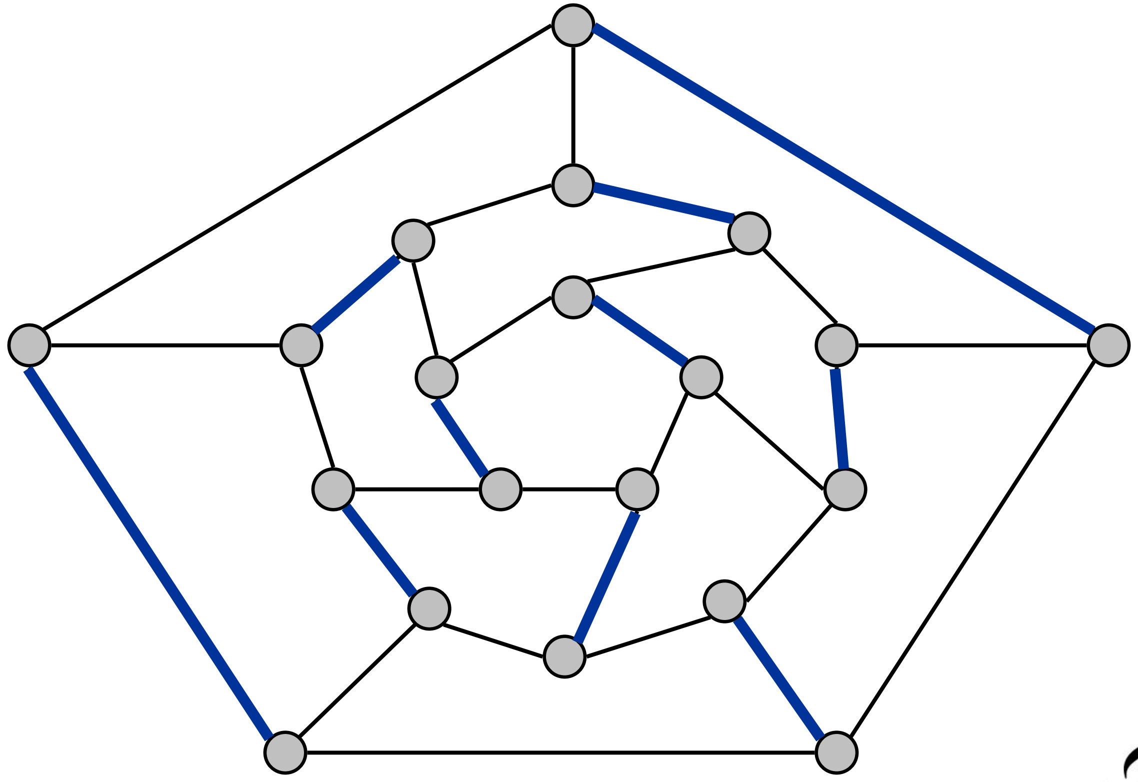
7.5 Network Flow: Bipartite Matching

- (Bipartite) Matching Problem
- Translation to Network Flow
- Perfect Matching

Matching.

- Input: undirected graph G = (V, E).
- M $\subseteq$ E is a matching if each node appears in at most one edge in M.
- Max matching: find a max cardinality matching.

Q. Given n nodes, m edges, what is upper bound on cardinality of M?

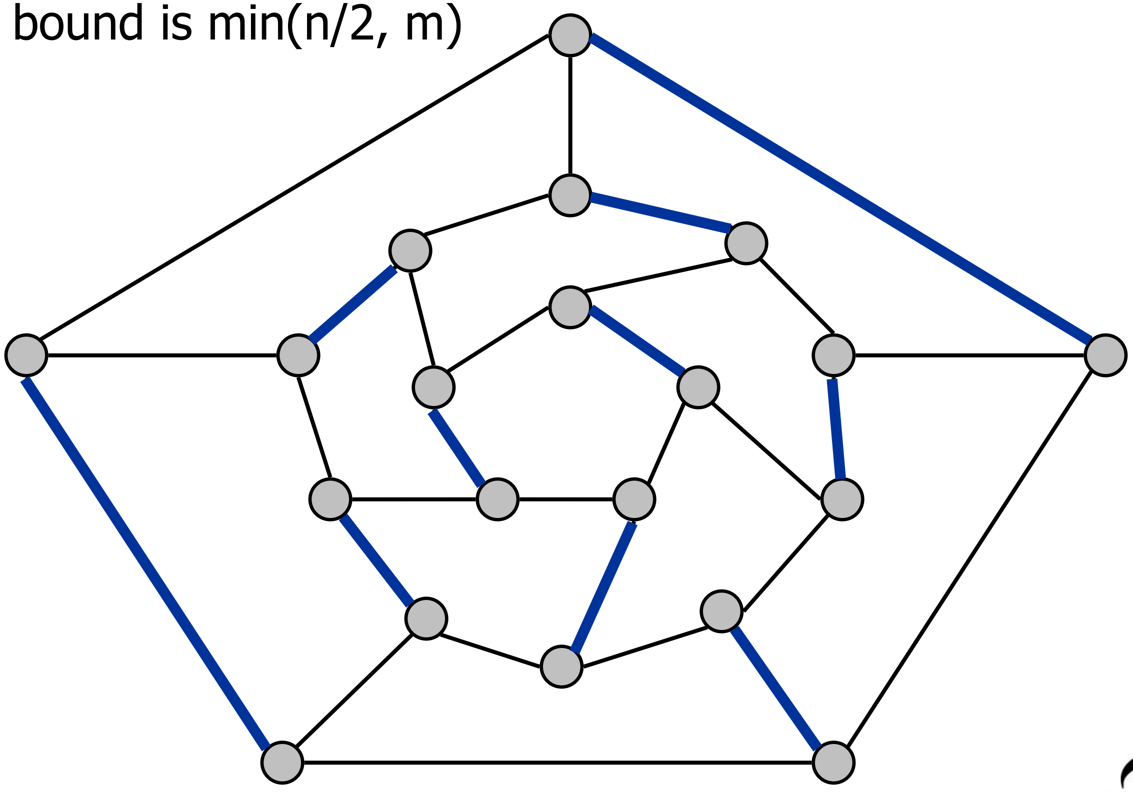


Matching.

- Input: undirected graph G = (V, E).
- M $\subseteq$ E is a matching if each node appears in at most one edge in M.
- Max matching: find a max cardinality matching.

Q. Given n nodes, m edges, what is upper bound on cardinality of M?

A. upper bound is min(n/2, m)



Matching Applications

Applications

- assignment of jobs to machines
- assignment of items to people
- assignment of lights to light switches (exercise 7.6)
- assignment of injured to hospitals (exercise 7.9)

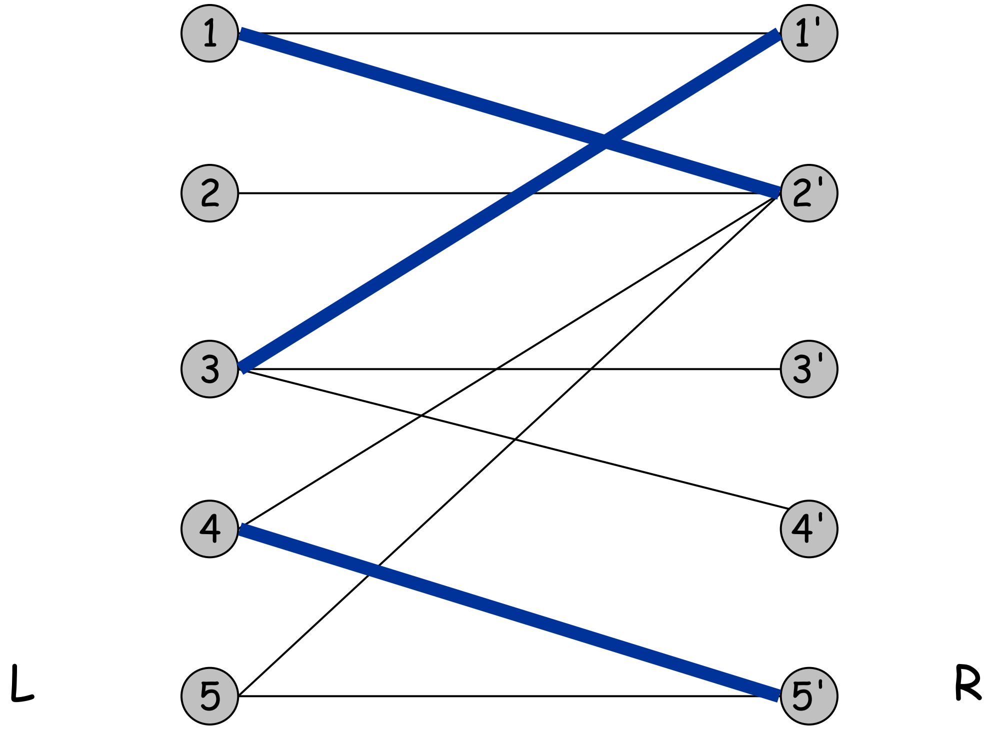
TU Delft

Bipartite Matching

Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Max matching: find a max cardinality matching.

Q. What is the cardinality of the max matching? (3,4,5)



matching

1-2', 3-1', 4-5'

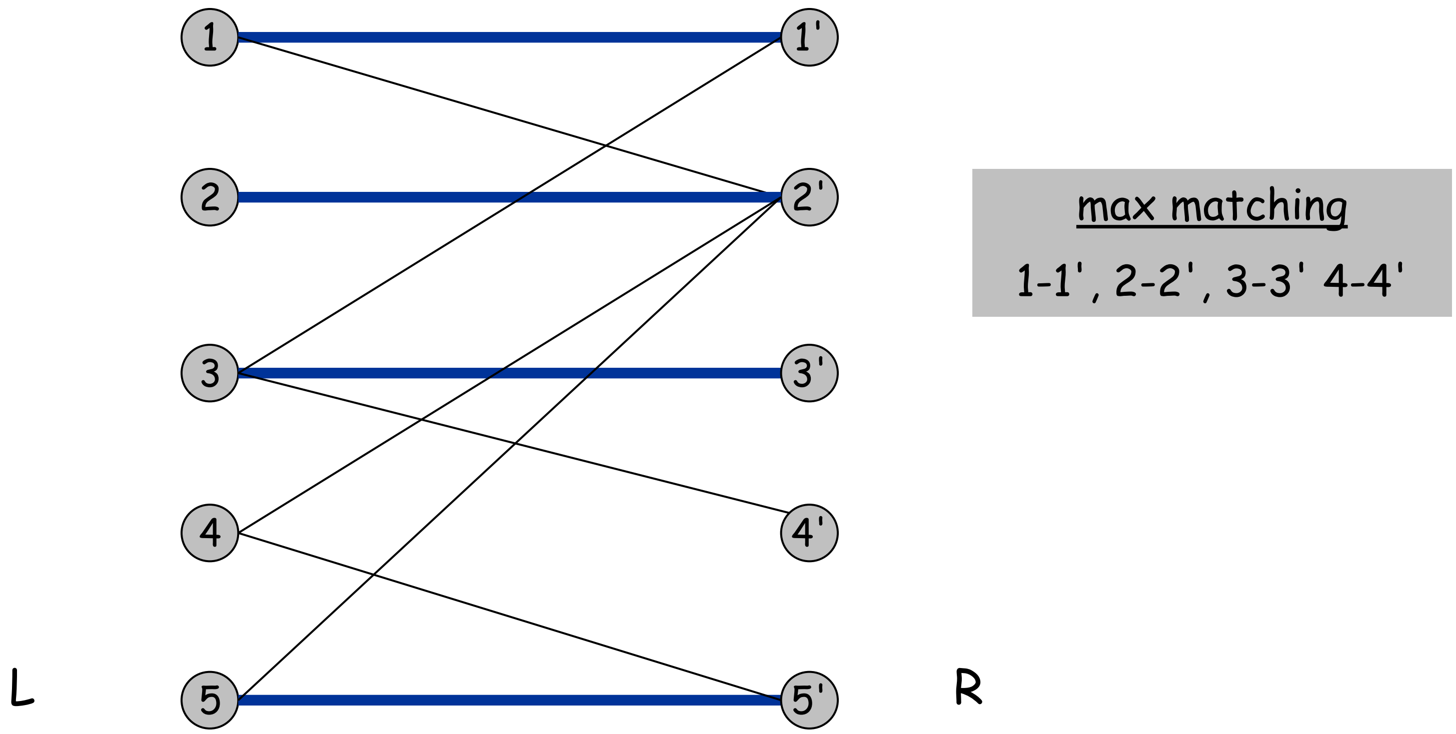
Bipartite Matching

Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Max matching: find a max cardinality matching.

Q. What is the difference from the stable matching problem?

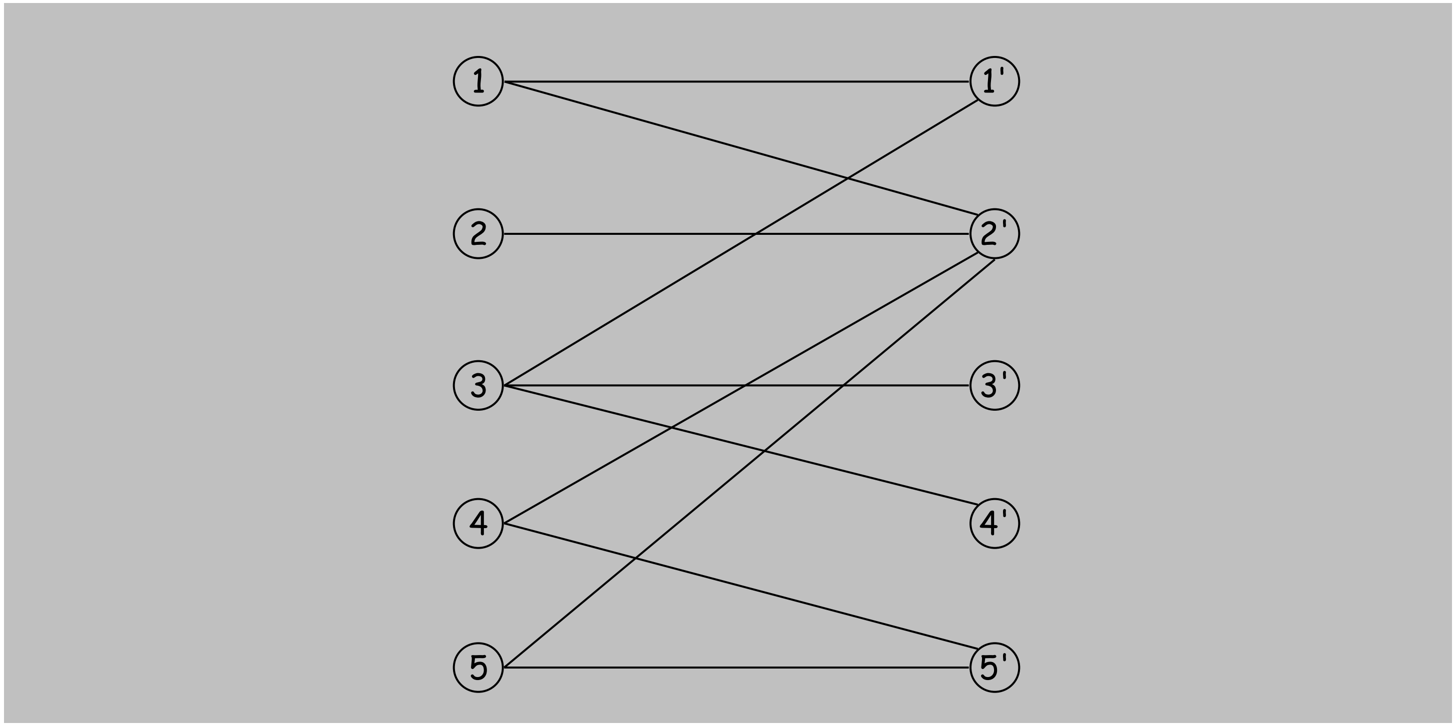
A. Stable matching: all connected, $|L|=|R|$, with preference lists



Bipartite Matching

Q. Formulate max bipartite matching as a max flow problem? (2 min)

1. Do you want the max matching to be the max flow or the min cut?
2. Which nodes should be s and t? (Existing or new?)
3. Do we need more edges, what should be the direction?
4. What should be the capacities?

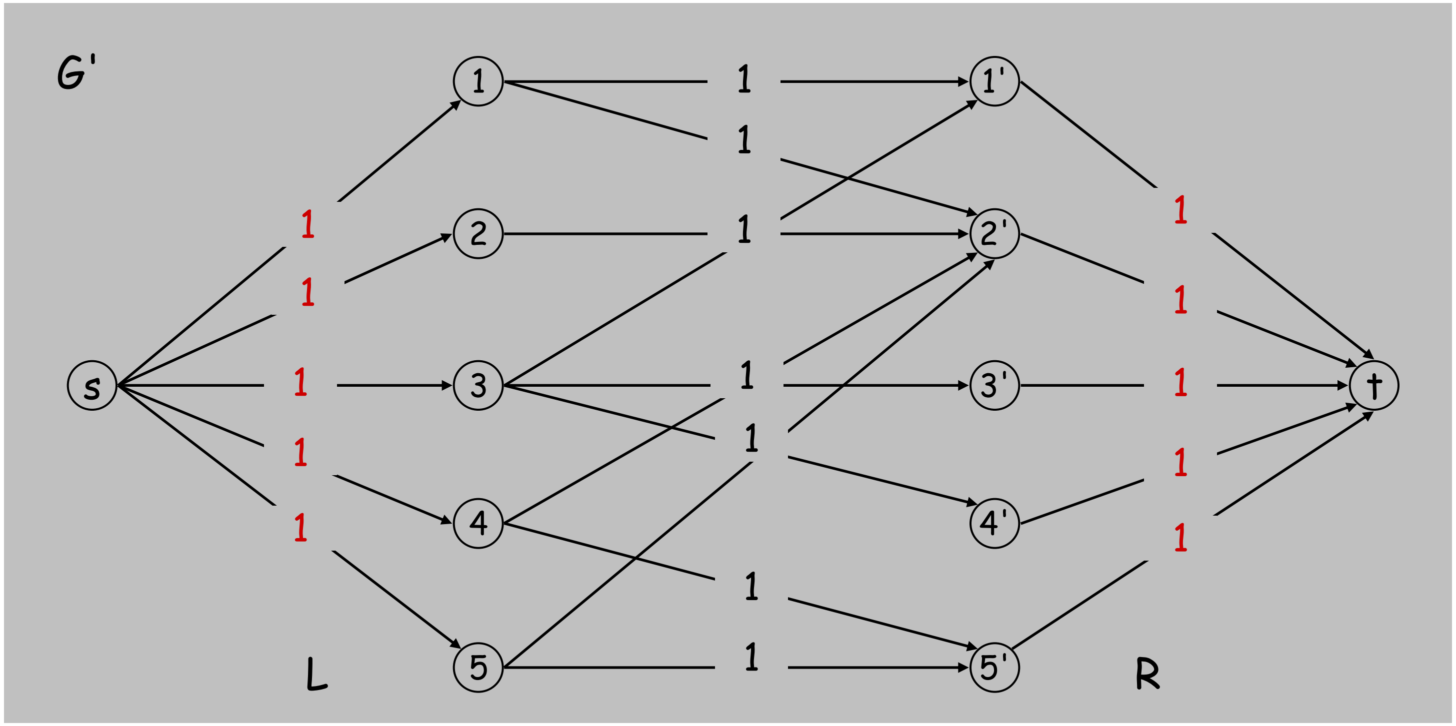


Bipartite Matching

Max flow formulation.

- Create digraph $G' = (L \cup R \cup \{s, t\}, E')$.
- Direct all edges from L to R, and assign unit (or infinite) capacity.
- Add source s, and unit capacity edges from s to each node in L.
- Add sink t, and unit capacity edges from each node in R to t.

Q. When do we have a max cardinality matching?

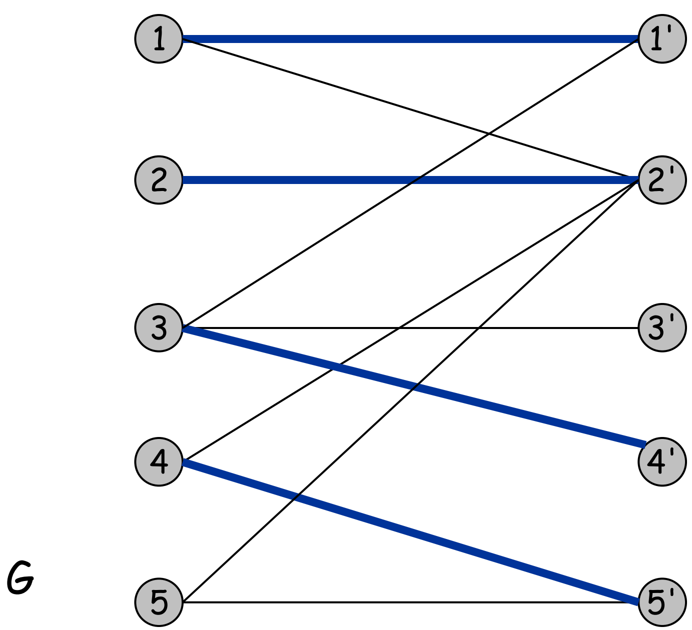


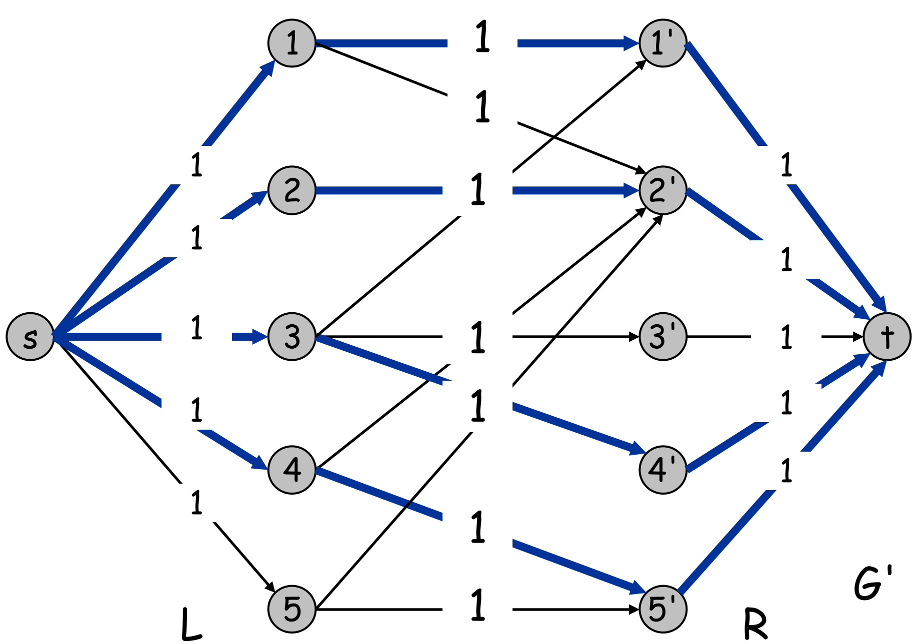
Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Pf.

Q. How to prove this equality?





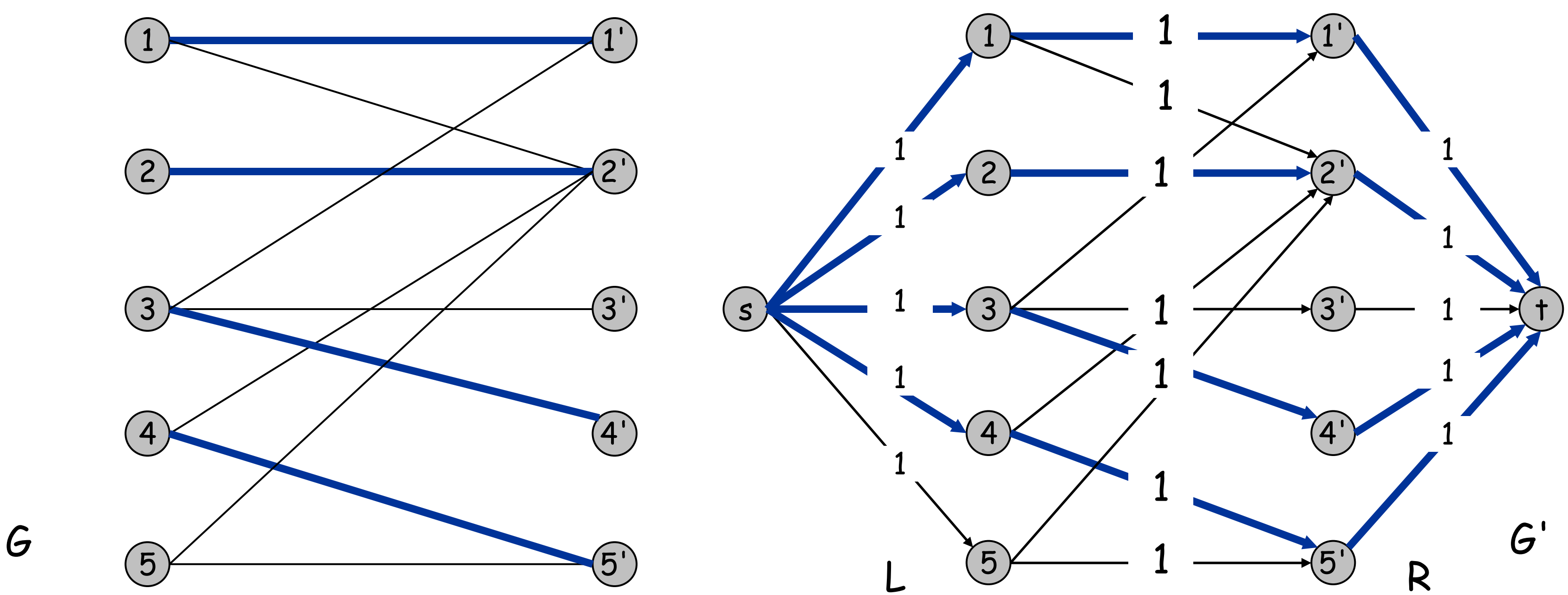
Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Pf.

Q. How to prove this equality?

A. Prove ≤ and ≥ separately.



Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

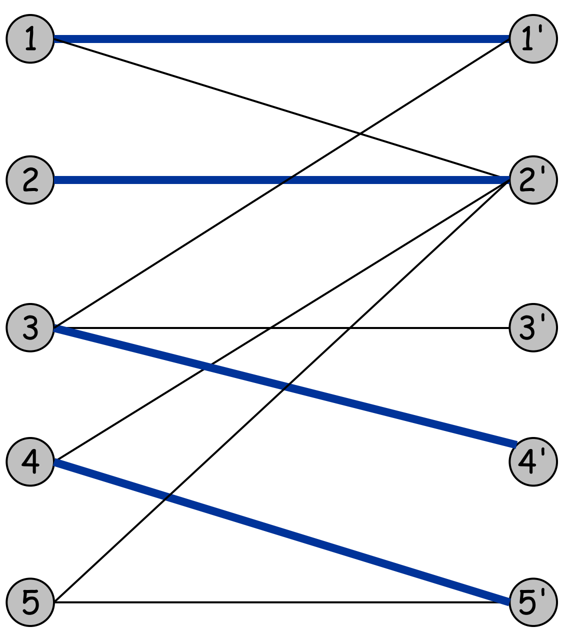
Pf. $\leq$

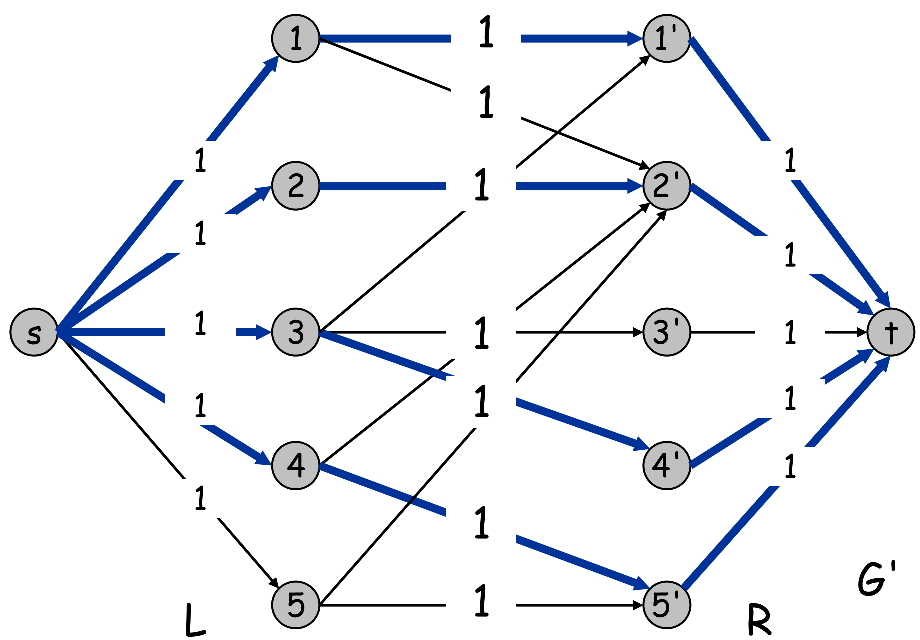
- Given max matching M in G of cardinality k.
- Consider flow f ...

Q. Which flow to consider?

- value of f is $\leq$ value of max flow in G' ▪

G



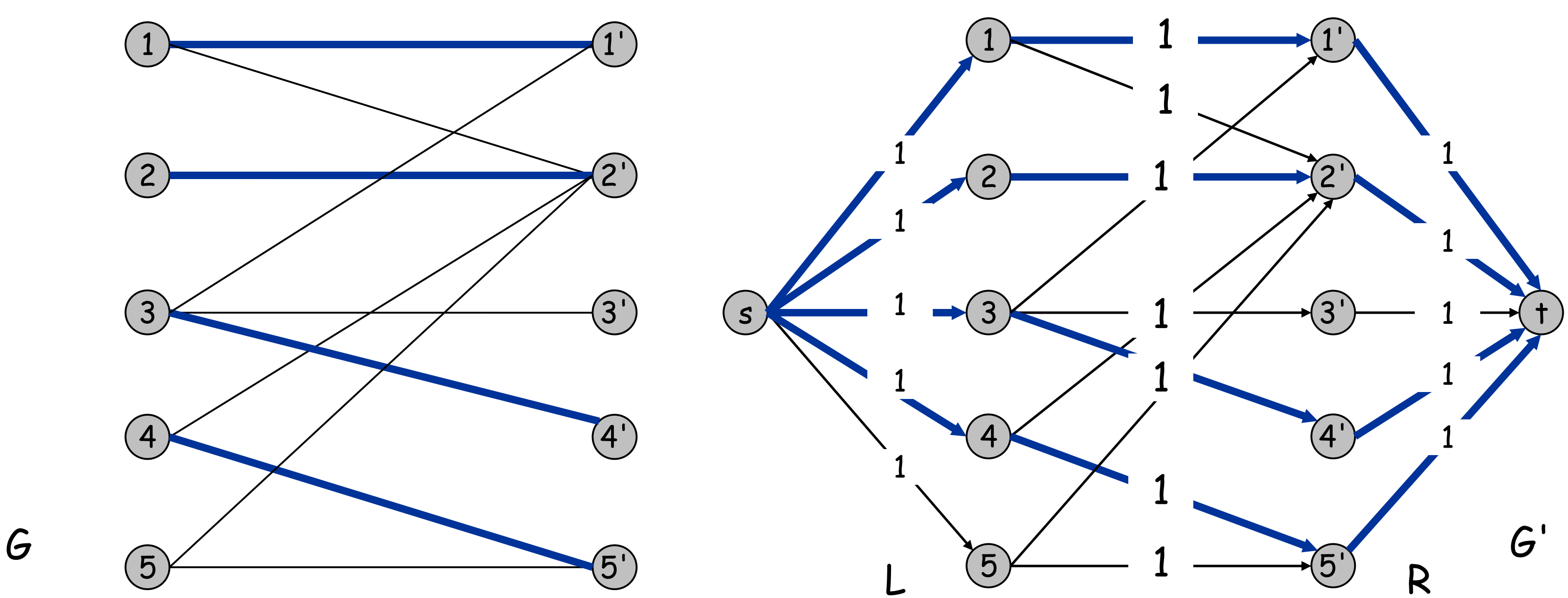


Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Pf. $\leq$

- Given max matching M in G of cardinality k.
- Consider flow f that sends 1 unit along k paths, one for each (l,r) in M: path is from s to l, from l to r, and from r to t.
- f is a flow, and has value k.
- value of f is $\leq$ value of max flow in G'. ▪



Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

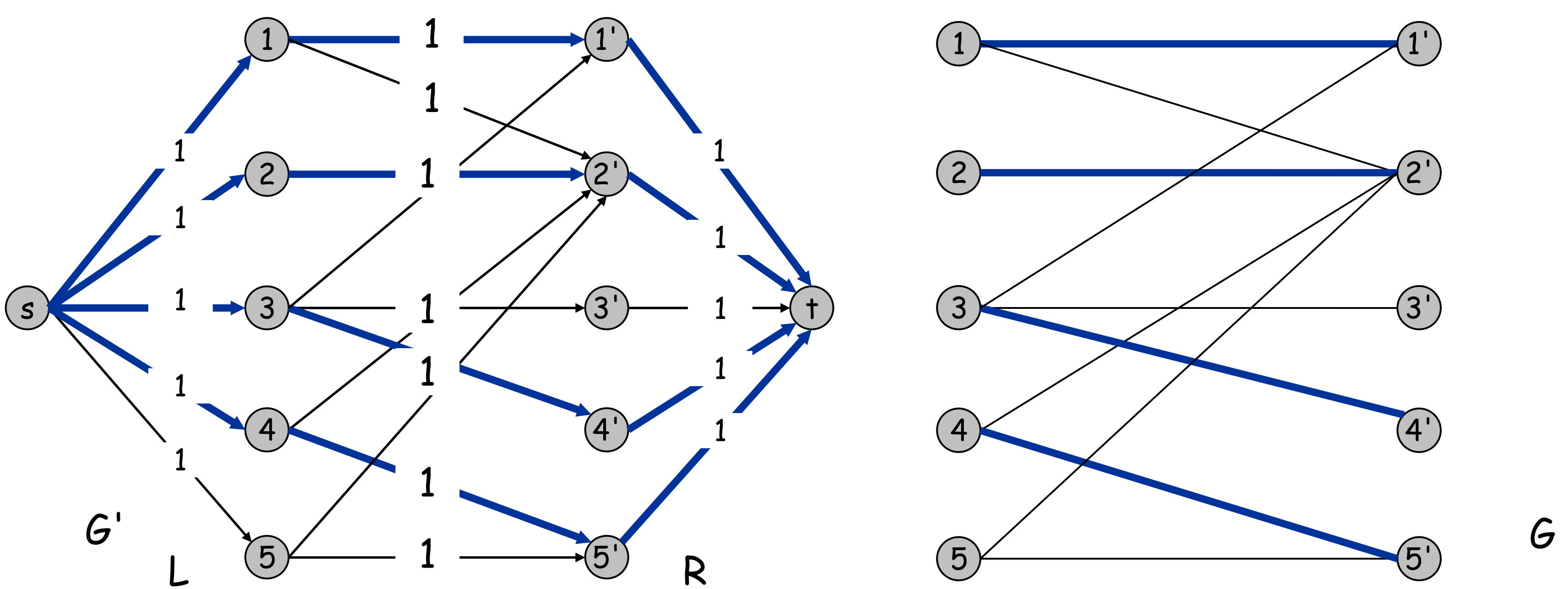
Pf. $\geq$

- Let f be a max flow in G' of value k.
- …

Q. Which matching M to consider?

– …

– $|M| = k$ ▪

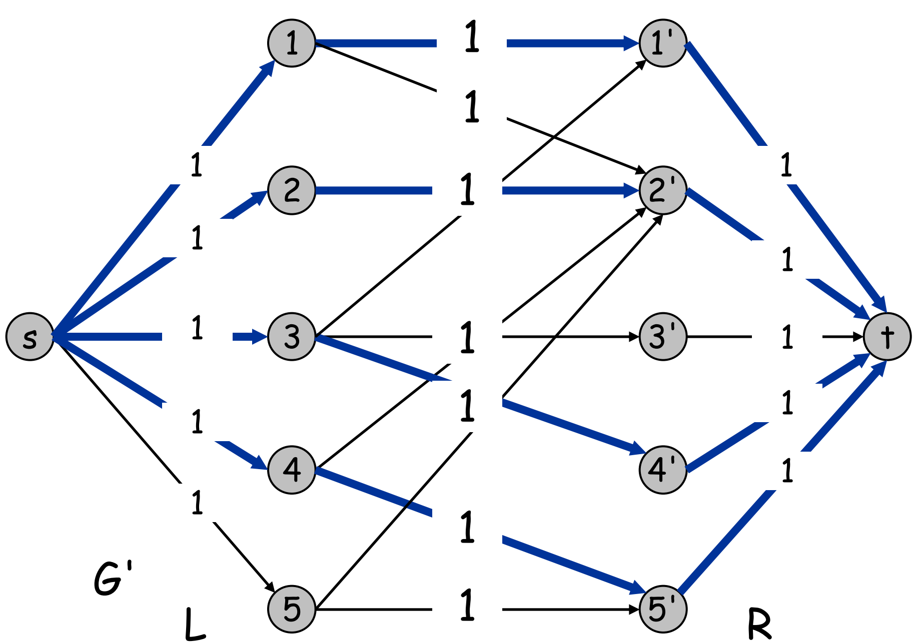


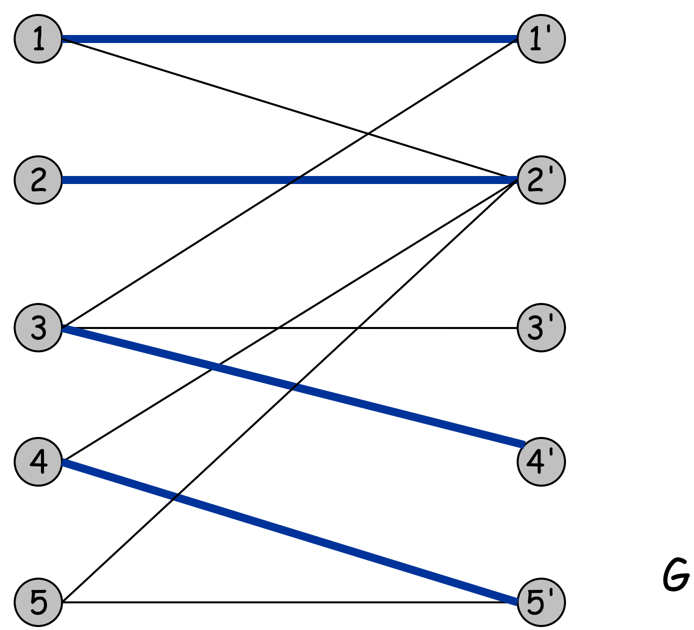
Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Pf. $\geq$

- Let f be a max flow in G' of value k.
- Integrality theorem $\Rightarrow$ k is integral and thus f is 0-1.
- Consider M = set of edges from L to R with f(e) = 1.
  - each node in L and R participates in at most one edge in M (because capacity of either all incoming or outgoing edges is at most 1)
  - |M| = k = v(f): consider flow over cut ($L \cup s$, $R \cup t$) ▪





This proof can be found on page 369.

Perfect Matching

Def. A matching $M \subseteq E$ is perfect if each node appears in exactly one edge in M.

Q*. When does a bipartite graph have a perfect matching? And can we indicate where the problem lies if not?

Structure of bipartite graphs with perfect matchings.

- Clearly we must have $|L| = |R|$.
- What other conditions are *necessary*? (i.e., perfect m. $\Rightarrow$ condition)
- What conditions are *sufficient*? (i.e., condition $\Rightarrow$ perfect m.)

Perfect Matching

Notation. Let S be a subset of nodes, and let N(S) be the set of nodes adjacent to nodes in S.

Q. How many neighbors should each node at least have for a perfect matching?

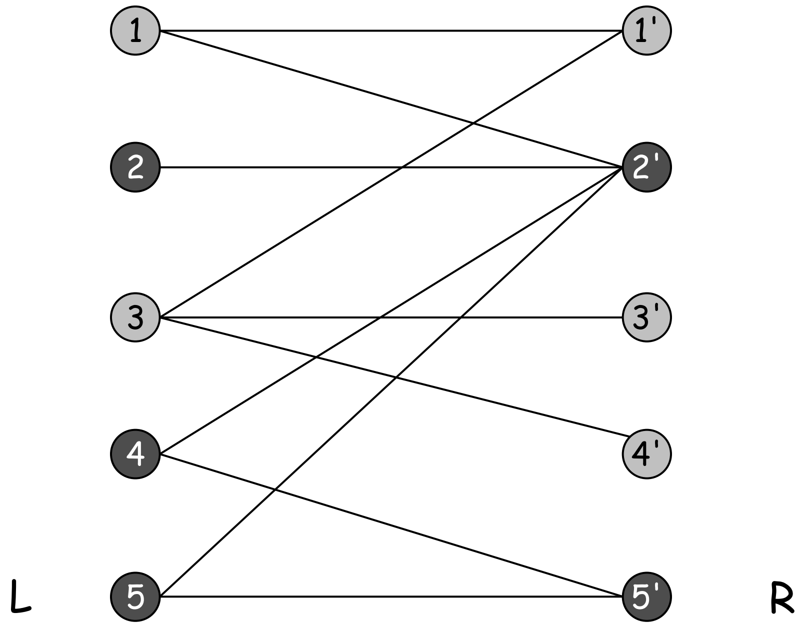
A. At least one.

Q. Generalize this to subsets of nodes S.

Perfect Matching

Notation. Let S be a subset of nodes, and let N(S) be the set of nodes adjacent to nodes in S.

Observation. If a bipartite graph $G = (L \cup R, E)$ has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$. (i.e., necessary condition)
Pf. Each node in S has to be matched to a different node in N(S).



No perfect matching:
S = { 2, 4, 5 }
N(S) = { 2', 5' }.

Marriage Theorem

Marriage Theorem. [Frobenius 1917, Hall 1935]
Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$.
G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. $\Rightarrow$ This was the previous observation.
Pf. $\Leftarrow$ By contrapositive, so to prove:
G has no perfect matching $\Rightarrow$ there is a set $S \subseteq L$ for which $|N(S)| < |S|$

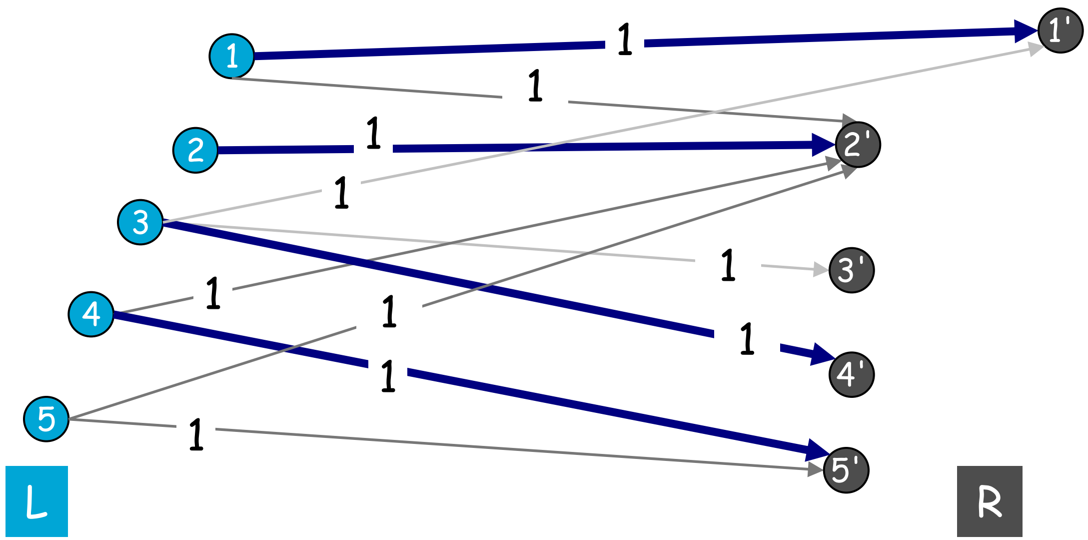
Proof of Marriage Theorem

Pf. $\Leftarrow$ By contrapositive. Suppose G does not have a perfect matching.

Q. Which set $S \subseteq L$ to choose?

- ...

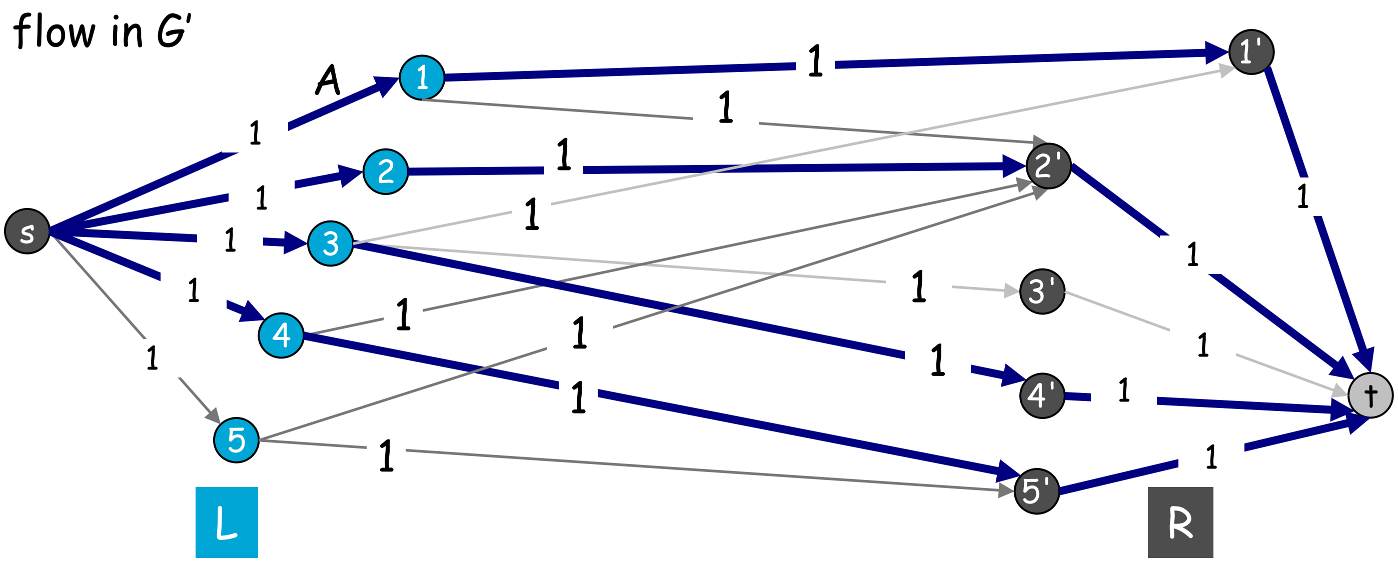
- Then $|N(S)| < |S|$. ▪



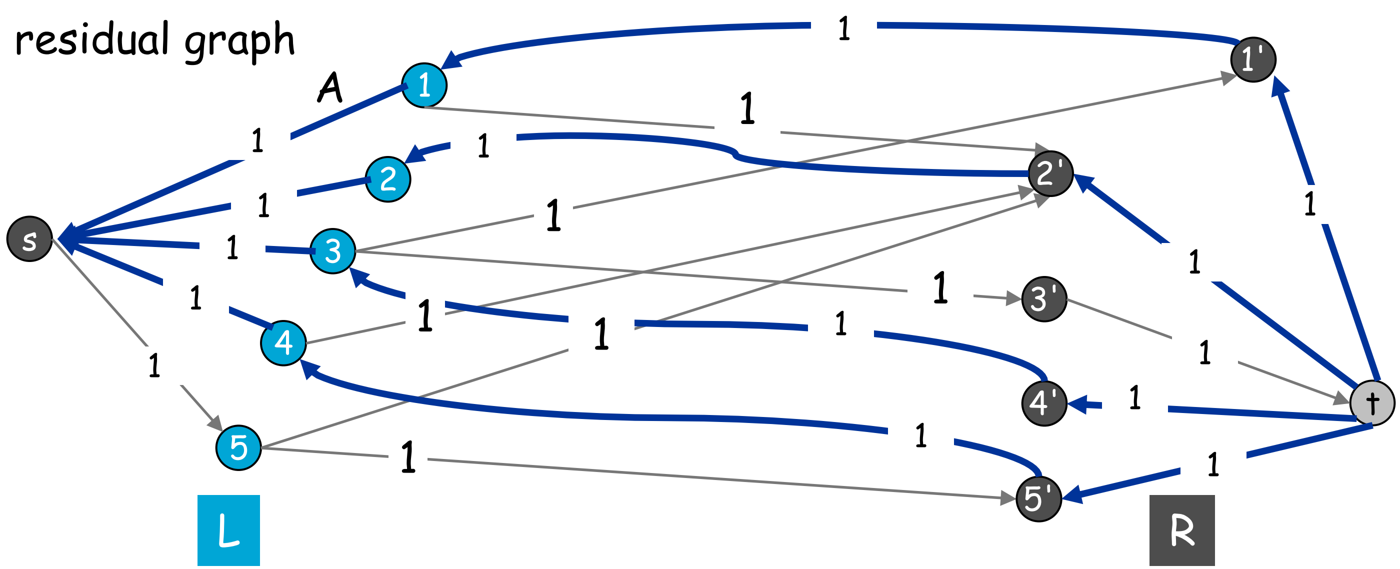
Proof of Marriage Theorem

Pf. $\Leftarrow$ By contrapositive. Suppose G does not have a perfect matching.

- Formulate as a max flow problem and let (A, B) be min cut in G′.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.
- …
- Choose $S = L_A$. Then $|N(S)| < |S|$. ▪



Proof of Marriage Theorem



Define:

min cut (A,B)

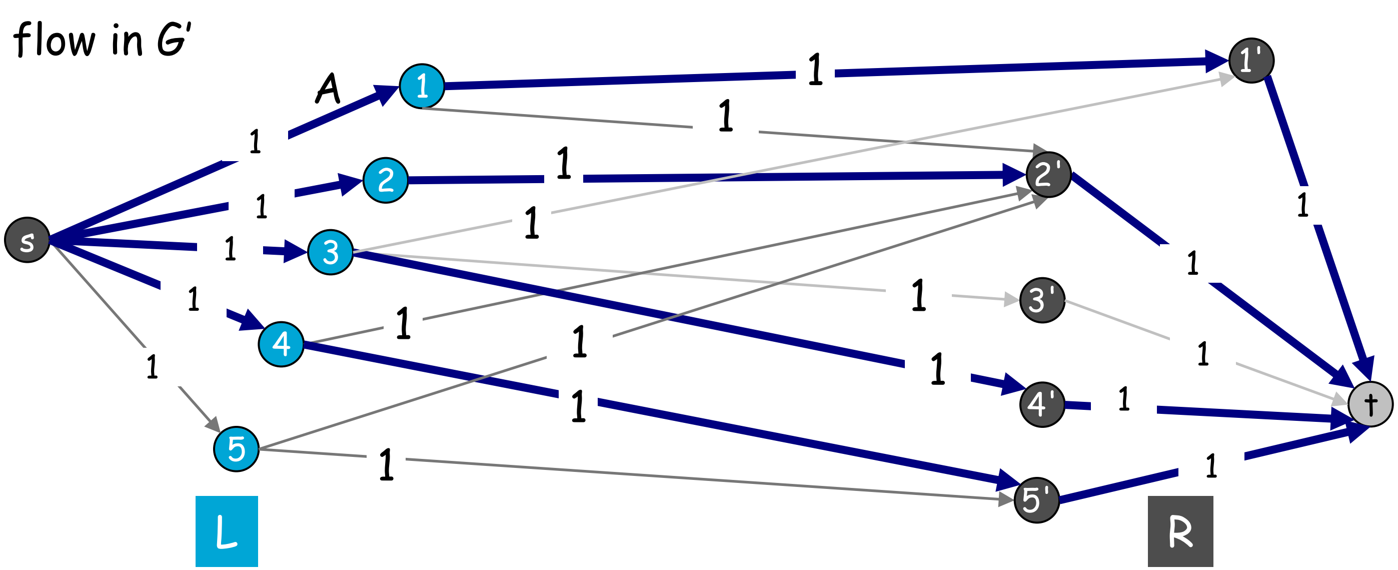
$L_A = L \cap A$

$L_B = L \cap B$

$R_A = R \cap A$

Show:

$|N(L_A)| < |L_A|$



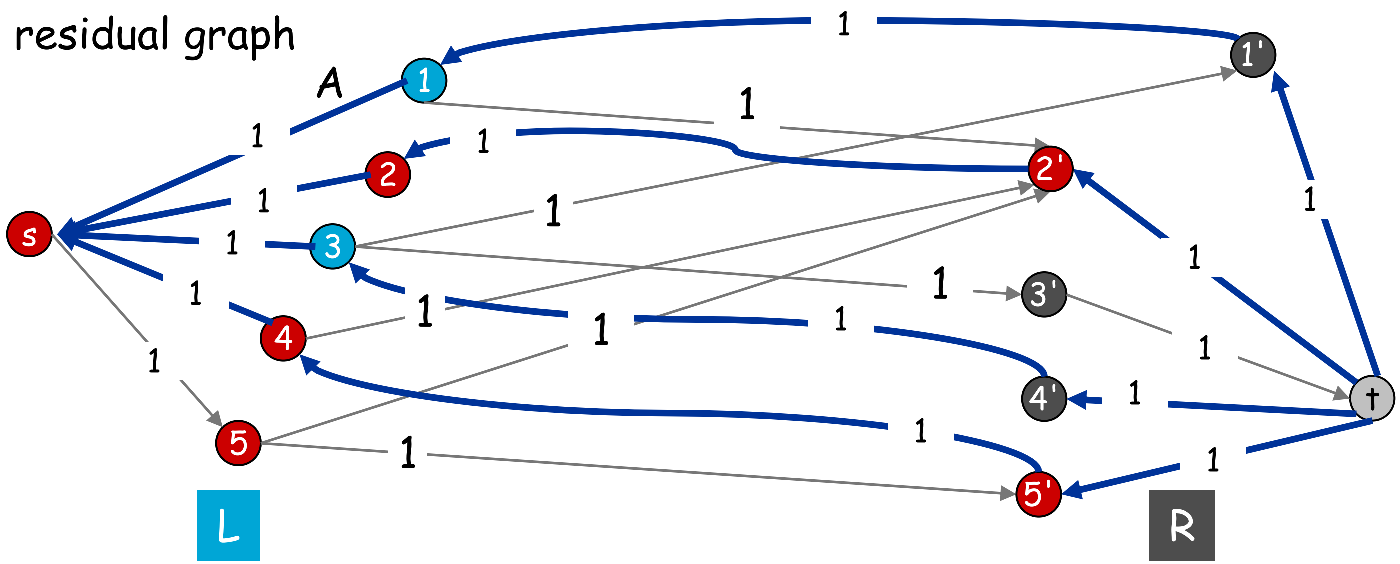
$L_A = \{2, 4, 5\}$

$L_B = \{1, 3\}$

$R_A = \{2', 5'\}$

$N(L_A) = \{2', 5'\}$

Proof of Marriage Theorem



Define:
min cut (A,B)
$L_A = L \cap A$
$L_B = L \cap B$
$R_A = R \cap A$

Show:
$|N(L_A)| < |L_A|$

Prop. No L-R edge (l,r) in min cut (A,B) with l in A and r in B.
Pf.
If there is no flow over (l,r), then r in A.
If there is a flow via (l,r), then l is not directly reachable from s in residual graph, so l is only in A if r in A.

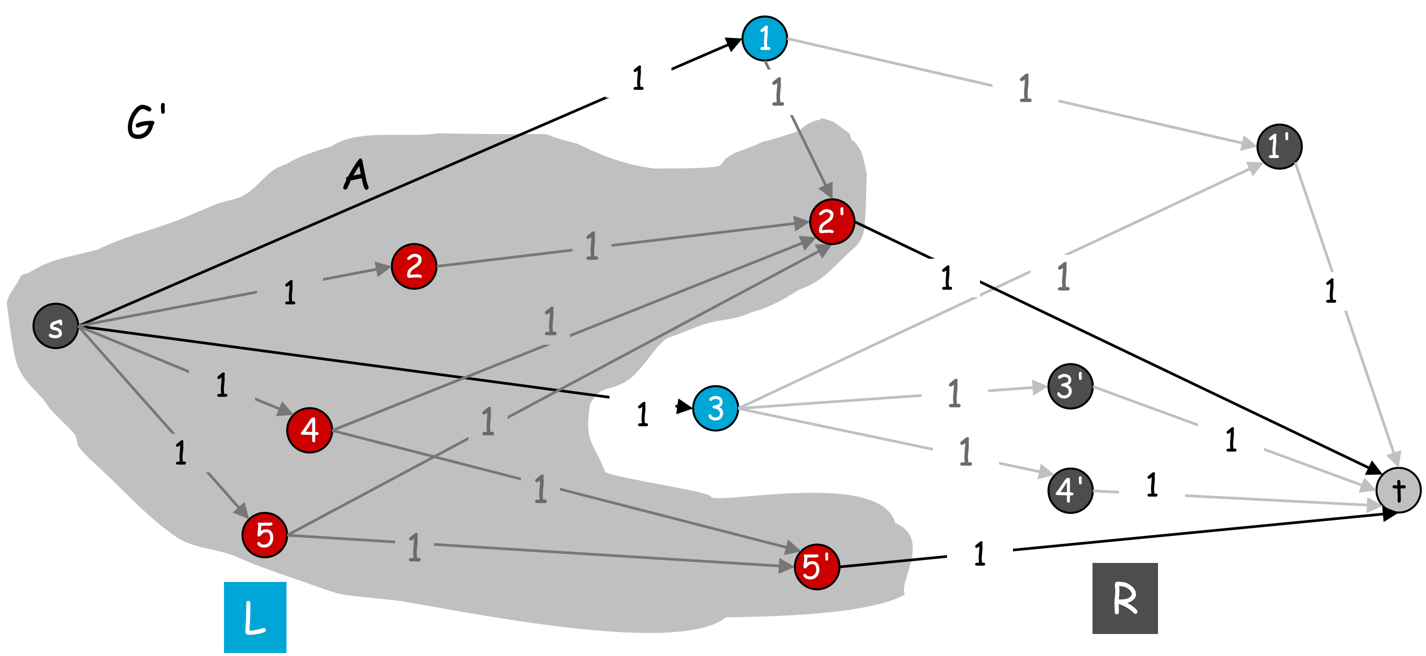
$L_A = \{2, 4, 5\}$
$L_B = \{1, 3\}$
$R_A = \{2', 5'\}$
$N(L_A) = \{2', 5'\}$

NB. This Proposition trivially holds if L-R edges have capacity ∞

Proof of Marriage Theorem

Pf. $\Leftarrow$ By contrapositive. Suppose G does not have a perfect matching.

- Formulate as a max flow problem and let (A, B) be min cut in G′.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.
- ...
- So $|N(L_A)| \leq \; ... = \; ... < \; ... = |L_A|$.
- Choose $S = L_A$. Then $|N(S)| < |S|$. ▪



$L_A = \{2, 4, 5\}$

$L_B = \{1, 3\}$

$R_A = \{2', 5'\}$

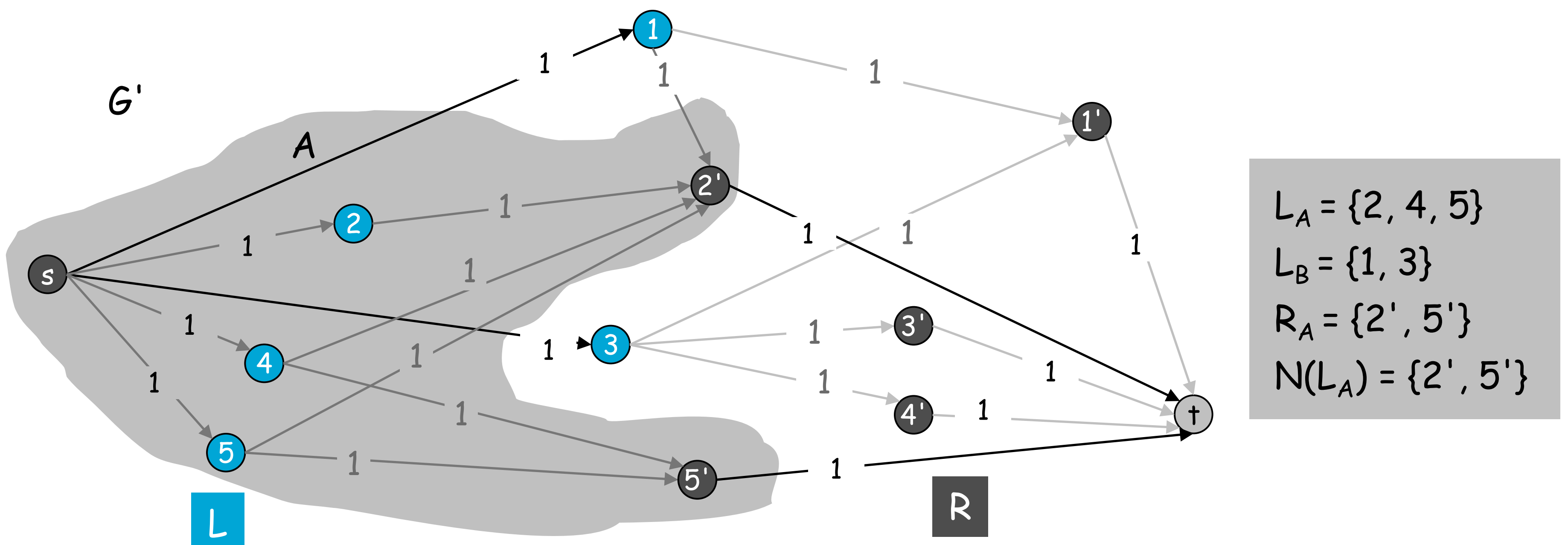
$N(L_A) = \{2', 5'\}$

Proof of Marriage Theorem

Pf. $\Leftarrow$ By contrapositive. Suppose G does not have a perfect matching.

- Formulate as a max flow problem and let (A, B) be min cut in G'.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.

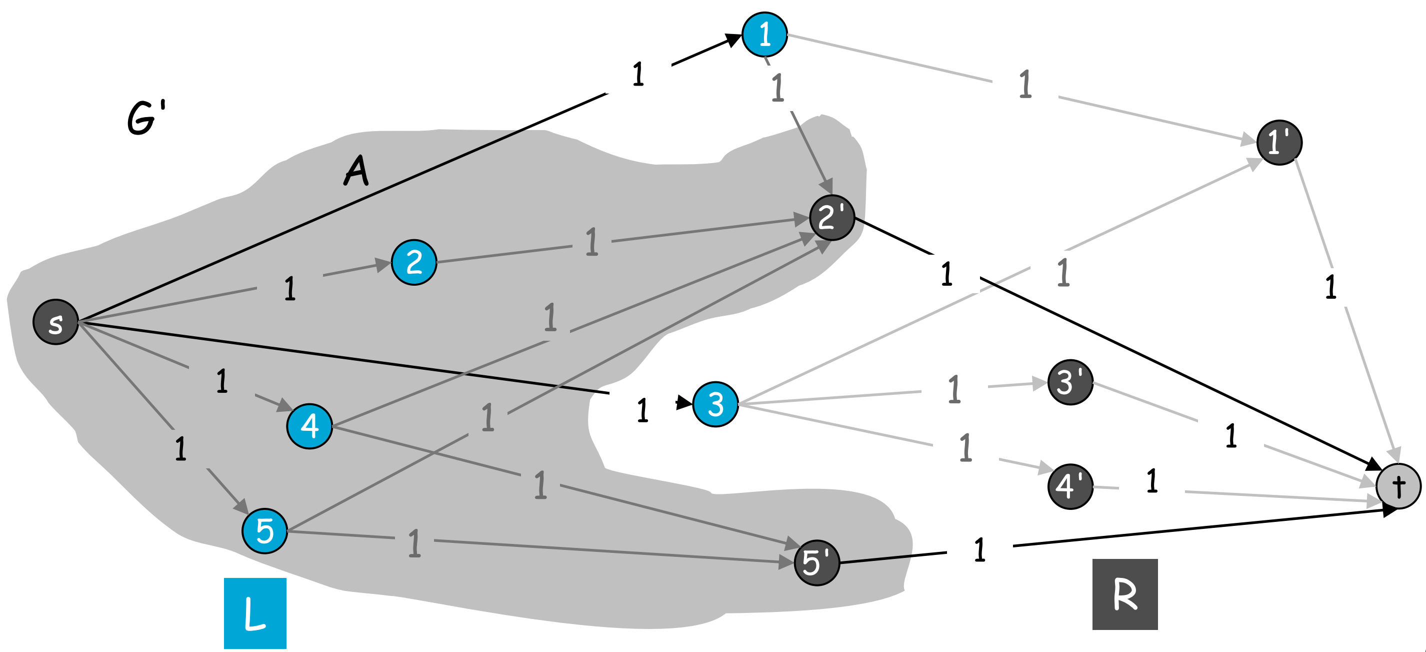
- No L-R edges from A to B in min cut: $cap(A, B) = | L_B | + | R_A |$.
- No L-R edges from A to B in min cut: $N(L_A) \subseteq R_A$.
- So $|N(L_A)| \leq | R_A | = cap(A, B) - | L_B | \ldots < | L | - | L_B | = | L_A |$.
- Choose $S = L_A$. Then $|N(S)| < |S|$. ▪



Proof of Marriage Theorem

Pf. $\Leftarrow$ By contrapositive. Suppose G does not have a perfect matching.

- Formulate as a max flow problem and let (A, B) be min cut in G'.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.
- Not perfect, so $v(f) < |L|$, so by max-flow min-cut, $cap(A, B) < |L|$.
- No L-R edges from A to B in min cut: $cap(A, B) = |L_B| + |R_A|$.
- No L-R edges from A to B in min cut: $N(L_A) \subseteq R_A$.
- So $|N(L_A)| \leq |R_A| = cap(A, B) - |L_B| < |L| - |L_B| = |L_A|$.
- Choose $S = L_A$. Then $|N(S)| < |S|$. ▪



$L_A = \{2, 4, 5\}$
$L_B = \{1, 3\}$
$R_A = \{2', 5'\}$
$N(L_A) = \{2', 5'\}$

This proof can be found on pages 372-373.

Marriage Theorem

Marriage Theorem. [Frobenius 1917, Hall 1935]
Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$.
G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

So if we cannot find perfect matching, we can now also explain why this is.

TU Delft

Bipartite Matching: Running Time

Which max flow algorithm to use for bipartite matching?

- Generic augmenting path: $O(mnc^*) = O(mn)$.
- Capacity scaling: $O(m^2 \log c^*) = O(m^2)$.
- Shortest augmenting path: $O(m\, n^{1/2})$.

Non-bipartite matching.

- Structure of non-bipartite graphs is more complicated, but well-understood. [Tutte-Berge, Edmonds-Galai]
- Blossom algorithm: $O(n^4)$. [Edmonds 1965]
- Best known: $O(m\, n^{1/2})$. [Micali-Vazirani 1980]

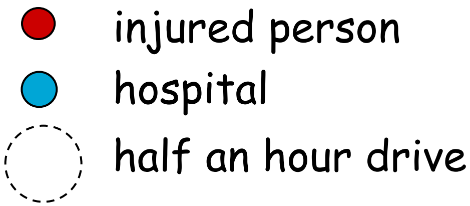
TU Delft

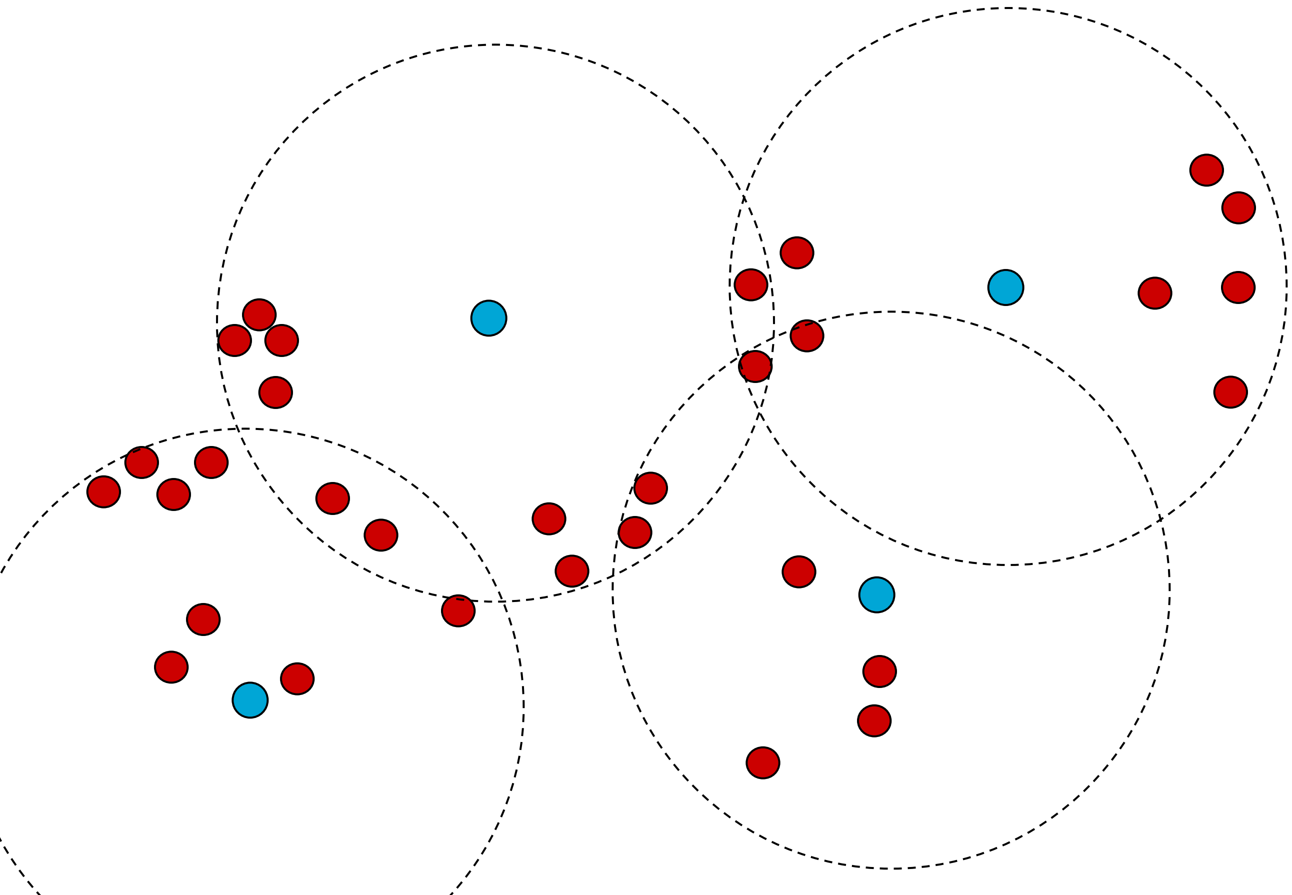
Exam exercise (7.9)

Q. Can n persons be distributed over k hospitals s.t.

- nobody travels more than half an hour, and
- each hospital doesn't get more than $\lceil n/k \rceil$ people?

(In this case: $n = 31$, $k = 4$, so $\lceil n/k \rceil = 8$.)





TU Delft