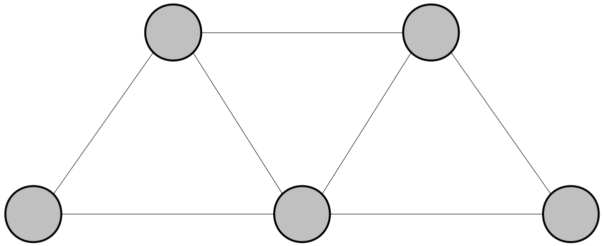
3.6 DAGs and Topological Ordering

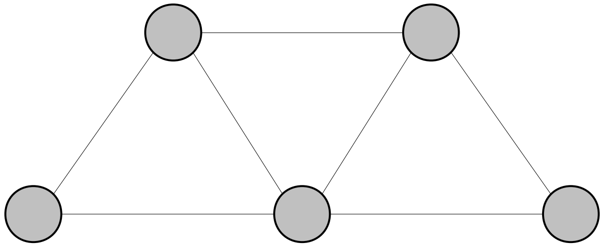
Connectivity in Directed Graphs

Q. How to determine if G is strongly connected, in $O(m + n)$ time?

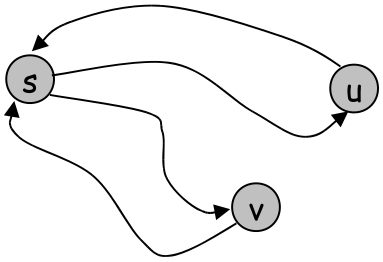


Connectivity in Directed Graphs

Q. How to determine if G is strongly connected, in $O(m + n)$ time?

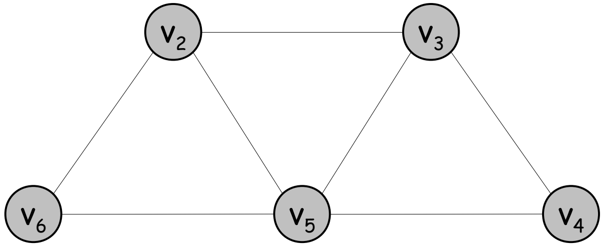


A. BFS(G,s) and BFS(G^{rev},s)



Connectivity in Directed Graphs

Q. Does this directed graph contain a cycle?

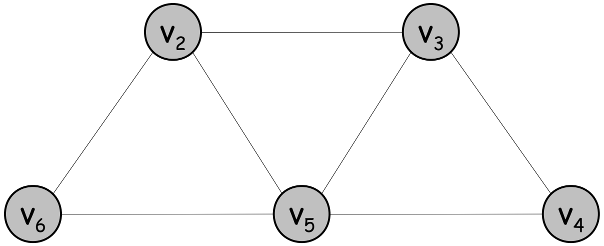


Q. How to define a directed cycle?

Connectivity in Directed Graphs

Def. A path in a *directed* graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by a *directed* edge (v_i, v_{i+1}) in E.

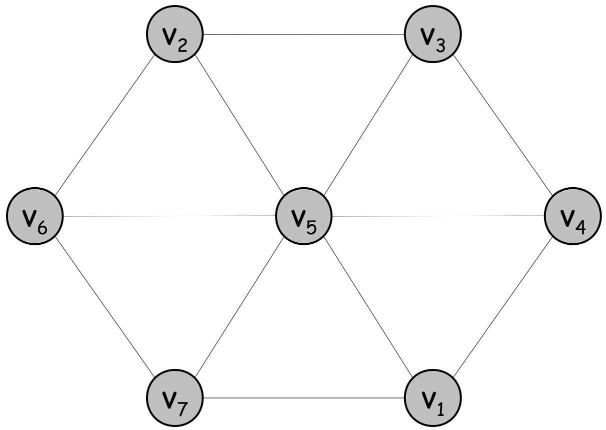
Def. A *directed* cycle in a directed graph G is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in G in which $v_1 = v_k$, $k > 2$, and the first $k-1$ nodes are all distinct.



Directed Acyclic Graphs

Def. A DAG is a directed graph that contains no directed cycles.

Ex. Precedence constraints: edge (v_i, v_j) means v_i must precede v_j.
Often used in planning algorithms.

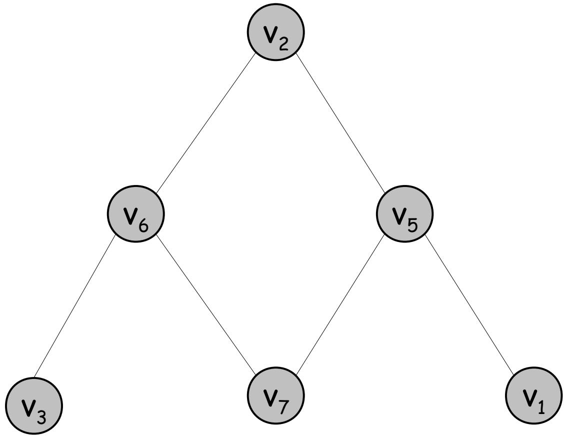


a DAG

Trees and DAGs

Q. Is a tree with directed edges a DAG?

Q. Is a DAG a tree?

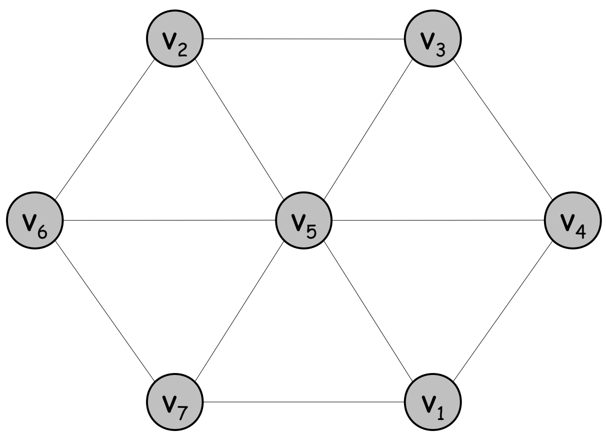


Topological Ordering

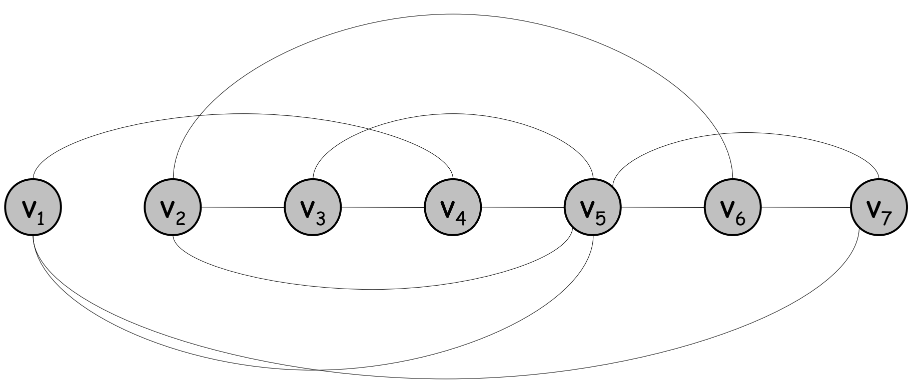
Def. A topological ordering of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.

[all arrows point to the right]

Ex. Check whether a set of precedence relations between (birth and death) events is chronologically consistent.



a DAG



a topological ordering

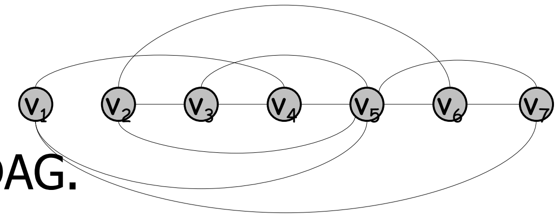
Precedence Constraints

Precedence constraints. Edge (v_i, v_j) means task v_i must occur before v_j.

Applications.

- Course prerequisite graph: course v_i must be taken before v_j.
- Compilation: module v_i must be compiled before v_j.
- Pipeline of computing jobs: output of job v_i needed to determine input of job v_j.
- Software development planning: some modules must be written before others (with durations $\rightarrow$ critical path)

Q. How can we find out whether a topological order exists?

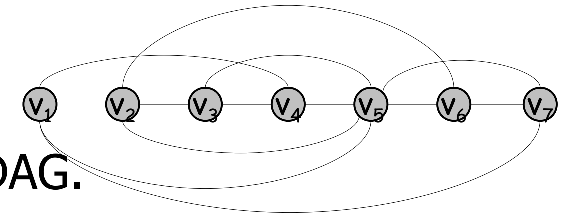


Lemma 3.18. If G has a topological order, then G is a DAG.

Pf.

Q. How to prove an "if …, then…"?

Directed Acyclic Graphs



Lemma 3.18. If G has a topological order, then G is a DAG.

Pf.

Suppose that G has a topological order $v_1, \ldots, v_n$.

Q. What proof technique to use now?

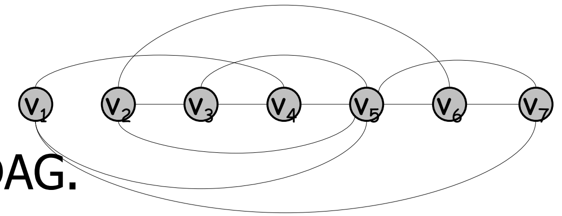
...

So G is a DAG. ▪

the supposed topological order: $v_1, \ldots, v_n$

TU Delft

Directed Acyclic Graphs



Lemma 3.18. If G has a topological order, then G is a DAG.

Pf. (by contradiction)

Suppose that G has a topological order $v_1, \ldots, v_n$.

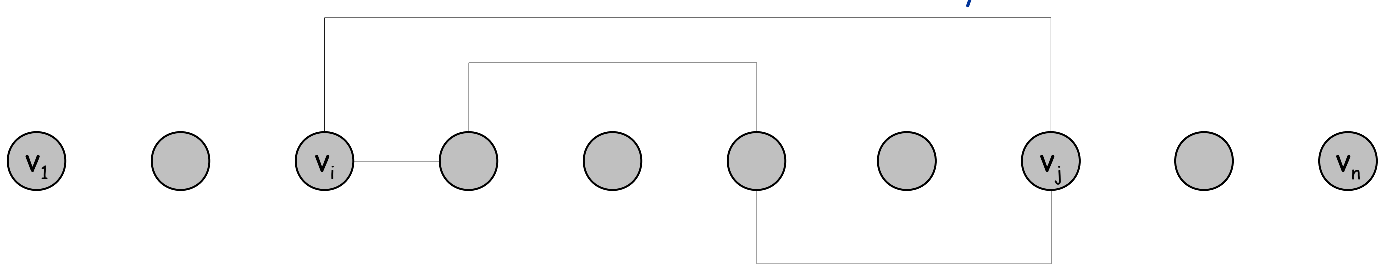
Suppose that G has a directed cycle C.

Q. How to derive a contradiction?

...

Contradiction.

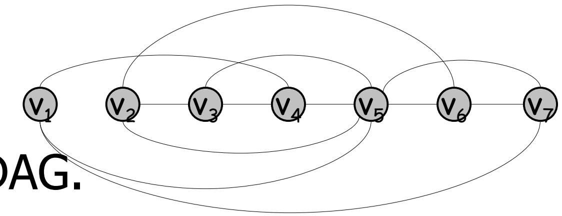
So G has no cycle. So G is a DAG. ▪



the supposed topological order: $v_1, \ldots, v_n$

TU Delft

Directed Acyclic Graphs



Lemma 3.18. If G has a topological order, then G is a DAG.

Pf. (by contradiction)

Suppose that G has a topological order $v_1, \ldots, v_n$.

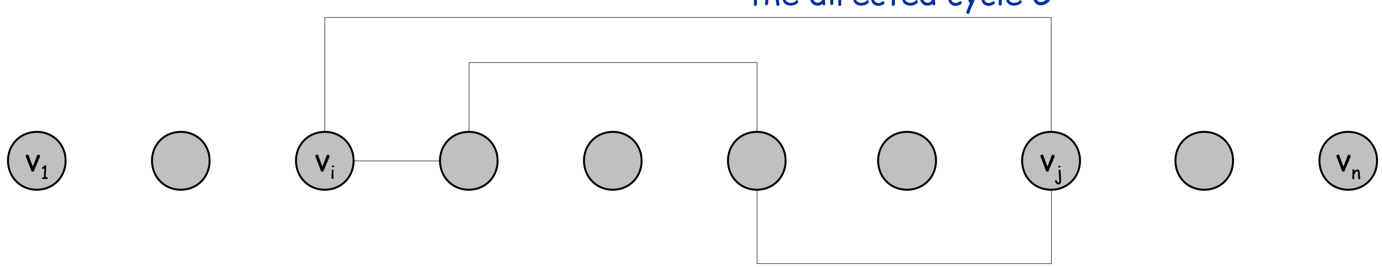
Suppose that G has a directed cycle C.

Let v_i be the lowest-indexed node in C, and let v_j be the node just before v_i in C; thus (v_j, v_i) is an edge.

By our choice of i, we have $i < j$.

On the other hand, since (v_j, v_i) is an edge and $v_1, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction.

So G has no cycle. So G is a DAG. ▪



the supposed topological order: $v_1, \ldots, v_n$

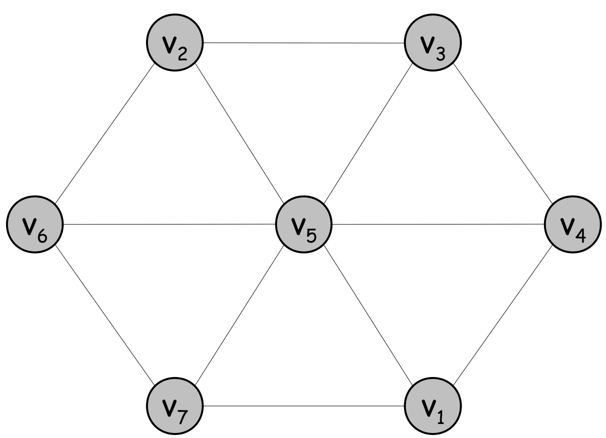
This proof can be found on page 101.

Directed Acyclic Graphs

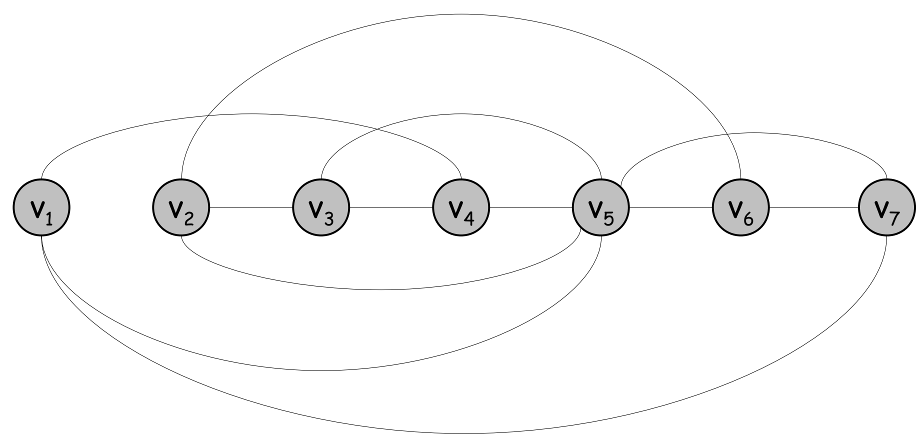
Lemma 3.18. If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?

Q. If so, how do we compute one?



a DAG



a topological ordering

Directed Acyclic Graphs

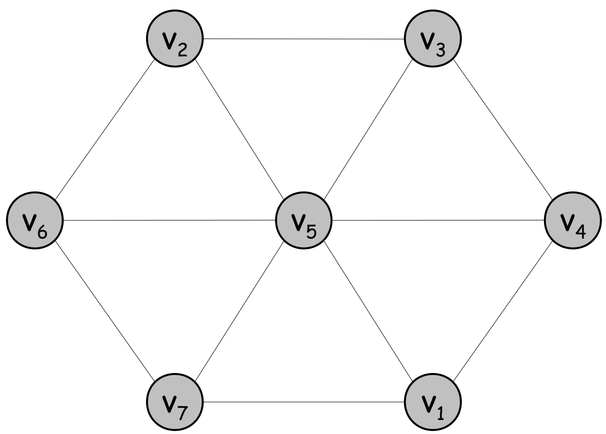
Lemma 3.18. If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?

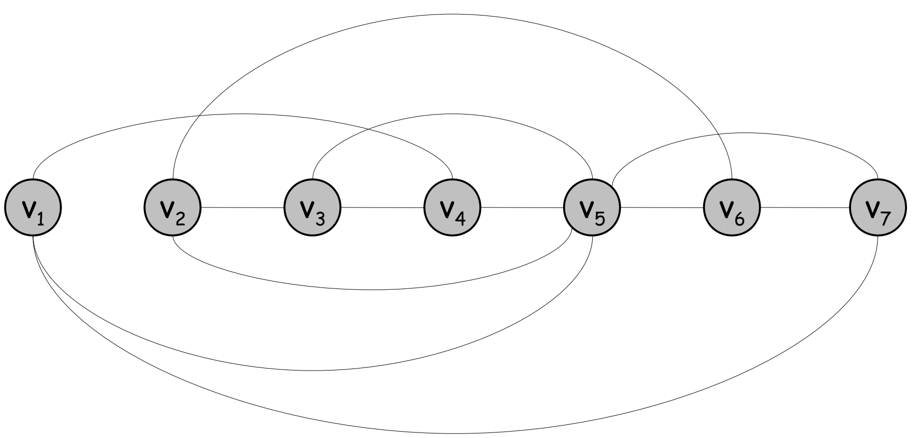
Q. If so, how do we compute one?

Idea: start with node with no incoming edge

Q. Does such a node always exist in a DAG?



a DAG



a topological ordering

Lemma 3.19. If G is a DAG, then G has a node with no incoming edges.

Pf.

Q. How to prove an “if …, then…”?

Directed Acyclic Graphs

Lemma 3.19. If G is a DAG, then G has a node with no incoming edges.

Pf.

Suppose that G is a DAG.

Q. What proof technique to use now?

…

So there must be a node with no incoming edges. ▪

Directed Acyclic Graphs

Lemma 3.19. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

Suppose that G is a DAG.

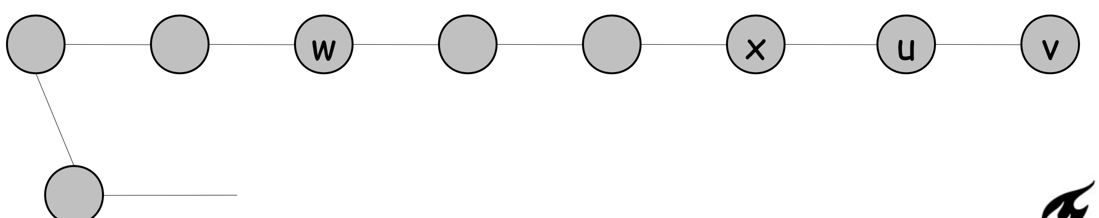
Suppose every node has at least one incoming edge.

Q. How to derive a contradiction?

…

Contradiction.

So there must be a node with no incoming edges. ▪



Lemma 3.19. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

Suppose that G is a DAG.

Suppose every node has at least one incoming edge.

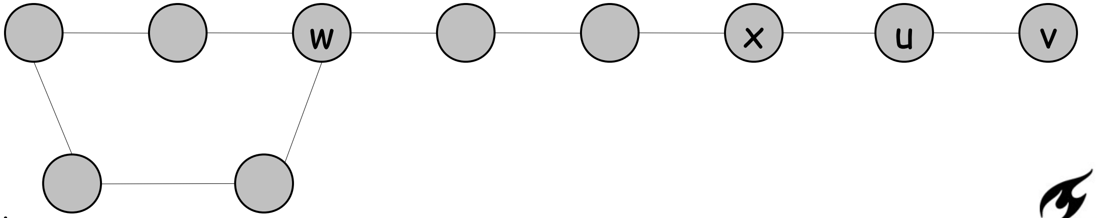
Pick any node v, and follow edges backward from v.

Repeat until we visit a node, say w, twice.

Let C denote the sequence of nodes encountered between successive visits to w. C is a cycle.

Contradiction.

So there must be a node with no incoming edges. ▪



This proof can be found on page 102.

Lemma 3.20. If G is a DAG, then G has a topological ordering.

Pf.

Idea of proof: add nodes without incoming edge one by one to topological ordering

Q. What proof technique can reflect this iterative procedure?

Lemma 3.20. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

Lemma 3.20. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

Base case:

Hypothesis:

Step:

Directed Acyclic Graphs

Lemma 3.20. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

Base case: true if n = 1, because topological ordering is G.

Hypothesis: If G is DAG of size ≤ n, then G has a topological ordering.

Step:

Q. What to prove here?

Directed Acyclic Graphs

Lemma 3.20. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

Base case: true if n = 1, because topological ordering is G.

Hypothesis: If G is DAG of size ≤ n, then G has a topological ordering.

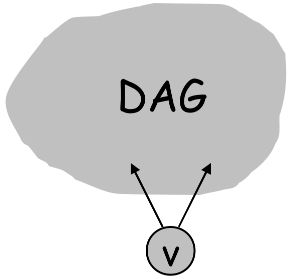
Step: Given DAG G′ with n+1 nodes.

… using inductive hypothesis (**IH**)

Create topological ordering for G:

- …

By induction the lemma is proven. ▪



Lemma 3.20. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

Base case: true if n = 1, because topological ordering is G.

Hypothesis: If G' is DAG of size ≤ n, then G' has topological ordering.

Step: Given DAG G with n+1 nodes.

Find a node v with no incoming edges.

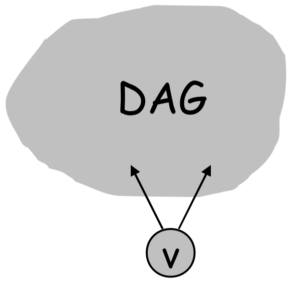
G - { v } is a DAG, since deleting v cannot create cycles.

By inductive hypothesis (**IH**), G - { v } has a topological ordering.

Create topological ordering for G:

- Place v first; then append topological ordering of G - { v }.
- This is valid since v has no incoming edges.

By induction the lemma is proven. ▪



This proof can be found on page 102.

Directed Acyclic Graphs

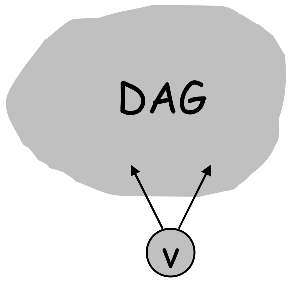
Q. Give an algorithm to return a topological ordering.

To compute a topological ordering of G:

Find a node v with no incoming edges and order it first

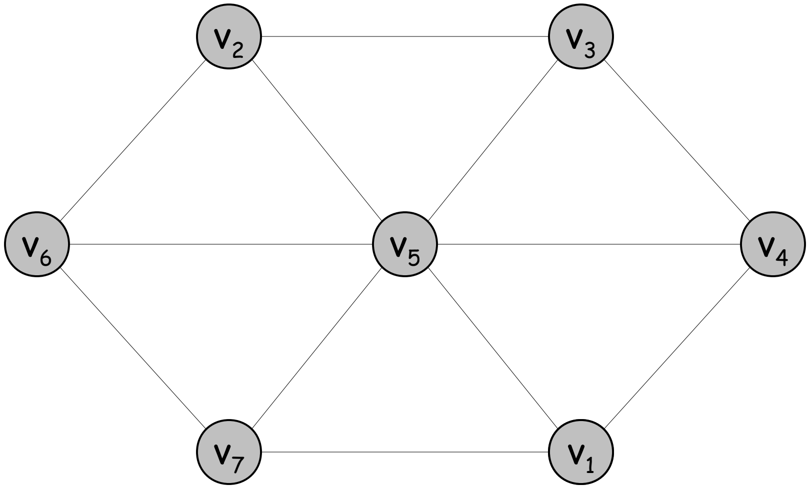
Delete v from G

Recursively compute a topological ordering of $G - \{v\}$
  and append this order after v



Topological Sorting Algorithm

Q. Give a topological sort of the following graph.



Topological Sorting Algorithm: Running Time

Theorem. Algorithm finds a topological order in $O(m + n)$ time.

Q. How to implement? Which information do you need to maintain?

Topological Sorting Algorithm: Running Time

Theorem. Algorithm finds a topological order in O(m + n) time.

Pf.

Maintain the following information:

- `count[w]` = remaining number of incoming edges in w
- S = set of remaining nodes with no incoming edges

Initialization: O(m + n) via single scan through graph.

Update: to delete v

- remove v from S
- decrement `count[w]` for all edges from v to w, and add w to S if `count[w]` hits 0
- this is O(1) per edge ▪