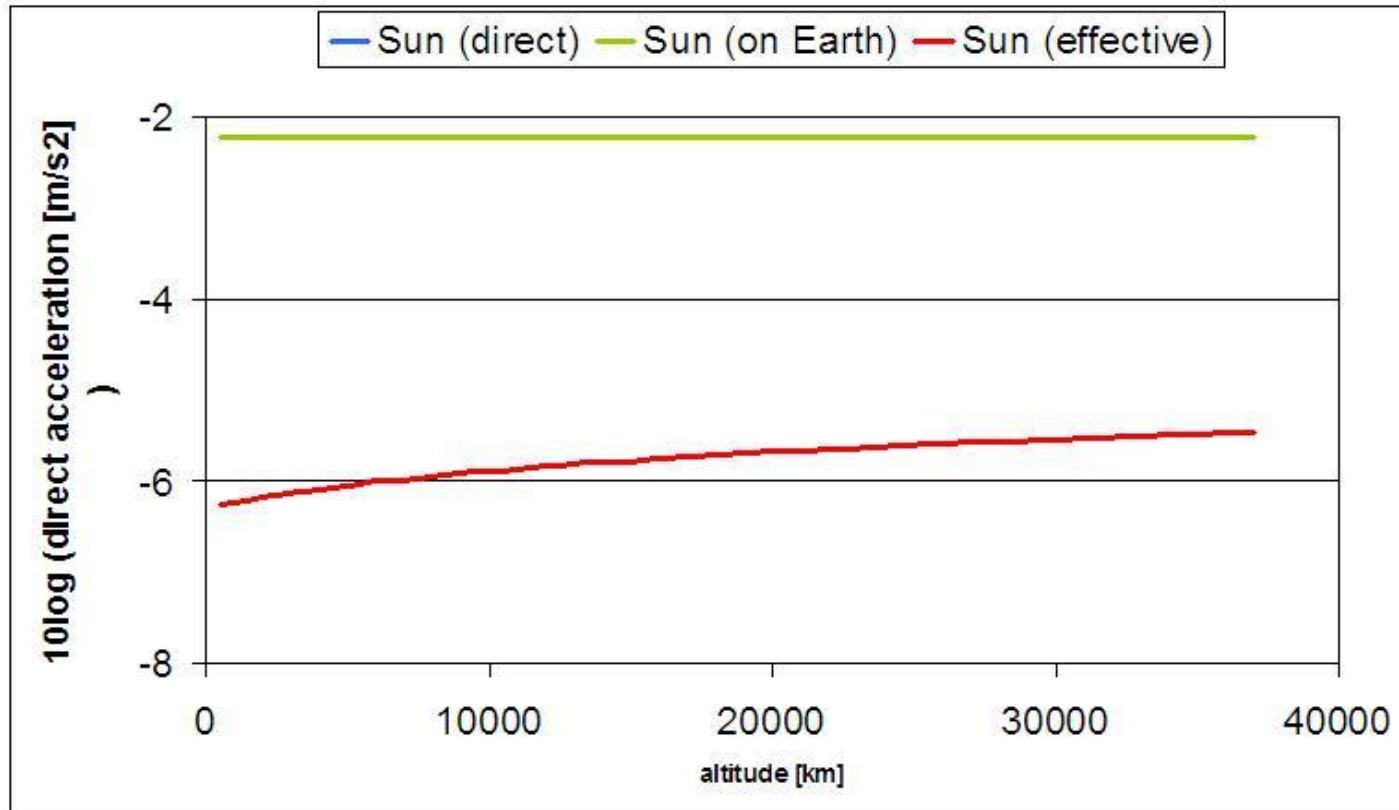
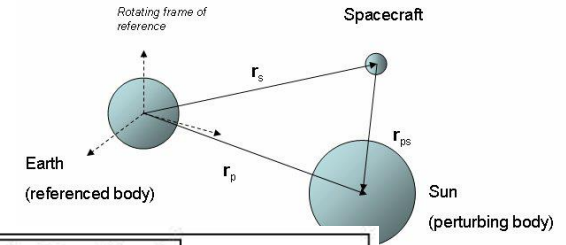
2 situations:

1. heliocentric (*i.e.*, orbits around Sun)
2. planetocentric (*i.e.*, orbits around Earth)

Third-body perturbations (cnt'd)



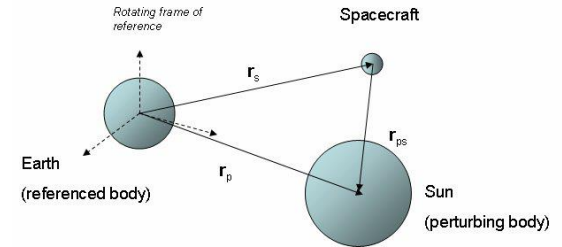
Third-body perturbations (cnt'd)



Conclusions:

- influence of Sun as 3rd body increases with distance from Earth
- effective 3rd body acceleration by Sun $O(10^{-6})$ m/s²

Third-body perturbations (cnt'd)



Question 1:

- a) Compute the dimension of the Sphere of Influence of the Earth (when the Sun is considered as the perturbing body). The SoI is given by the following general equation:

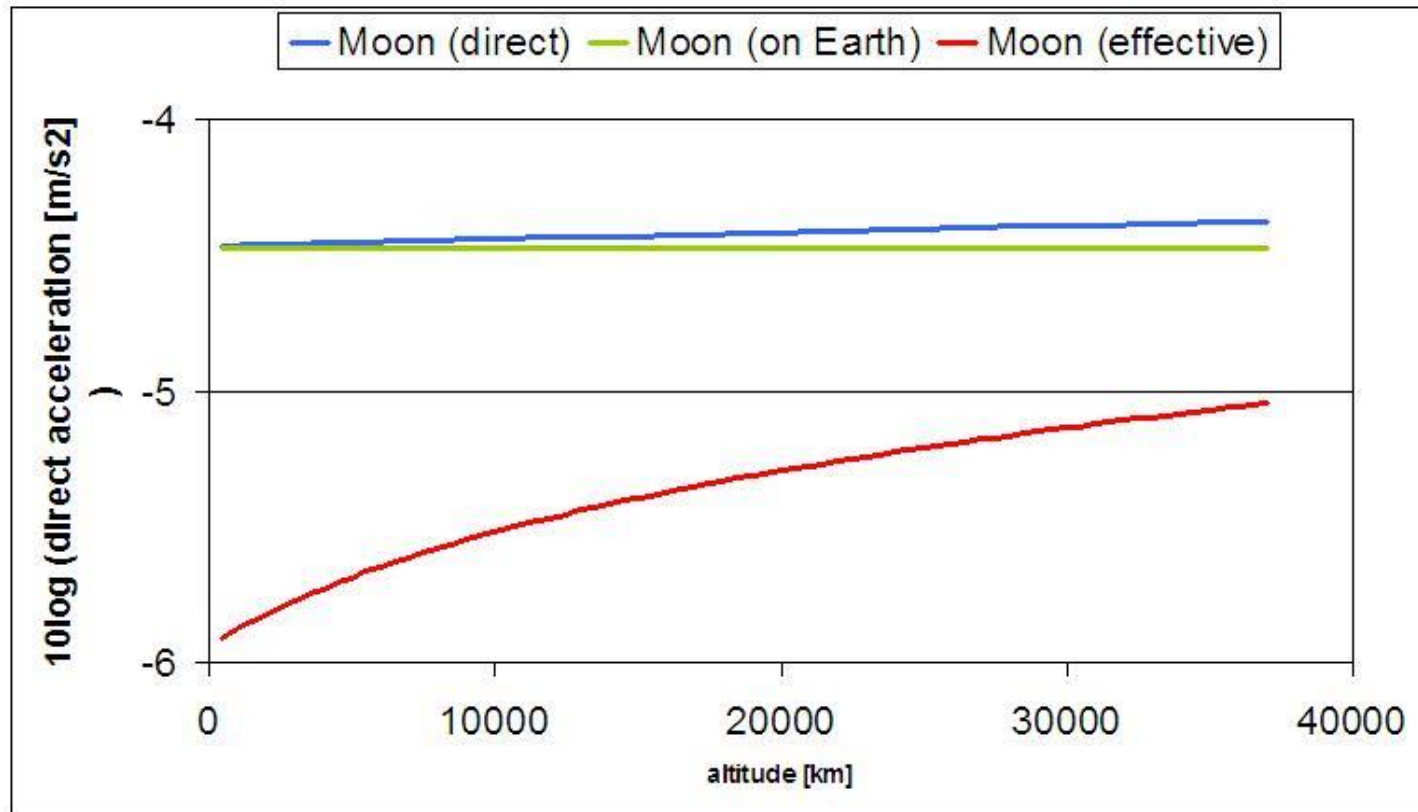
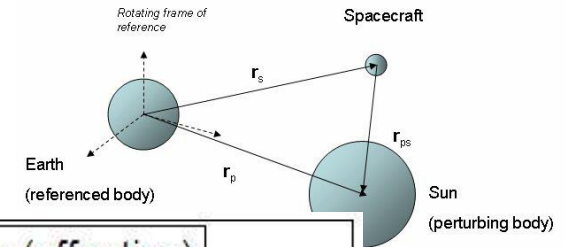
$$r_{SoI} = r_{3rd} \left(\frac{M_{main}}{M_{3rd}} \right)^{0.4}$$

- b) What is the value of the radial attraction exerted by the Earth at this distance? (if you were unable to make the question a, use a value of 1×10^6 km for this position).
- c) What is the effective gravitational acceleration by the Sun at this position? Assume that Earth, Sun and satellite are on a straight line.
- d) What is the relative perturbation of the solar attraction, compared to that of the main attraction of the Earth?

Data: $1 \text{ AU} = 149.6 \times 10^6 \text{ km}$, $m_{\text{Sun}} = 2.0 \times 10^{30} \text{ kg}$, $m_{\text{Earth}} = 6.0 \times 10^{24} \text{ kg}$, $\mu_{\text{Sun}} = 1.3271 \times 10^{11} \text{ km}^3/\text{s}^2$, $\mu_{\text{Earth}} = 398600 \text{ km}^3/\text{s}^2$.

Answers: see footnotes below **(BUT TRY YOURSELF FIRST!!)**

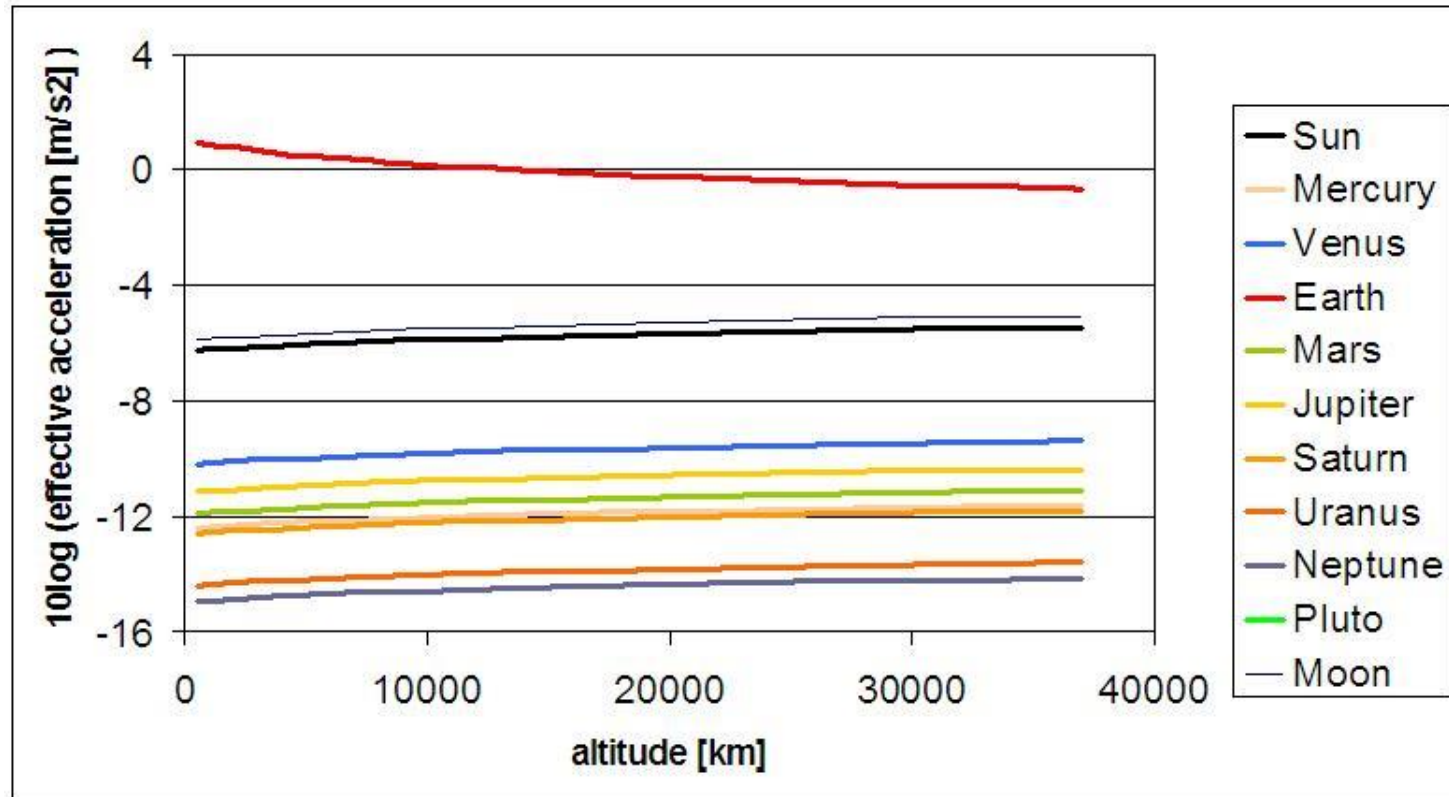
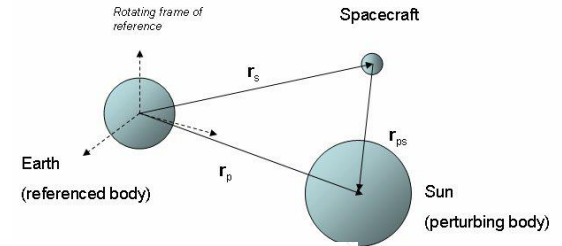
Third-body perturbations (cnt'd)



Conclusions:

- influence of Moon as 3rd body increases with distance from Earth
- effective 3rd body acceleration by Moon $O(10^{-5}) \text{ m/s}^2$ at GEO

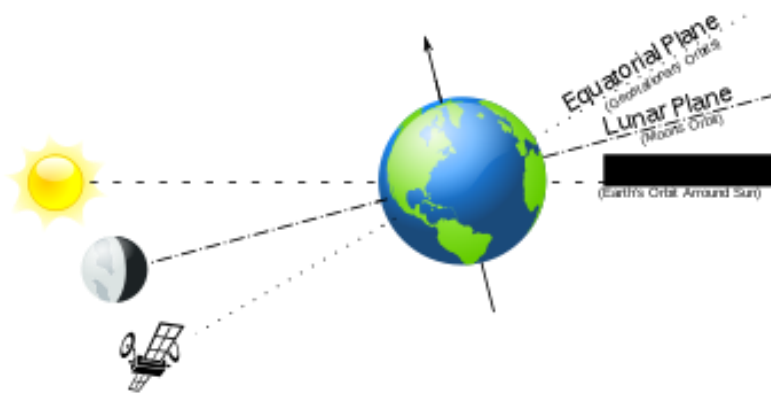
Third-body perturbations (cnt'd)



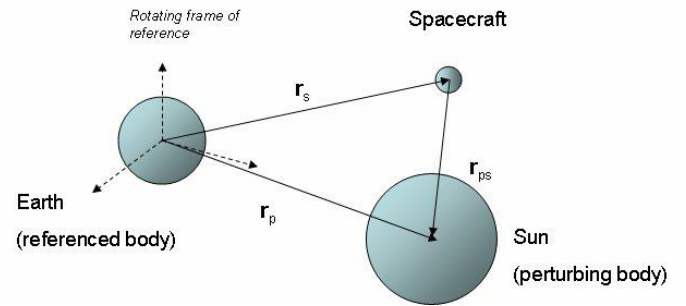
Conclusions:

- Earth dominant (central body; within SoI)
- Moon most important 3rd body; Sun directly after
- effect of planets about 4 orders of magnitude smaller

Third-body perturbations (cnt'd)

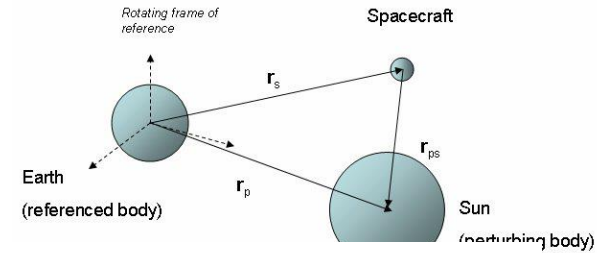


[Wikimedia, 2010]



celestial body	ΔV -budget for GEO sat [m/s/yr]
<i>Moon</i>	36.93
<i>Sun</i>	14.45

Third-body perturbations (cnt'd)



Question 2:

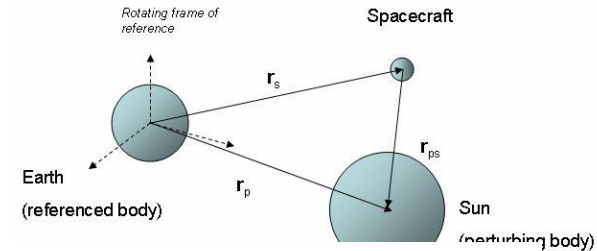
Consider a hypothetical planet X with mass 5×10^{25} kg, orbiting the Sun in a circular orbit with radius 3 AU. The orbital plane coincides with the ecliptic (*i.e.*, the orbital plane of the Earth).

- Make a sketch of the situation when the gravitational attraction of this planet X on satellites around the Earth is largest.
- Idem for the case when this would be smallest.
- Compute the maximum and minimum perturbing acceleration due to this planet X, on a geostationary satellite (radius orbit is 42200 km).

Data: $G = 6.673 \times 10^{-11} \text{ m}^3/\text{kg}/\text{s}^2$; $\mu_{\text{Earth}} = 398600 \text{ km}^3/\text{s}^2$; $1 \text{ AU} = 149.6 \times 10^6 \text{ km}$

Answers: see footnotes below **(BUT TRY YOURSELF FIRST!!)**

Third-body perturbations (cnt'd)



Question 3:

The treatment of the motion of a satellite is driven by the fact whether the vehicle is inside the Sphere of Influence (SoI) or not.

- Describe the concept of the SoI, and give its mathematical (underlying) definition.
- The dimension of the SoI can be approximated by the equation given below. Consider the Earth-Moon system: $\mu_{\text{Earth}} = 398600 \text{ km}^3/\text{s}^2$, $\mu_{\text{Moon}} = 4903 \text{ km}^3/\text{s}^2$, average distance Earth-Moon = 384,000 km. What is the radius of the SoI of the Moon?

$$r_{\text{SoI}} = r_{3rd} \left(\frac{M_{\text{main}}}{M_{3rd}} \right)^{0.4}$$

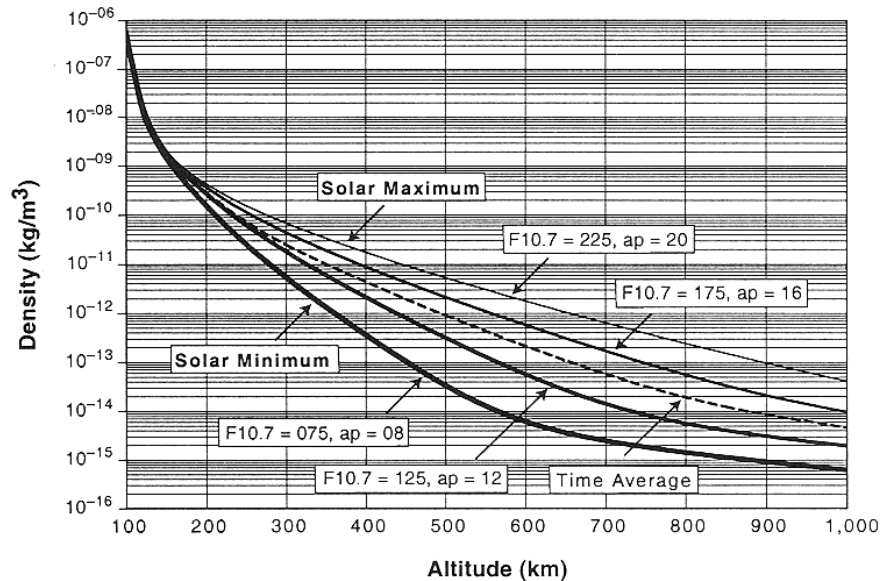
- Suppose that the Earth and Moon have equal masses. What would now be the radius of the SoI? What would it be from a physical point of view? Discuss the results.

Perturbations:

- **Irregularities gravity field**
- **Third body**
- **Atmospheric drag**
- **Solar radiation pressure**
- **Thrust**

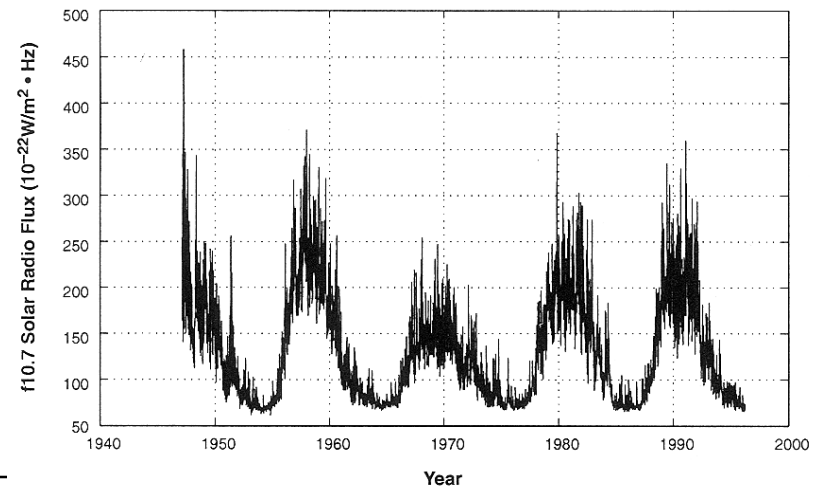
Atmosphere

$$\mathbf{a}_{\text{drag}} = -\frac{C_D S}{m} \frac{1}{2} \rho V^2 \frac{\mathbf{V}}{V} ; \quad \rho = \rho_0 \exp(-\Delta h / H)$$



[Wertz, 2009]

[Wertz, 2009]



Atmosphere (cnt'd)

after
[Wertz, 2009]:

Altitude [km]	Atmospheric density [kg/m ³]	
	minimum	maximum
200	1.8×10^{-10}	3.5×10^{-10}
300	8.2×10^{-12}	4.0×10^{-11}
400	7.3×10^{-13}	7.6×10^{-12}
500	9.0×10^{-14}	1.8×10^{-12}
600	1.7×10^{-14}	4.9×10^{-13}
700	5.7×10^{-15}	1.5×10^{-13}
800	3.0×10^{-15}	4.4×10^{-14}
900	1.8×10^{-15}	1.9×10^{-14}
1000	1.2×10^{-15}	8.8×10^{-15}
1250	4.7×10^{-16}	2.6×10^{-15}
1500	2.3×10^{-16}	1.2×10^{-15}

Atmosphere (cnt'd)

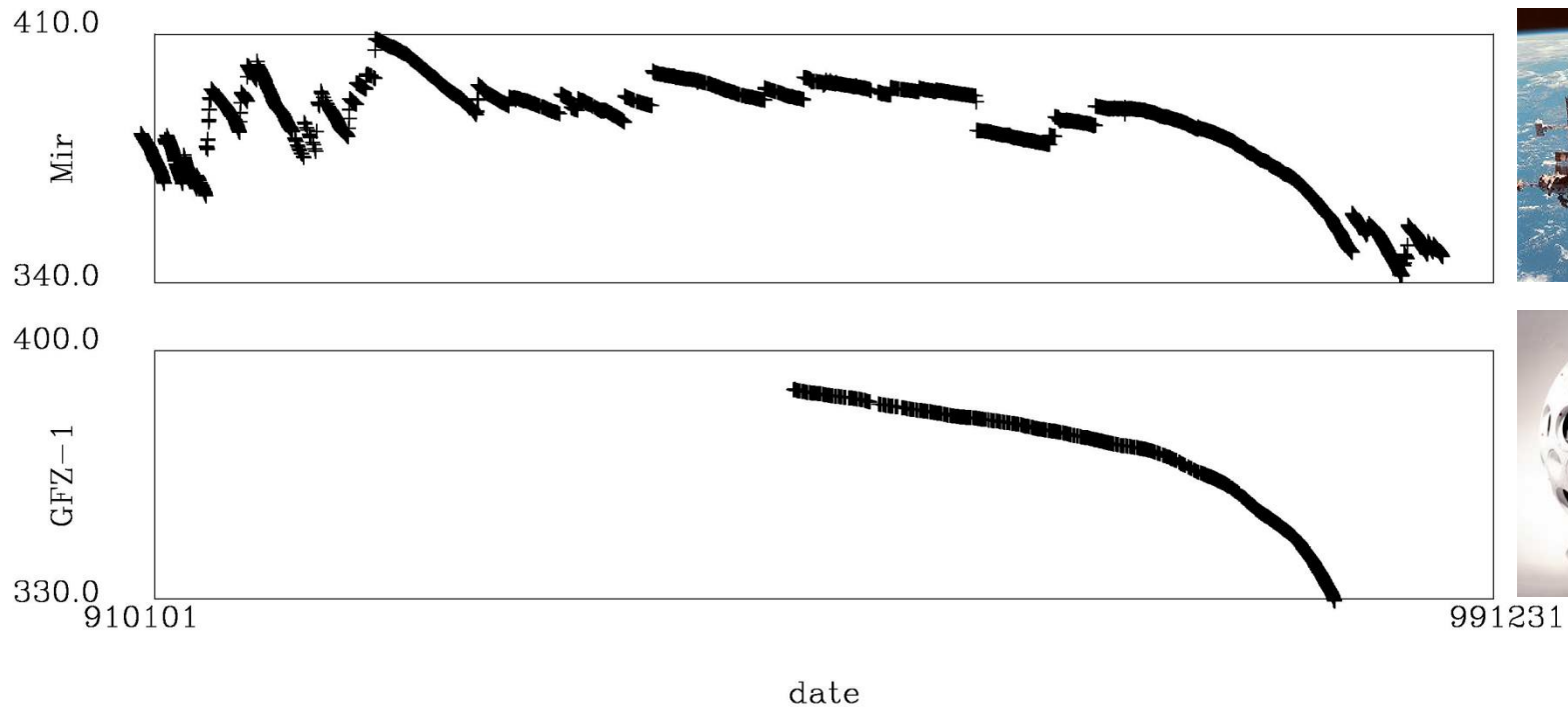
[Wertz, 2009]:

Satellite	Mass (kg)	Shape	Max. XA (m ²)	Min. XA (m ²)	Max. Drag Coef.	Min. Drag Coef.	Max. Ballistic Coef. (kg/m ²)	Min. Ballistic Coef. (kg/m ²)	Type of Mission
Oscar-1	5	box	0.075	0.0584	4	2	42.8	16.7	Comm.
Intercos.-16	550	cylind.	2.7	3.16	2.67	2.1	82.9	76.3	Scientific
Viking	277	octag.	2.25	0.833	4	2.6	128	30.8	Scientific
Explorer-11	37	octag.	0.18	0.07	2.83	2.6	203	72.6	Astronomy
Explorer-17	188.2	sphere	0.621	0.621	2	2	152	152	Scientific
Sp. Teles.	11,000	cylind.*	112	14.3	3.33	4	192	29.5	Astronomy
OSO-7	634	9-sided	1.05	0.5	3.67	2.9	437	165	Solar Physics
OSO-8	1,063	cylind.*	5.99	1.81	3.76	4	147	47.2	Solar Physics
Pegasus-3	10,500	cylind.*	264	14.5	3.3	4	181	12.1	Scientific
Landsat-1	891	cylind.*	10.4	1.81	3.4	4	123	25.2	Rem. Sens.
ERS-1	2,160	box*	45.1	4	4	4	135	12.0	Rem. Sens.
LDEF-1	9,695	12-face	39	14.3	2.67	4	169	93.1	Environment
HEAO-2	3,150	hexag.	13.9	4.52	2.83	4	174	80.1	Astronomy
Vanguard-2	9.39	sphere	0.2	0.2	2	2	23.5	23.5	Scientific
SkyLab	76,136	cylind.*	462	46.4	3.5	4	410	47.1	Scientific
Echo-1	75.3	sphere	731	731	2	2	0.515	0.515	Comm.
Extrema							437	0.515	

*With solar arrays

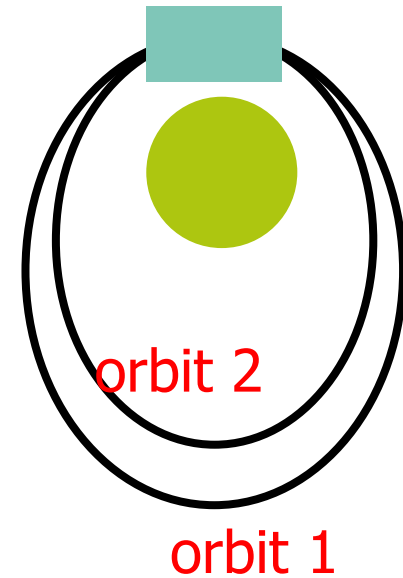
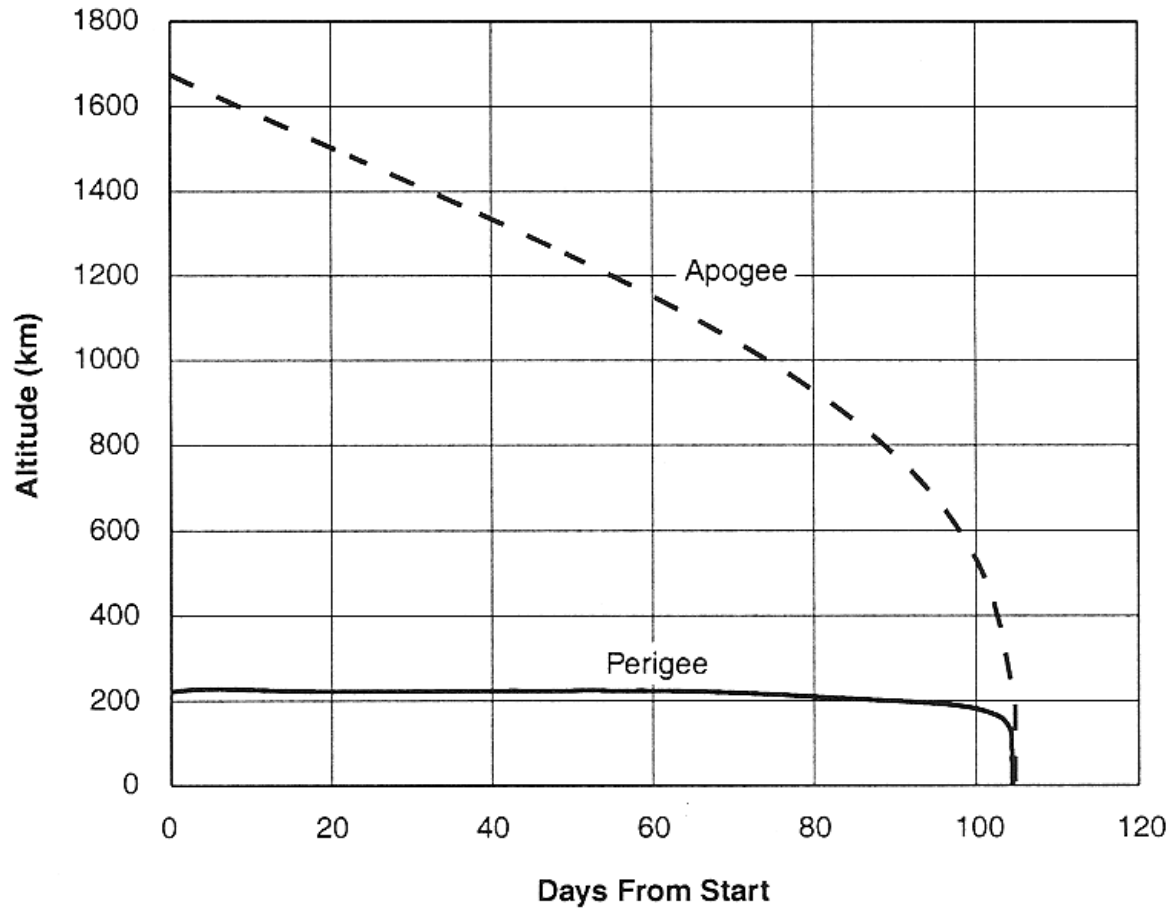
Atmosphere (cnt'd)

altitudes of Mir and GFZ-1:



Atmosphere (cnt'd)

[Wertz, 2009]:



Atmosphere (cnt'd)

circular orbits:

$$\Delta a_{2\pi} = -2\pi (C_D A/m) \rho a^2 \quad [\text{m}]$$

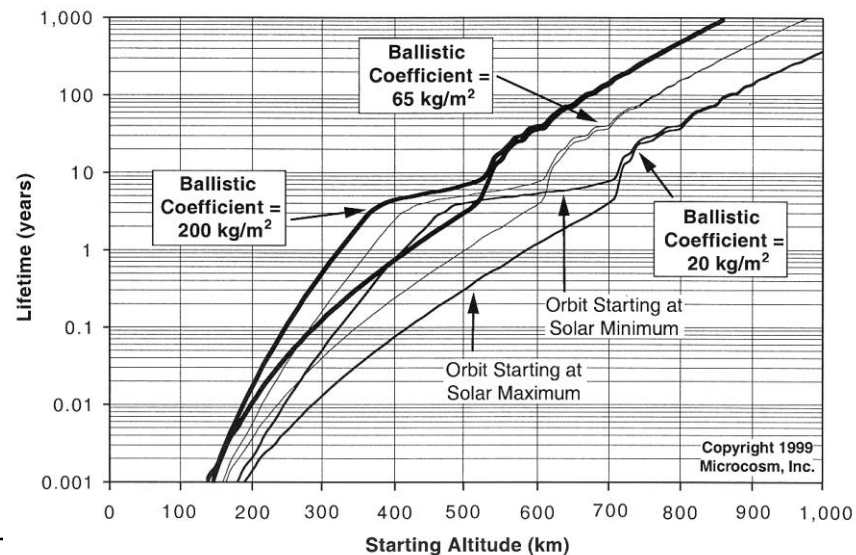
$$\Delta T_{2\pi} = -6\pi^2 (C_D A/m) \rho a^2 / V \quad [\text{s}]$$

$$\Delta V_{2\pi} = \pi (C_D A/m) \rho a V \quad [\text{m/s}]$$

$$\Delta e_{2\pi} = 0 \quad [-]$$

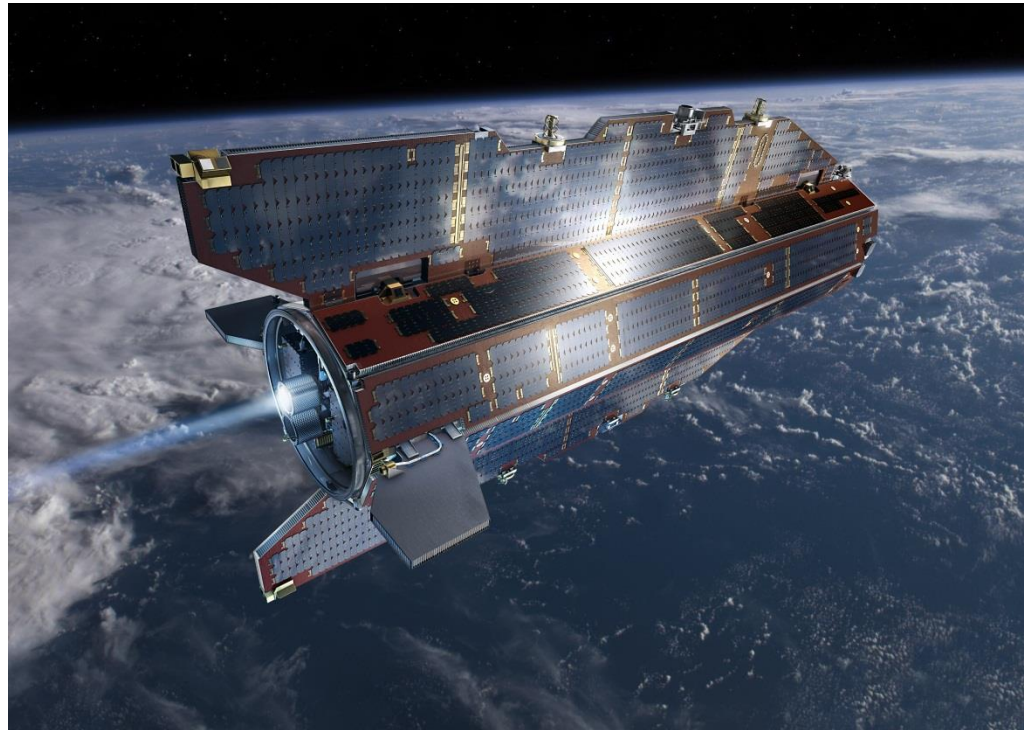
$$L = -H / \Delta a_{2\pi} \quad [\text{rev}]$$

[Wertz, 2009]:



Atmosphere (cnt'd)

GOCE:



[ESA, 2010]

- **launch date: March 17, 2009**
- **altitude: 250 km**
- **phase solar cycle?**
- **.....**

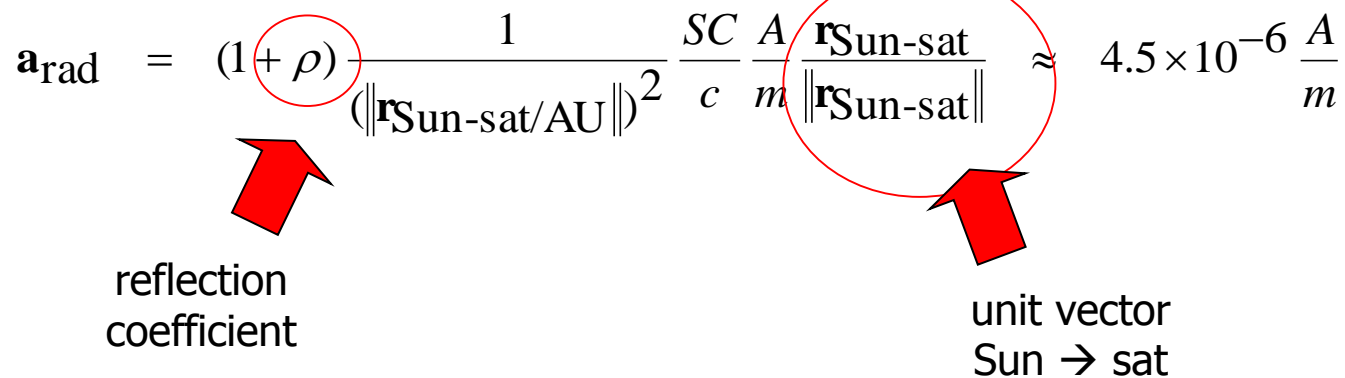
Perturbations:

- **Irregularities gravity field**
- **Third body**
- **Atmospheric drag**
- **Solar radiation pressure**
- **Thrust**

Solar radiation pressure

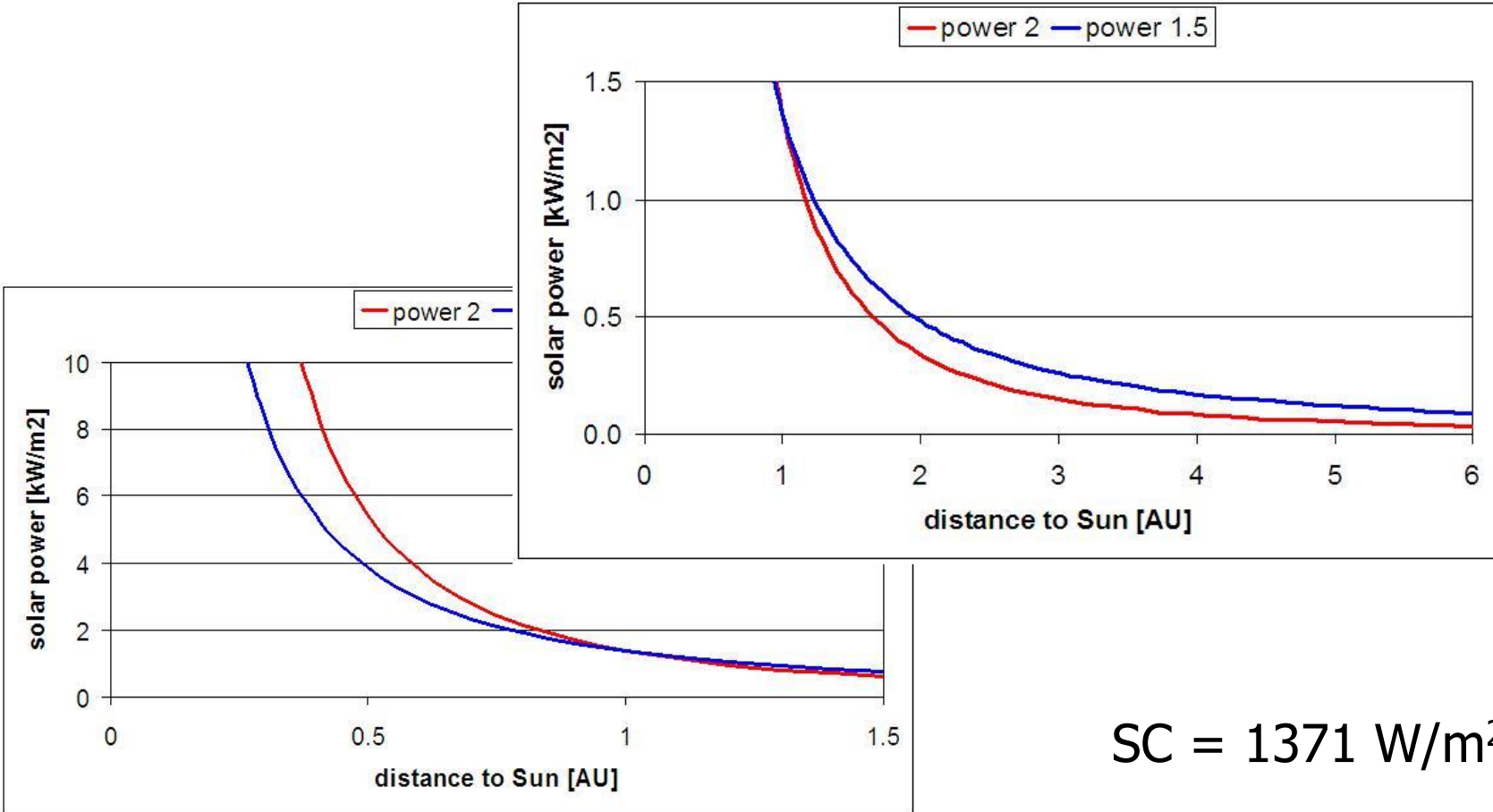
- amount of energy emitted by Sun at 1 AU distance: $\approx 1371 \text{ W/m}^2$
- value hardly dependent on solar activity \rightarrow "Solar Constant" (SC)
- Solar radiation pressure [N/m^2]: SC / c
- energy reduces with distance w.r.t. Sun: $\text{energy}(r) = \text{SC}/r^2$

$$\mathbf{a}_{\text{rad}} = (1 + \rho) \frac{1}{(\|\mathbf{r}_{\text{Sun-sat}}/\text{AU}\|)^2} \frac{\text{SC}}{c} \frac{A}{m} \frac{\mathbf{r}_{\text{Sun-sat}}}{\|\mathbf{r}_{\text{Sun-sat}}\|} \approx 4.5 \times 10^{-6} \frac{\text{A}}{\text{m}}$$



reflection coefficient unit vector
Sun \rightarrow sat

Solar radiation pressure (cnt'd)



$$SC = 1371 \text{ W/m}^2$$

Solar radiation pressure (cnt'd)



ISS (100-1000 m²; 500 ton)

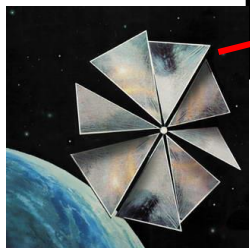


ENVISAT

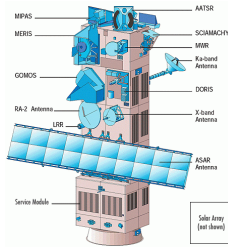
LAGEOS

Echo-1

solar sail (100x100 m; 300 kg)



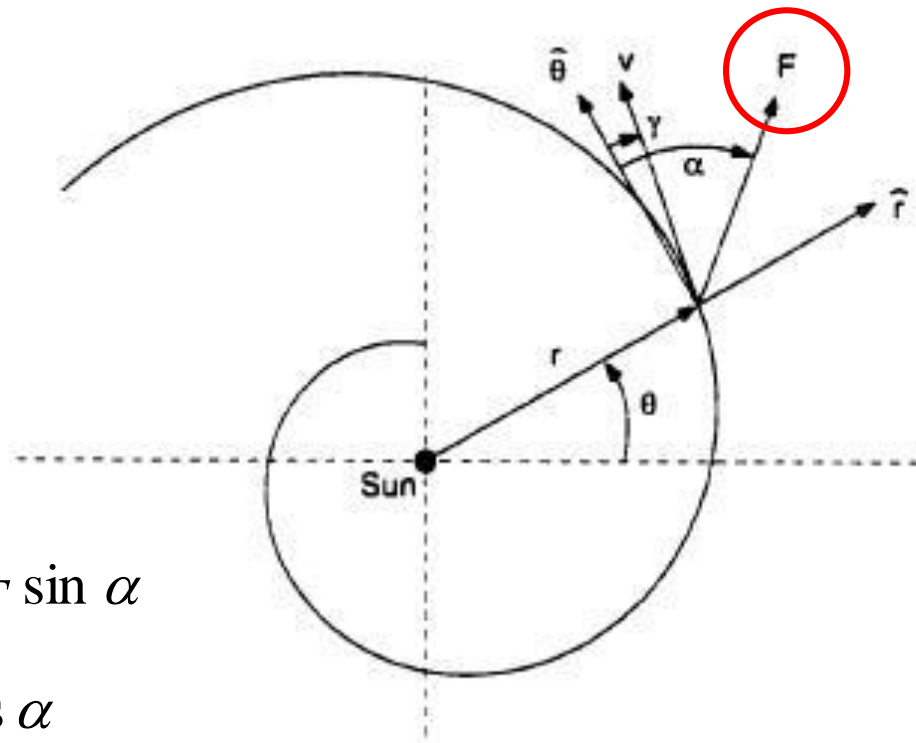
satellite	a_{rad} [m/s ²]
<i>ISS (100-1000 m²; 500 ton)</i>	$9 - 90 \times 10^{-10}$
<i>ENVISAT</i>	$8.3 - 83 \times 10^{-9}$
<i>LAGEOS</i>	3×10^{-9}
<i>Echo-1</i>	4.4×10^{-5}
<i>solar sail (100x100 m; 300 kg)</i>	1.5×10^{-4}



Perturbations:

- **Irregularities gravity field**
- **Third body**
- **Atmospheric drag**
- **Solar radiation pressure**
- **Thrust**

Thrust



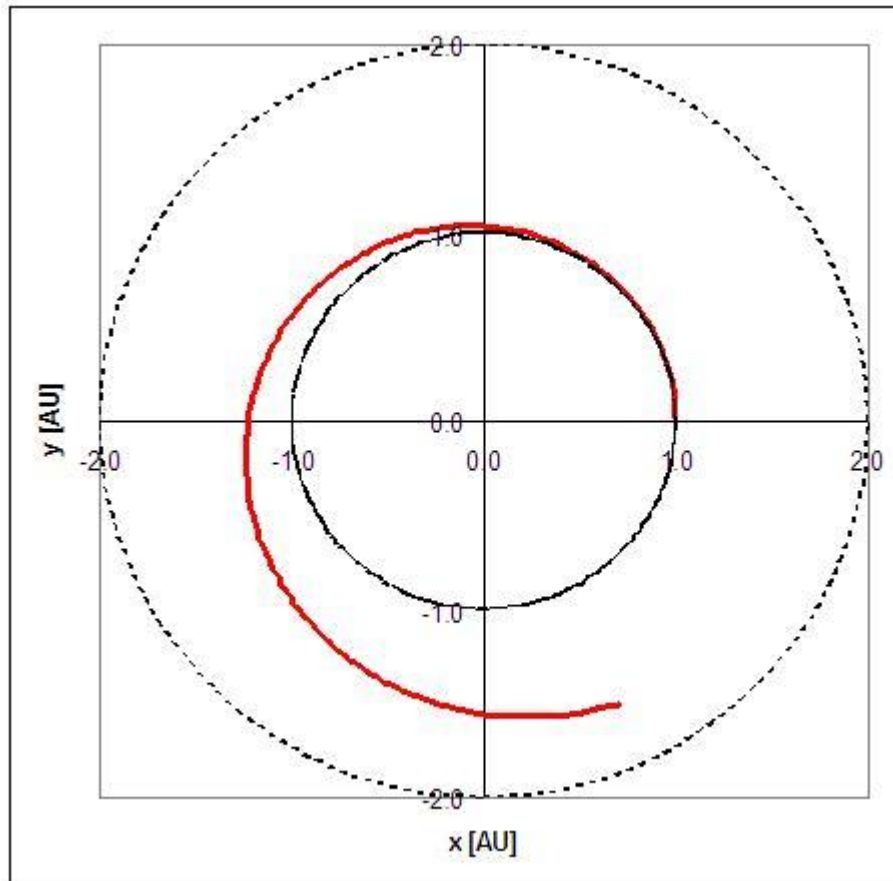
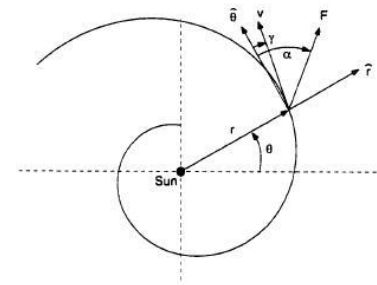
$$\ddot{r} - r\dot{\theta}^2 + \mu/r^2 = a_T \sin \alpha$$

$$\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = a_T \cos \alpha$$

$$(or: \ddot{\theta} r + 2\dot{\theta} \dot{r} = a_T \cos \alpha)$$

[Petropoulos 2001]

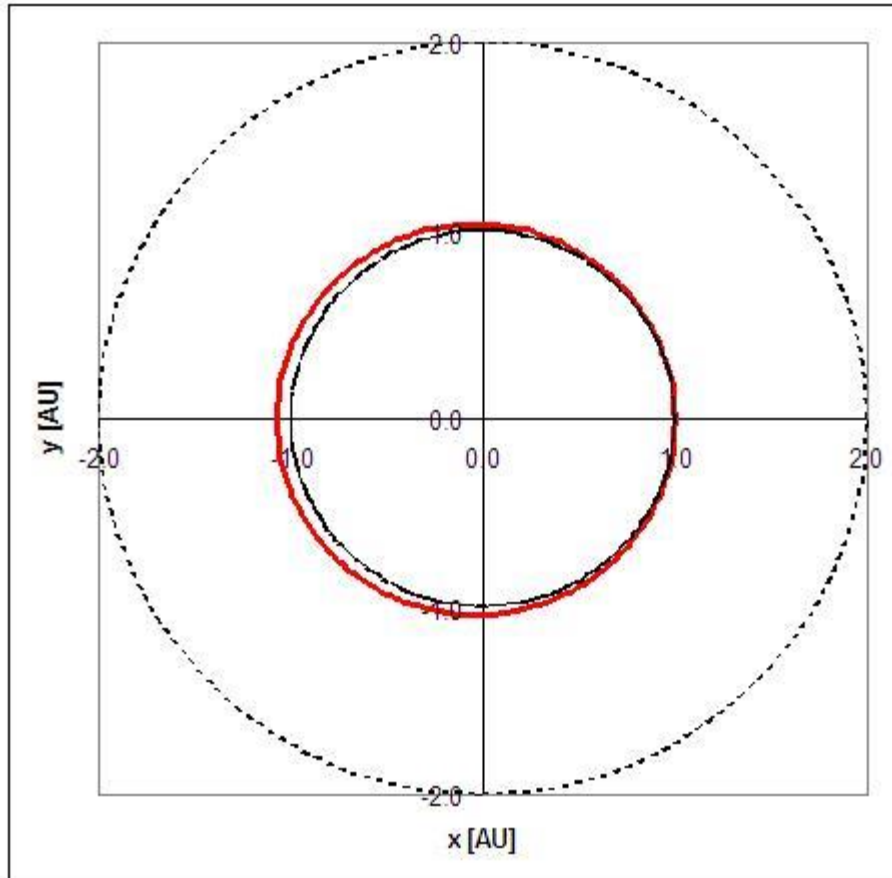
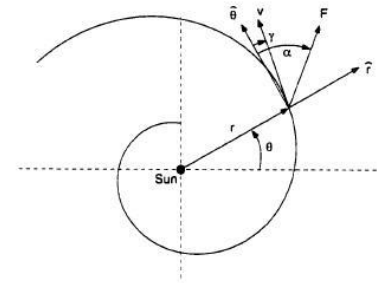
Thrust (cnt'd)



Example 1:

acceleration of 0.190 mm/s^2 in along-track direction (*i.e.*, $\alpha=0^\circ$)

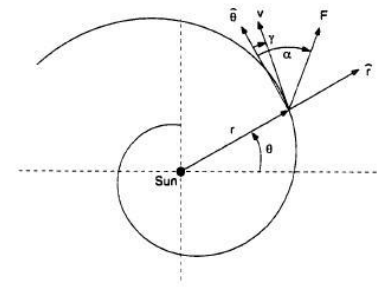
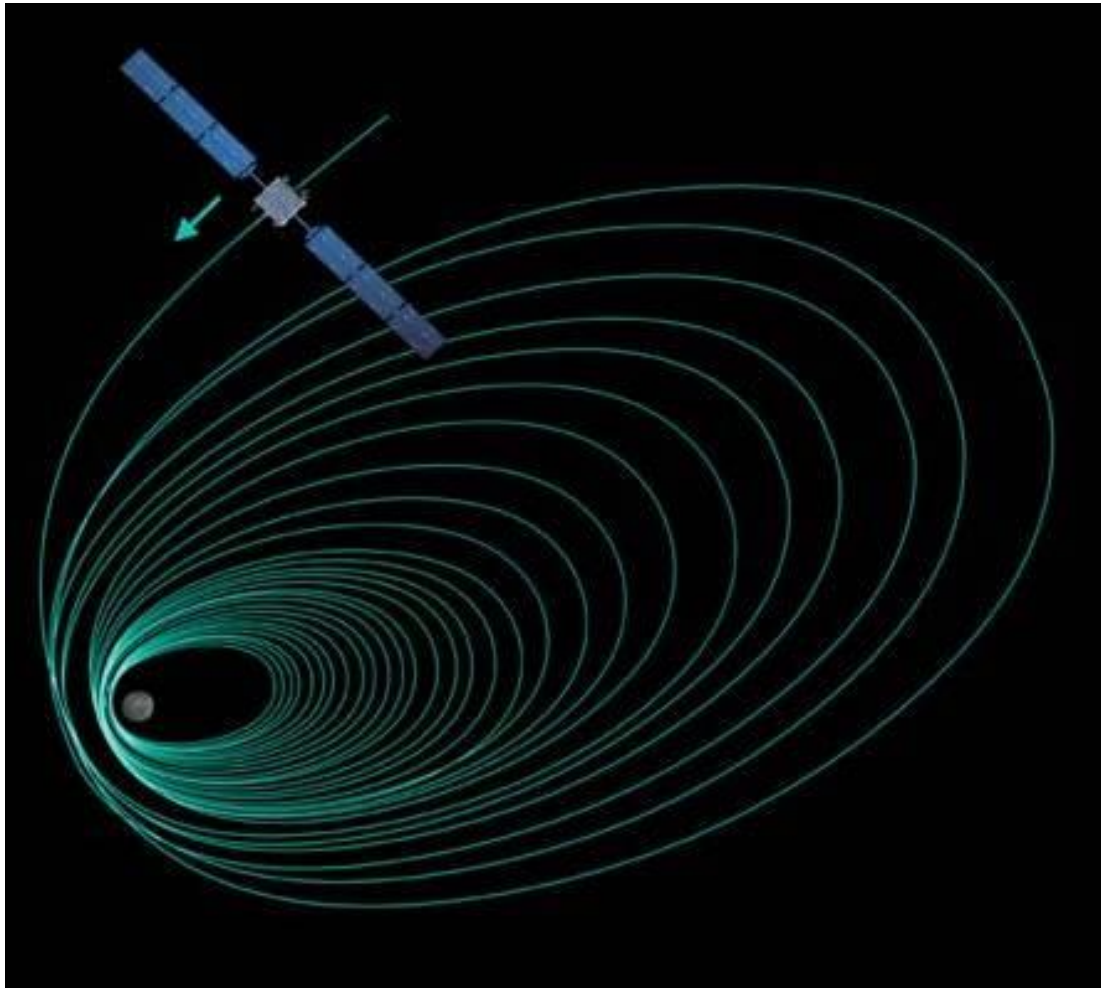
Thrust (cnt'd)



Example 2:

acceleration of 0.190 mm/s^2 in radial direction (*i.e.*, $\alpha=90^\circ$)

Thrust (cnt'd)



Example 3:
orbit of SMART-1
[ESA, 2010]

Thrust (cnt'd)

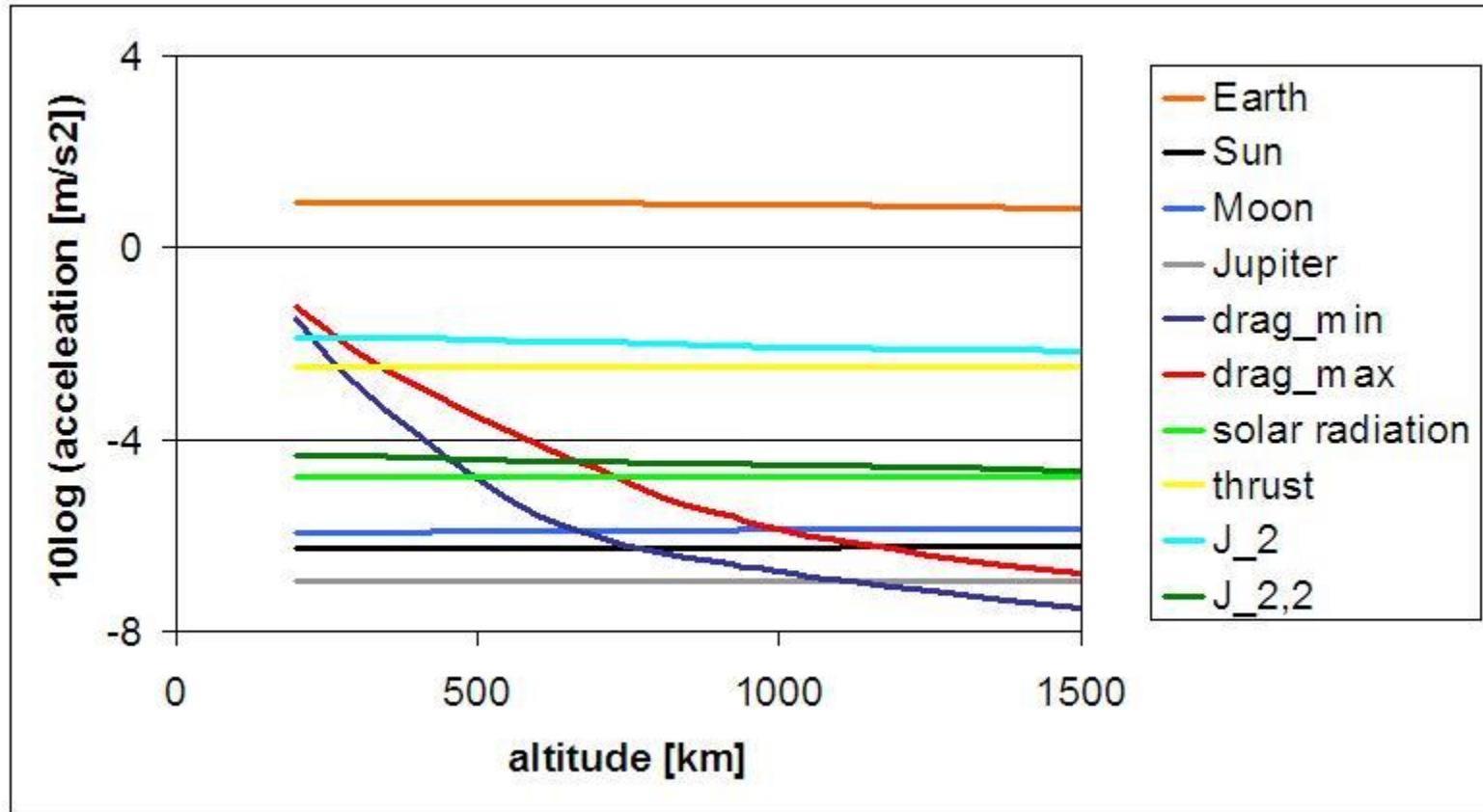
High thrust:

- can compete against central gravity (*i.e.*, launch)
- instantaneous velocity changes (orbits around Earth + interplanetary orbits)

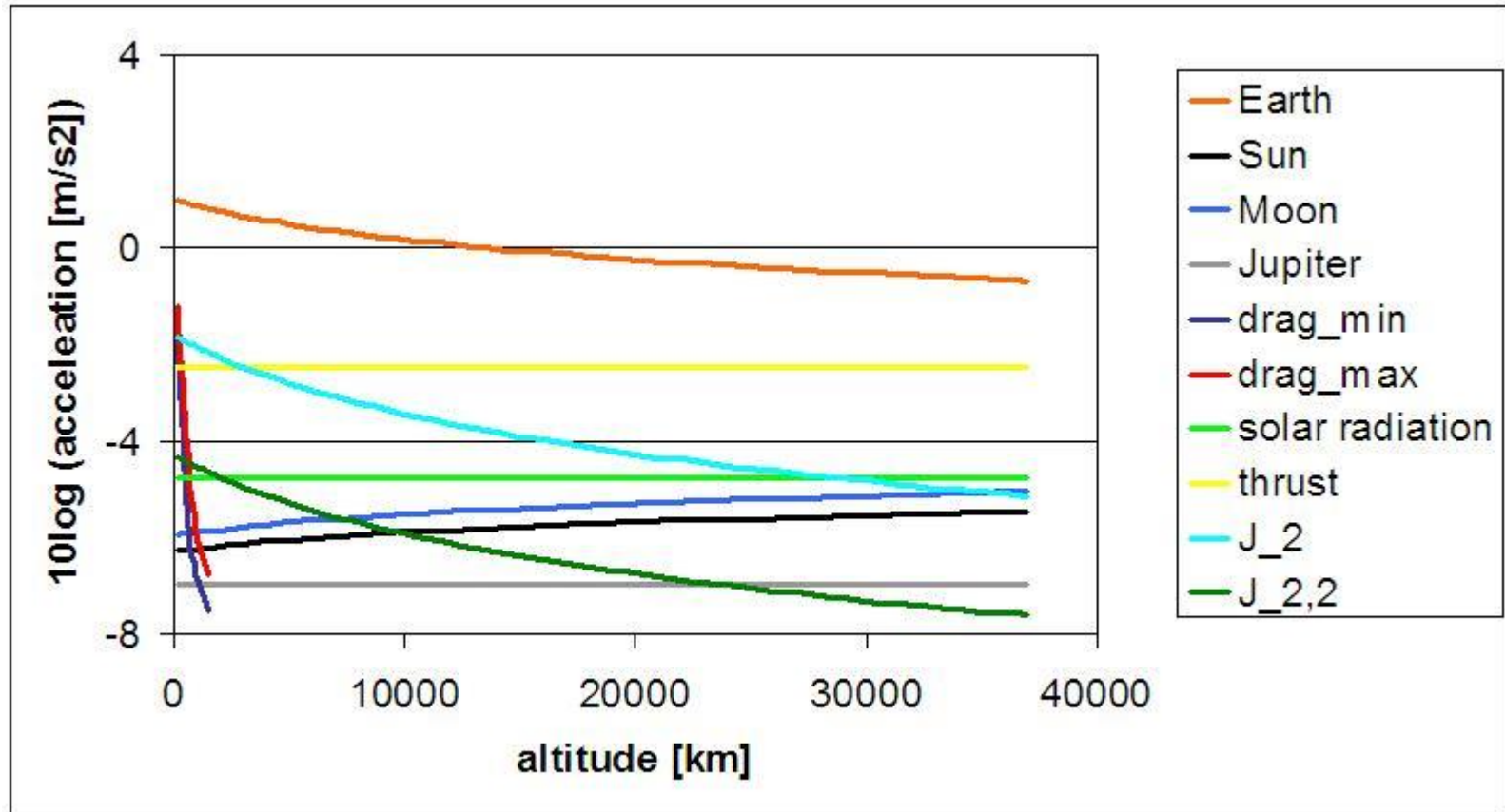
Low thrust:

- attractive since high I_{sp}
- primary propulsion for interplanetary missions
- station-keeping
- since 2012: transfer LEO → GEO

Summary: Low Earth Orbit (1)



Summary: Low Earth Orbit (2)

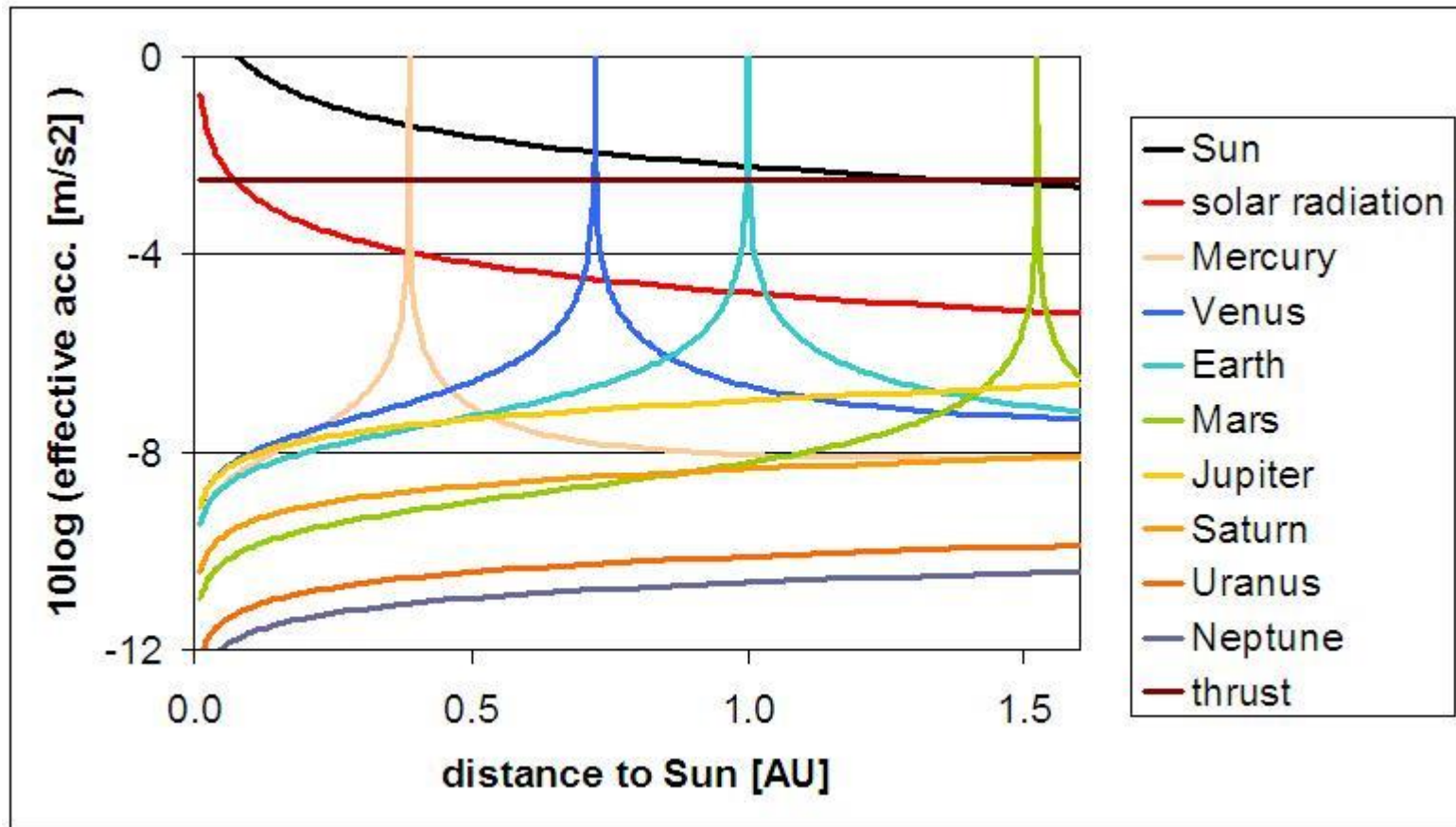


Summary: Low Earth Orbit (3)

Conclusions near-Earth orbits:

- J_2 acceleration is dominant perturbation for all LEO
- low thrust already important
- atmospheric drag dominant perturbation at very low altitudes
- Solar, lunar and $J_{2,2}$ accelerations very small, but build up for GEO
- Kepler orbit very good 1st-order approximation (2-4 orders of magnitude Δ)

Summary: interplanetary orbit (1)



Summary: interplanetary orbit (2)

Conclusions interplanetary orbits:

- low thrust important
- Solar radiation can be important
- Kepler orbit very good 1st-order approximation (4-6 orders of magnitude Δ)

Summary perturbing forces

Question 4:

- a) Mention the 5 different categories of perturbing forces that may act on an arbitrary vehicle. Describe each category briefly (about 5 lines per item).
- b) Give a brief description of the (relative) importance of these categories.

Summary perturbing forces

Question 5:

- a) Mention the 5 different categories of perturbing forces that may act on an arbitrary vehicle.
- b) Compute the magnitude of the main gravitational force exerted by the Earth's gravity field.
- c) Compute the magnitude of the various perturbing sources.

Data: $\mu_{\text{Earth}} = 398600 \text{ km}^3/\text{s}^2$; $\mu_{\text{Moon}} = 4903 \text{ km}^3/\text{s}^2$; $\mu_{\text{Sun}} = 1.327 \times 10^{11} \text{ km}^3/\text{s}^2$; $R_e = 6378 \text{ km}$; $h_{\text{GOCE}} = 250 \text{ km}$; $\rho_{\text{atm,avg}} = 6.2 \times 10^{-11} \text{ kg/m}^3$; $\text{AU} = 149.6 \times 10^6 \text{ km}$; $\text{SC} = 1371 \text{ W/m}^2$; $c = 3 \times 10^5 \text{ km/s}$; $m_{\text{GOCE}} = 1050 \text{ kg}$; $S_{\text{GOCE}} = 1.0 \text{ m}^2$; $C_{D,\text{GOCE}} = 2.2$; $C_{R,\text{GOCE}} = 1.2$; $T_{\text{GOCE}} = 10 \text{ mN}$; $J_2 = 1082 \times 10^{-6}$; $J_{2,2} = 1.816 \times 10^{-6}$; $\lambda_{2,2} = -14.9^\circ$