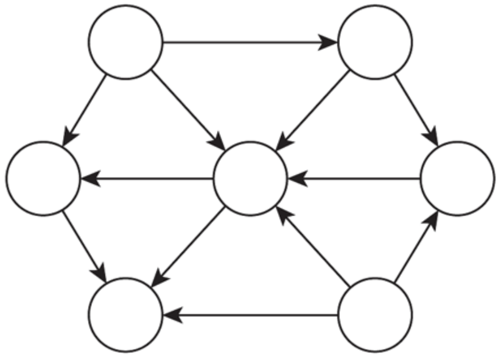
3.5 Connectivity in Directed Graphs

Directed Graphs

Directed graph. G = (V, E)

Edge (u, v) goes from node u to node v.



Ex. Web graph - hyperlink points from one web page to another.

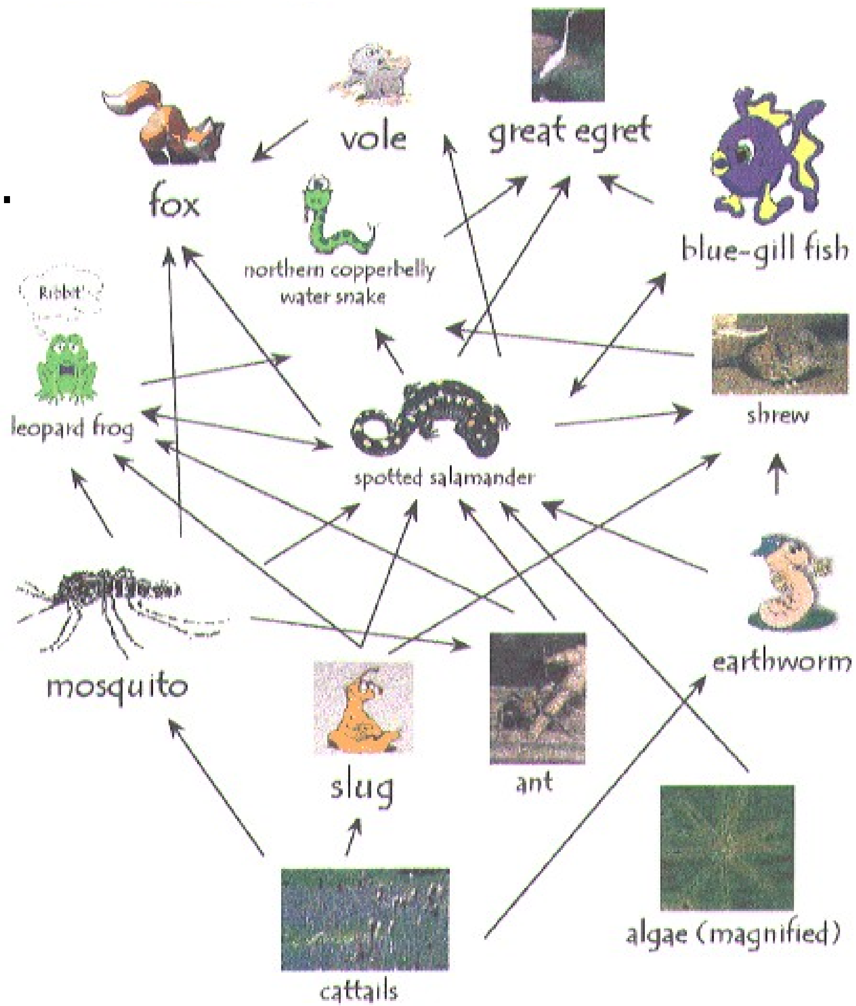
Directedness of graph is crucial.

Modern web search engines exploit hyperlink structure to rank web pages by importance.

Ecological Food Web

Food web graph.

- Node = species.
- Edge = from prey to predator.



Reference: http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.giff

(Directed) Graph Search

Problems in directed graphs…

Directed reachability. Given a node s, find all nodes reachable from s.

Directed s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

(Directed) Graph search. BFS extends naturally to directed graphs.

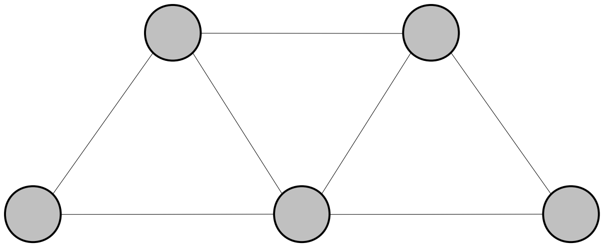
Web crawler. Start from web page s. Find all web pages linked from s, either directly or indirectly.

Strong Connectivity

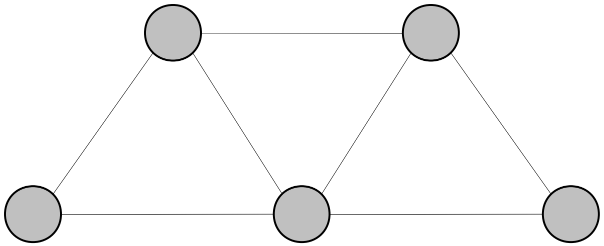
Def. Node u and v are mutually reachable if there is a path from u to v and also a path from v to u.

Def. A graph is strongly connected if every pair of nodes is mutually reachable.

Q. Which graph is strongly connected?



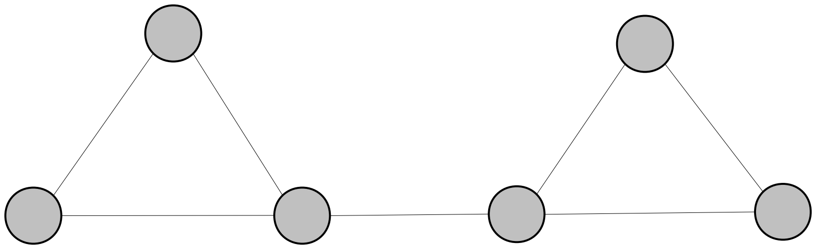
G_1



G_2

Strong Connectivity

Q. Is this graph strongly connected?



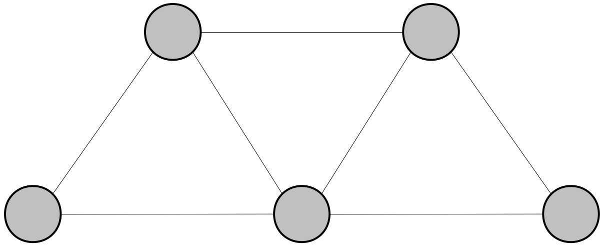
Ex. Web of trust (eg PGP-key ring):

I trust some friends by signing their keys.

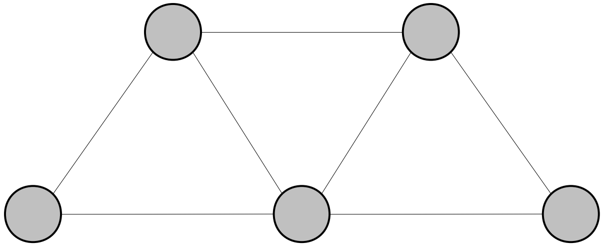
If web of trust is strongly connected → I can trust everyone and everyone trusts me!

Strong Connectivity: Algorithm

Q. How to determine if G is strongly connected, in O(m + n) time? (1 min)



strongly connected



not strongly connected

Strong Connectivity

Q. How to determine if G is strongly connected, in O(m + n) time?

Lemma. Let s be any node.

G is strongly connected ⇔ every node is reachable from s, and s is reachable from every node.

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Pf. ⇒

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Strong Connectivity

Q. How to determine if G is strongly connected, in $O(m + n)$ time?

Lemma. Let s be any node.
G is strongly connected $\Leftrightarrow$ every node is reachable from s, and s is reachable from every node.

Pf. $\Rightarrow$: Suppose strongly connected.
To prove: every node reachable from s, and s reachable from every node.
Q. Why does this hold?

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Q. Why does this hold?
A. Follows from definition of strongly connected graph (every pair of nodes is mutually reachable).

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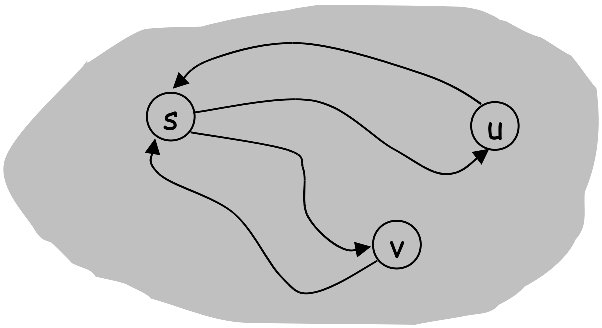
Strong Connectivity

Q. How to determine if G is strongly connected, in $O(m + n)$ time?

Lemma. Let s be any node.
G is strongly connected ⇔ every node is reachable from s, and s is reachable from every node.
Pf. ⇒ Follows from definition (every pair of nodes is mutually reachable).
Pf. ⇐ Suppose every node reachable from s, s reachable from every node.
To prove: G is strongly connected.
To prove: Every two nodes are mutually reachable.
Let two nodes u and v be given.
Q. Why is u reachable from v? And v reachable from u?



Strong Connectivity

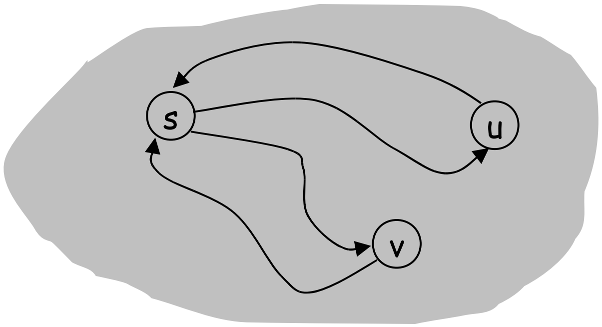
Q. How to determine if G is strongly connected, in O(m + n) time?

Lemma. Let s be any node.
G is strongly connected ⇔ every node is reachable from s, and s is reachable from every node.

Pf. ⇒ Follows from definition (every pair of nodes is mutually reachable).

Pf. ⇐ Path from u to v: concatenate u-s path with s-v path.
Path from v to u: concatenate v-s path with s-u path. ▪

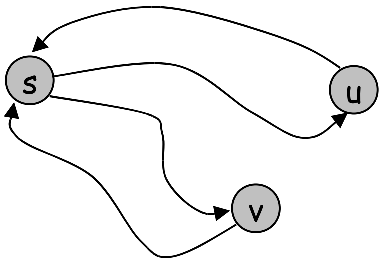
ok if paths overlap



Strong Connectivity

Lemma. Let s be any node.
G is strongly connected $\Leftrightarrow$ every node is reachable from s, and s is reachable from every node.

Q. How to determine if G is strongly connected, in O(m + n) time?



Strong Connectivity: Algorithm

Theorem. Can determine if G is strongly connected in O(m + n) time.

Pf.

- Pick any node s.
- Run BFS from s in G.
- Run BFS from s in G^{rev}. ← reverse orientation of every edge in G
- Return true iff all nodes reached in both BFS executions.
- Correctness follows immediately from lemma. ▪

strongly connected

not strongly connected