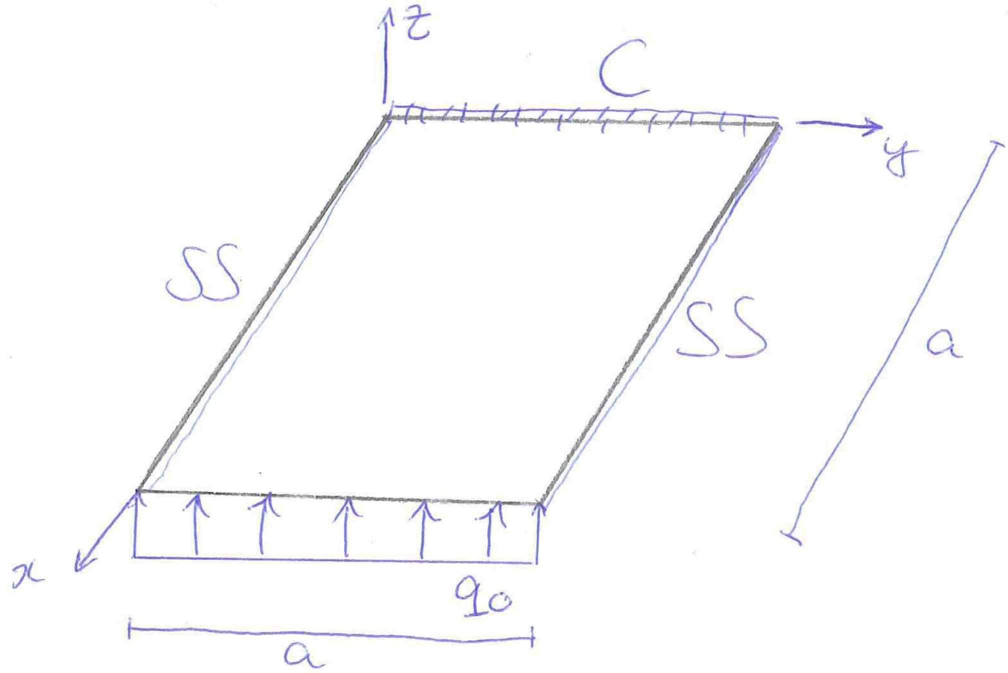
Take Home Exam

Problem 1



1a) We have: $w = \sum_n \varphi_n(x) \sin \frac{n\pi y}{a} = \sum_n \varphi_n(x) \sin \alpha y$ (1)

with $\alpha = \frac{n\pi}{a}$

$$\Pi = U + V$$

with: $U = \int_0^a \int_0^a \frac{D}{2}\left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx} w_{,yy} + 2(1-\nu) w_{,xy} \right] dx dy$ (2)

$V = -\int_0^a \int_0^a q(x,y)\, w \, dx dy$ (3)

From $w_n = \varphi_n(x) \sin\alpha y$:

$w_{n,xx} = \varphi_n'' \sin\alpha y$

$w_{n,yy} = -\varphi_n \alpha^2 \sin\alpha y$

$w_{n,xy} = \varphi_n' \alpha \cos\alpha y$

$\Longrightarrow$

$w_{n,xx}^2 = \varphi_n''^2 \sin^2 \alpha y$

$w_{n,yy}^2 = \varphi_n^2 \alpha^4 \sin^2 \alpha y$ (4)

$w_{n,xy}^2 = \varphi_n'^2 \alpha^2 \cos^2 \alpha y$

$w_{n,xx} w_{n,yy} = -\varphi_n \varphi_n'' \alpha^2 \sin^2 \alpha y$

Now use:

$$\int_0^a \sin^2 \alpha y \, dy = \int_0^a \cos^2 \alpha y \, dy = \frac{a}{2} \qquad (5)$$

$$\int_0^a \sin \frac{n\pi y}{a} \sin \frac{m\pi y}{a} \, dy = \int_0^a \cos \frac{n\pi y}{a} \cos \frac{m\pi y}{b} \, dy = 0 \quad \text{for } m \neq n$$

Hence: $U = \sum_n U_n$, with (using (2,4,5))

$$U_n = \frac{Da}{4} \int_0^a \left(\varphi_n''^2 + \varphi_n^2 \alpha^4 - 2\nu \varphi_n \varphi_n'' \alpha^2 + 2(1-\nu) \varphi_n'^2 \alpha^2 \right) dx$$

Also for $V = \sum_n V_n$ with

$$V_n = - \int_0^a \int_0^a q(x,y) \, w_n \, dx dy \quad , \quad \text{with } q(x,y) = q_0 \delta(x-a)$$

$$= - \int_0^a q_0 \delta(x-a) \varphi_n(x) \, dx \int_0^a \sin \alpha y \, dy$$

$$= - q_0 \varphi_n(a) \cdot \frac{1}{\alpha} \left[1 - (-1)^n \right]$$

$$= \begin{cases} - \dfrac{2 q_0 \varphi_n(a)}{\alpha} & \text{for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

So $\Pi = U + V = \sum_n U_n + V_n = \sum_n \Pi_n$, with

$$\Pi_n = U_n + V_n \qquad (6)$$

$$= \frac{Da}{4} \int_0^a \left(\varphi_n'' + \varphi_n^2 \alpha^4 - 2\nu \varphi_n \varphi_n'' \alpha^2 + 2(1-\nu) \varphi_n' \alpha^2 \right) dx + - \frac{2 q_0}{\alpha} \varphi_n(a)$$

1 b) $\delta\Pi = 0 \Rightarrow \delta\Pi_n = 0$ for all $n = 1, 3, 5, \ldots$

From (6) and using $\delta\varphi_n''^2 = 2\varphi_n''\delta\varphi_n''$, etc.:

$$\delta\Pi_n = \frac{Da}{4}\int_0^a \Big(2\varphi_n''\delta\varphi_n'' + 2\alpha^4\varphi_n\delta\varphi_n - 2\nu\alpha\left(\varphi_n\delta\varphi_n'' + \varphi_n''\delta\varphi_n\right) + 2(1-\nu)\alpha^2\cdot 2\varphi_n'\delta\varphi_n'\Big)dx - \frac{2q_0}{\alpha}\delta\varphi_n(a)$$

$$= \frac{Da}{2}\int_0^a\Big[\left(\alpha^4\varphi_n - \nu\alpha^2\varphi_n''\right)\delta\varphi_n + \underbrace{\left(\varphi_n'' - \nu\alpha^2\varphi_n\right)\delta\varphi_n''}_{\mathrm{I}} + \underbrace{2(1-\nu)\alpha^2\varphi_n'\delta\varphi_n'}_{\mathrm{II}}\Big]dx \underbrace{- \frac{2q_0}{\alpha}\delta\varphi_n(a)}_{B.T.}$$

Use integration by parts: (use $\delta\varphi_n(0) = \delta\varphi_n'(0) = 0$)

$$\mathrm{I}: \int_0^a \left(\varphi_n'' - \nu\alpha^2\varphi_n\right)\delta\varphi_n'' dx = \Big[\left(\varphi_n'' - \nu\alpha^2\varphi_n\right)\delta\varphi_n'\Big]_0^a - \int_0^a\left(\varphi_n''' - \nu\alpha^2\varphi_n'\right)\delta\varphi_n' dx$$

$$= \Big[\left(\varphi_n'' - \nu\alpha^2\varphi_n\right)\delta\varphi_n'\Big]_{x=a} - \Big[\left(\varphi_n''' - \nu\alpha^2\varphi_n'\right)\delta\varphi_n\Big]_0^a + \int_0^a\left(\varphi_n'''' - \nu\alpha^2\varphi_n''\right)\delta\varphi_n dx$$

$$= \underbrace{\Big[\left(\varphi_n'' - \nu\alpha^2\varphi_n\right)\delta\varphi_n' - \left(\varphi_n''' - \nu\alpha^2\varphi_n'\right)\delta\varphi_n\Big]_{x=a}}_{B.T.\ \text{(Boundary Terms)}} + \int_0^a\left(\varphi_n'''' - \nu\alpha^2\varphi_n''\right)\delta\varphi_n dx$$

$$\mathrm{II}: \int_0^a 2(1-\nu)\alpha^2\varphi_n'\delta\varphi_n' = \Big[2(1-\nu)\alpha^2\varphi_n'\delta\varphi_n\Big]_0^a - \int_0^a 2(1-\nu)\alpha^2\varphi_n''\delta\varphi_n dx$$

$$= \underbrace{\Big[2(1-\nu)\alpha^2\varphi_n'\delta\varphi_n\Big]_{x=a}}_{B.T.} - \int_0^a 2(1-\nu)\alpha^2\varphi_n''\delta\varphi_n dx$$

Collecting results from integration by parts and substituting these into $\delta\Pi_n$:

$$\delta\Pi_n = \frac{Da}{2}\int_0^a \left(\alpha^4\varphi_n - \nu\alpha^2\varphi_n'' + \varphi_n'''' - \nu\alpha^2\varphi_n'' - 2(1-\nu)\alpha^2\varphi_n''\right)\delta\varphi_n\,dx + B.T.$$

$$= \frac{Da}{2}\int_0^a \left(\alpha^4\varphi_n - 2\alpha^2\varphi_n'' + \varphi_n''''\right)\delta\varphi_n\,dx + B.T.$$

$$\Rightarrow \text{So: } \delta\Pi = 0 \Rightarrow \boxed{\varphi_n'''' - 2\alpha^2\varphi_n'' + \alpha^4\varphi_n = 0} \qquad (7)$$

Boundary conditions :

We already used the essential B.C.

$$\varphi_n(0) = 0 \qquad (BC\ 1)$$

$$\varphi_n'(0) = 0 \qquad (BC\ 2)$$

The reamaing two B.C. are derived from the B.T. in the expression of $\delta\Pi_n$

$$B.T. = \frac{Da}{2}\left[\left(\varphi_n'' - \nu\alpha^2\varphi_n\right)\delta\varphi_n' + \left(-\varphi_n''' + \nu\alpha^2\varphi_n' + 2(1-\nu)\alpha^2\varphi_n'\right)\delta\varphi_n\right]_{x=a} - \frac{2q_0}{\alpha}\delta\varphi_n(a)$$

$$= \frac{Da}{2}\left[\left(\varphi_n'' - \nu\alpha^2\varphi_n\right)\delta\varphi_n' + \left(\varphi_n''' - (2-\nu)\alpha^2\varphi_n' + \frac{4q_0}{Da\alpha}\right)\delta\varphi_n\right]_{x=a}$$

So, from $\delta\Pi_n = 0$ we get :

$$\varphi_n''(a) - \nu\alpha^2 \varphi_n(a) = 0 \qquad \text{(BC3)}$$

$$\varphi_n'''(a) - (2-\nu)\alpha^2 \varphi_n'(a) = -\frac{4q_0}{Da\alpha} = -\frac{4q_0}{Dn\pi} \quad \text{for } n \text{ odd (BC4)}$$

1c) Assume $\varphi_n = C_n e^{\lambda_n x} \Rightarrow \varphi_n' = C_n \lambda_n e^{\lambda_n x}$, $\varphi_n'' = \ldots$ etc

Into (7):

$$(\lambda_n^4 - 2\alpha^2\lambda_n^2 + \alpha^4) C_n e^{\lambda_n x} = 0$$

$$(\lambda_n^2 - \alpha^2)^2 = 0$$

$$\lambda_n = \pm \alpha$$

So:

$$\varphi_n = C_1 e^{\alpha x} + C_2 e^{-\alpha x} + C_3 x e^{\alpha x} + C_4 x e^{-\alpha x}$$

$$\varphi_n' = C_1\alpha e^{\alpha x} - C_2 \alpha e^{-\alpha x} + C_3(e^{\alpha x} + \alpha x e^{\alpha x}) + C_4(e^{-\alpha x} - \alpha x e^{-\alpha x}) \qquad (8)$$

$$\left.\begin{array}{l}\varphi_n'' = \ldots \\ \varphi_n''' = \ldots\end{array}\right\} \text{Use Maple / MATLAB}$$

Substitute (8) into (BC1)-(BC4) to solve for C_1, C_2, C_3, C_4:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ \alpha & -\alpha & 1 & 1 \\ & \ldots & \ldots & \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{4q_0}{Dn\pi} \end{pmatrix}$$

Solve using MATLAB/Maple

Substitute into (1) and show the plots.

1d) $\sigma_{vm} = \sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$

In order to find σ_x, σ_y, τ_{xy}, use:

$$M_x = -D(w_{,xx} + \nu w_{,yy})$$

$$M_y = -D(w_{,yy} + \nu w_{,xx})$$

$$M_{xy} = -D(1-\nu)\, w_{,xy}$$

Assuming a linear stress distribution through the thickness of the plate, we get:

$$\sigma_{x,max} = 6\,\frac{M_x}{t^2}$$

$$\sigma_{y,max} = 6\,\frac{M_y}{t^2}$$

$$\tau_{xy,max} = 6\,\frac{M_{xy}}{t^2}$$

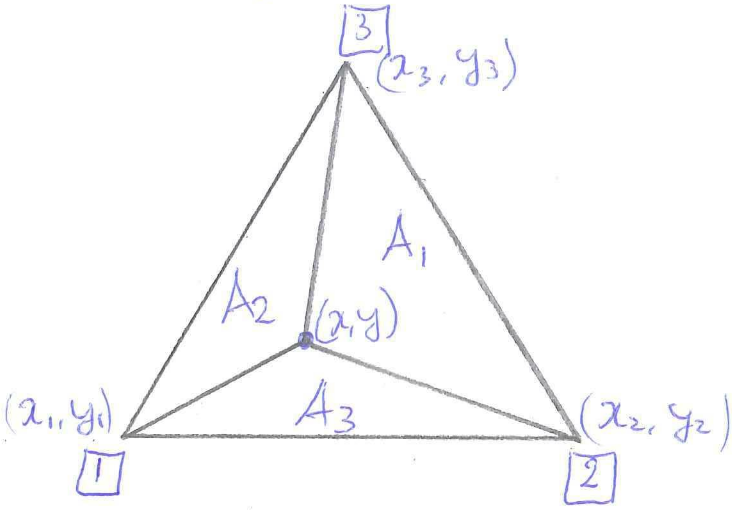
Use these, together with the derivatives of (1) to compute σ_{vm}, and show the plots.

Problem 2

2a) One way of showing that a general point (x,y) inside the triangle can be located by

$$x = \sum_{i=1}^{3} x_i \xi_i \quad , \quad y = \sum_{i=1}^{3} y_i \xi_i$$

is as follows:



We have $A_1 + A_2 + A_3 = A$, with A the area of the large triangle

$$\Rightarrow \frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = 1 \qquad (1)$$

With:

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x \\ y_2 & y_3 & y \end{vmatrix}, \quad A_2 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix}, \quad A_3 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix}$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Now, let $\xi_1 \equiv \frac{A_1}{A}$, $\xi_2 \equiv \frac{A_2}{A}$, $\xi_3 = \frac{A_3}{A}$.

Then it can be shown that indeed

$$x = \sum_{i=1}^{3} x_i \xi_i \quad , \quad y = \sum_{i=1}^{3} y_i \xi_i$$

and with ('), we also get:

$$\xi_1 + \xi_2 + \xi_3 = 1$$

To find the expressions for ξ_1, ξ_2, ξ_3 as functions of x, x_i, y, y_i, use:

$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

Use: $\xi_3 = 1 - \xi_1 - \xi_2$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

Inverting gives:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & x_3 - x_2 \\ y_3 - y_1 & x_1 - x_3 \end{bmatrix} \begin{pmatrix} x - x_3 \\ y - y_3 \end{pmatrix}, \quad \text{with } 2A = \begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix}$$

2b) Simply supported means:

$\varphi_i = 0$ on all edges

$\varphi_i'' \big|_{\text{normal to edge}} = 0$ on all edges

Consider the edge [2]-[3], which is characterized by $\xi_1 = 0$. So at this edge:

$$\varphi_i = \xi_1^{p_{1i}} \xi_2^{p_{2i}} \xi_3^{p_{3i}} = 0$$

For the bending moments ($\varphi'' = 0$), we have:

$$\frac{\partial^2 \varphi_i}{\partial \xi_1^2} = 0$$ as this denotes the normal direction to the edge

So:

$$\varphi_i = \xi_1^{p_{1i}} \xi_2^{p_{2i}} \xi_3^{p_{3i}} = \xi_1^{p_{1i}} \xi_2^{p_{2i}} (1-\xi_1-\xi_2)^{p_{3i}}$$

$$\frac{\partial \varphi_i}{\partial \xi_1} = p_{1i}\, \xi_1^{p_{1i}-1} \xi_2^{p_{2i}} (1-\xi_1-\xi_2)^{p_{3i}} - p_{3i}\, \xi_1^{p_{1i}} \xi_2^{p_{2i}} (1-\xi_1-\xi_2)^{p_{3i}-1}$$

$$\frac{\partial^2 \varphi_i}{\partial \xi_1^2} = p_{1i}(p_{1i}-1)\, \xi_1^{p_{1i}-2} \xi_2^{p_{2i}} (1-\xi_1-\xi_2)^{p_{3i}} - 2 p_{1i} p_{3i}\, \xi_1^{p_{1i}-1} \xi_2^{p_{2i}} (1-\xi_1-\xi_3)^{p_{3i}-1}$$
$$+ p_{3i}(p_{3i}-1)\, \xi_1^{p_{1i}} \xi_2^{p_{2i}} (1-\xi_1-\xi_2)^{p_{3i}-2}$$

From this we see that $\frac{\partial^2 \varphi_i}{\partial \xi_1^2} = 0$ for $p_{1i}, p_{2i}, p_{3i} \geq 3$

The same argumentation can be applied to the other two edges.

2c) Assume $w = \sum_i c_i \varphi_i$

From the lecture notes we get for buckling analysis the following governing equation:

$$\left(\underline{\underline{K}} - \lambda \underline{\underline{K}}_g\right) \underset{\sim}{v} = \underline{0} \qquad (9)$$

with

$$K_{ij} = \iint_A D\left[\varphi_{i,xx}\varphi_{j,xx} + \varphi_{i,yy}\varphi_{j,yy} + \nu\left(\varphi_{i,xx}\varphi_{j,yy} + \varphi_{j,xx}\varphi_{i,xx}\right) + 2(1-\nu)\varphi_{i,xy}\varphi_{j,xy}\right] dA \qquad (10)$$

$$K_{g_{ij}} = -\iint_A \left[N_x \varphi_{i,x}\varphi_{j,x} + N_y \varphi_{i,y}\varphi_{j,y}\right] dA \qquad (\text{for } N_{xy} = 0) \quad (11)$$

For equal biaxial loading we can assume $N_x = N_y = 1$ such that λ in (9) becomes the buckling load, and $\underset{\sim}{v}$ in (9) becomes the eigenvector, i.e. the collection of c_i that makes up the mode.

From Q. 2a we know that:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{\underline{J}} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

with $$\underline{\underline{J}} = \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} \end{bmatrix} \qquad (12)$$

Therefore :

$$\begin{pmatrix} \varphi_{i,x} \\ \varphi_{i,y} \end{pmatrix} = \underline{\underline{J}}^{-1} \begin{pmatrix} \dfrac{\partial \varphi_i}{\partial \xi_1} \\ \dfrac{\partial \varphi_i}{\partial \xi_2} \end{pmatrix} \qquad (13)$$

which can be easily evaluated using $\varphi_i = \xi_1^{p_{1i}} \xi_2^{p_{2i}} (1-\xi_1-\xi_2)^{p_{3i}}$.

Then also:

$$\varphi_{i,xx} = \underline{\underline{J}}^{-1}(1,:) \begin{pmatrix} \dfrac{\partial \varphi_{i,x}}{\partial \xi_1} \\ \dfrac{\partial \varphi_{i,y}}{\partial \xi_2} \end{pmatrix} \qquad (14)$$

First row of $\underline{\underline{J}}^{-1}$

Can be evaluated from (13), as this equation gives $\varphi_{i,x}$ as function of ξ_1, ξ_2

Similarly:

$$\varphi_{i,yy} = \underline{\underline{J}}^{-1}(2,:) \begin{pmatrix} \dfrac{\partial \varphi_{i,y}}{\partial \xi_1} \\ \dfrac{\partial \varphi_{i,y}}{\partial \xi_2} \end{pmatrix} \qquad (15)$$

$$\varphi_{i,xy} = \underline{\underline{J}}^{-1}(1,:) \begin{pmatrix} \dfrac{\partial \varphi_{i,y}}{\partial \xi_1} \\ \dfrac{\partial \varphi_{i,y}}{\partial \xi_2} \end{pmatrix} \qquad (16)$$

Also:

$$dA = \det(J)\, d\xi_1 d\xi_2 = 2A\, d\xi_1 d\xi_2 \qquad (17)$$

Substitution of (13)-(17) into (10) & (11) gives the stiffness matrices. The integration boundaries are:

$$\int_0^1 \int_0^{1-\xi_1} [\ldots] \, d\xi_2 \, d\xi_1$$

The number of modes is determined by the permutations satisfying $p_0 \leq p_{1i} + p_{2i} + p_{3i} \leq p$, with $p_0 = 9$ and $p_{1i}, p_{2i}, p_{3i} \geq 3$. So:

$p = 9$:

p_{1i}	p_{2i}	p_{3i}
3	3	3

$p = 10$:

p_{1i}	p_{2i}	p_{3i}
3	3	3
4	3	3
3	4	3
3	3	4

$p = 11$: ...

$p = 12$: ...

Now ~~everything~~ $\underline{\underline{K}}$ and $\underline{\underline{K}}_g$ are fully defined for a given p and (9) can be solved.

The values for λ need to be presented in the report together with the mode shapes obtained from $\underline{v}$ is (9).

Concergence of λ has to be discussed.