7.6 Disjoint Paths

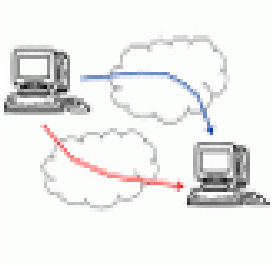
- Disjoint Paths Problem
  - Max-flow/min-cut formulation (inc. proof)
- Network connectivity problem
  - Max-flow/min-cut formulation (inc. proof)

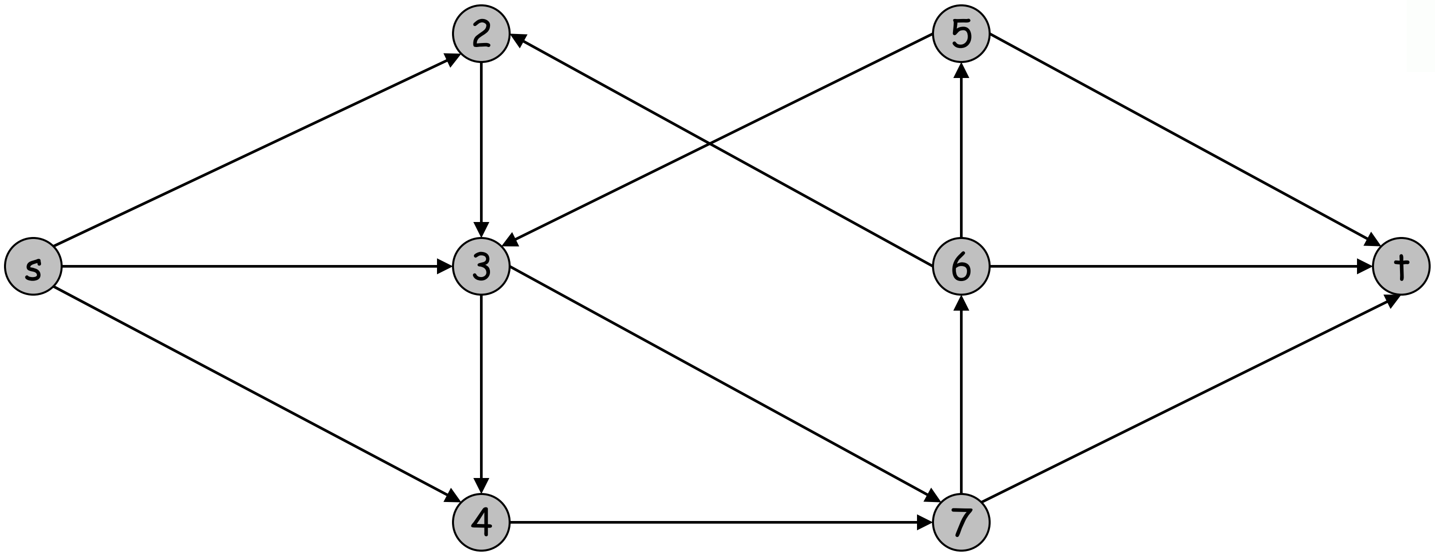
Edge Disjoint Paths

Disjoint path problem. Given a digraph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths.

Def. Two paths are edge-disjoint if they have no edge in common.

Q. How many edge disjoint paths are possible here? (1, 2, or 3)





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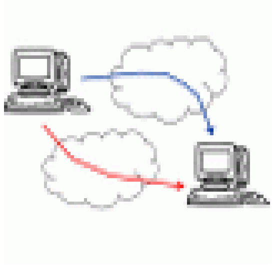
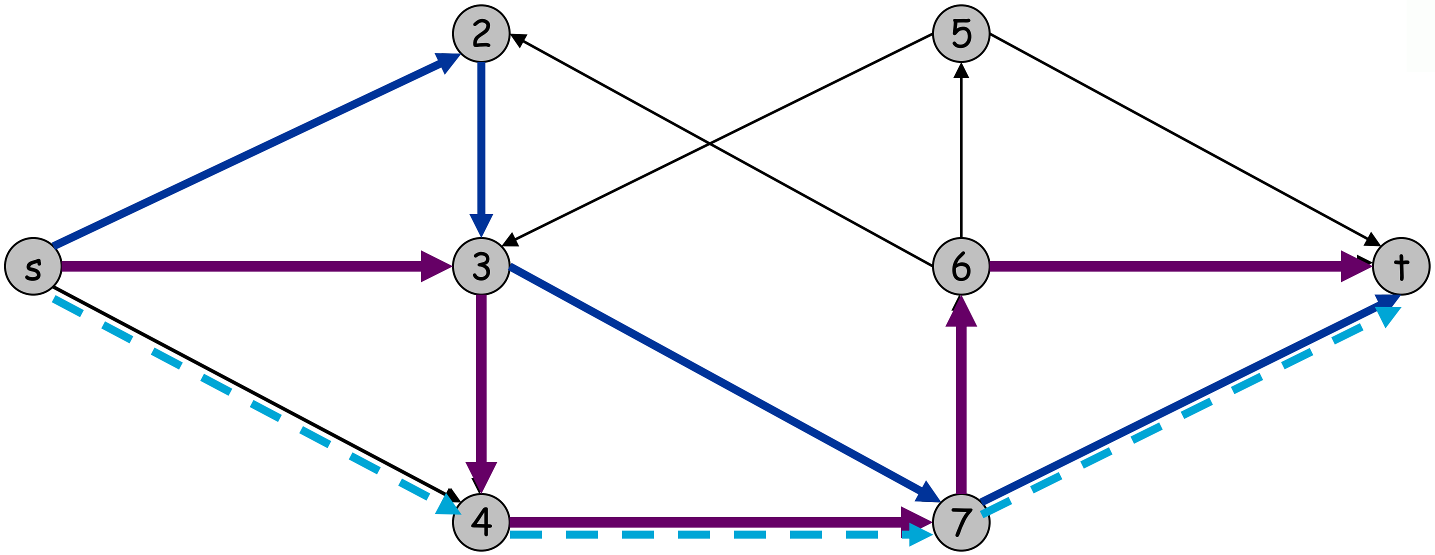
A. Two.



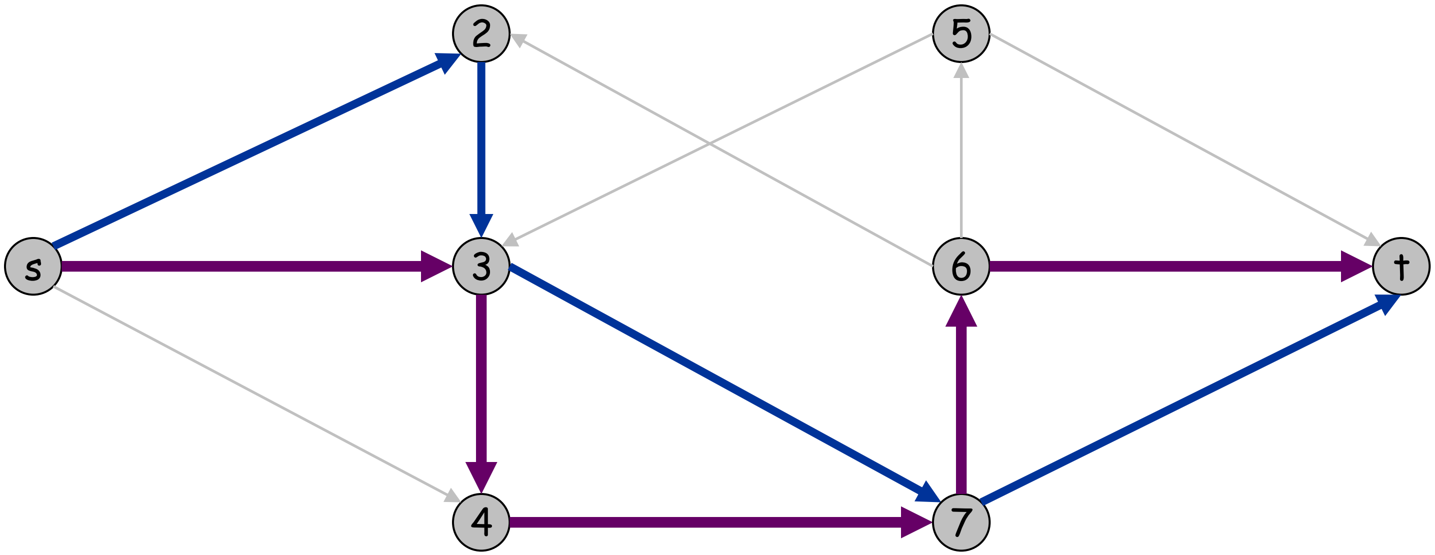


Edge Disjoint Paths

Disjoint path problem. Given a digraph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths.

Def. Two paths are edge-disjoint if they have no edge in common.

Q. How can we formulate this as a maximum flow problem? (1 min)



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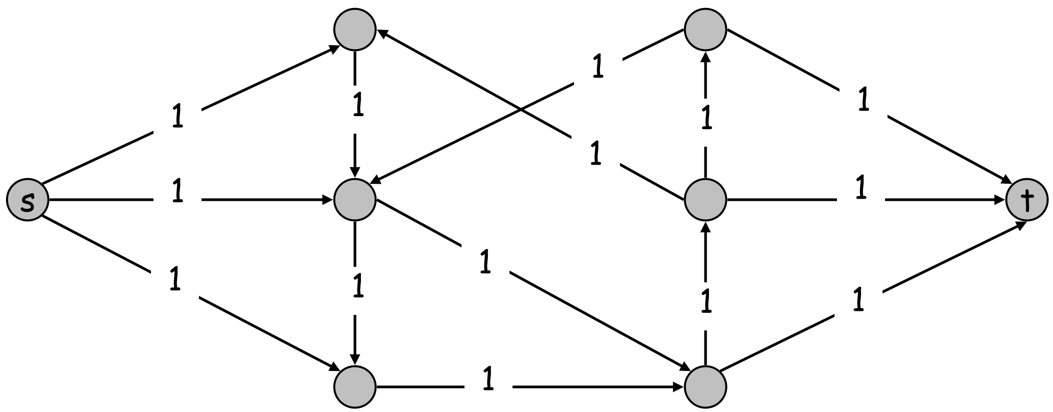
Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



Q. How to compute the max number of edge-disjoint s-t paths?

Edge Disjoint Paths

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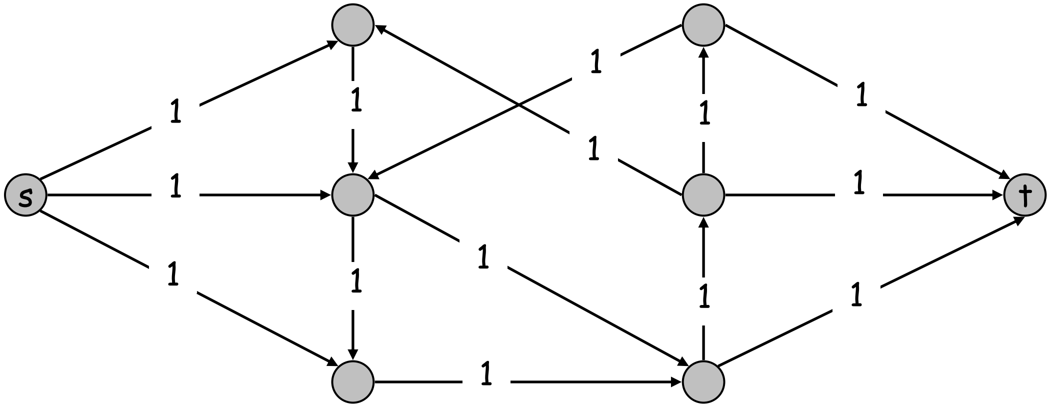


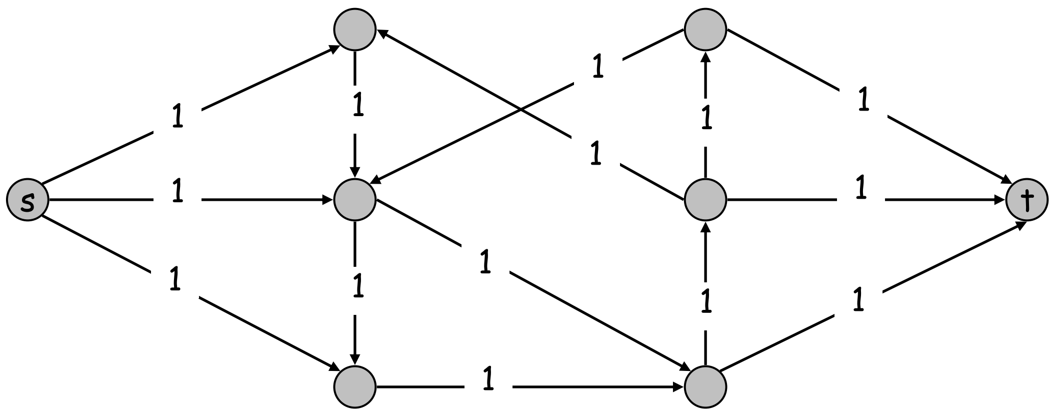
Q. How to compute the max number of edge-disjoint s-t paths?

Theorem. Max number edge-disjoint s-t paths = max flow value.

Pf.

Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



Q. How to compute the max number of edge-disjoint s-t paths?

Theorem. Max number edge-disjoint s-t paths = max flow value.

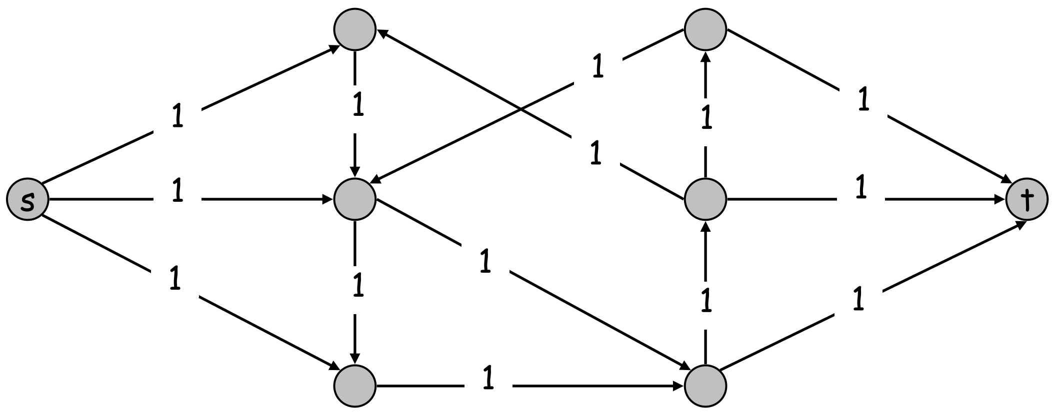
Pf. $\leq$

- Suppose there are max k edge-disjoint s-t paths $P_1, \ldots, P_k$.
- Set f(e) =

- f is less than or equal than the maximum flow. ▪

Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



Q. How to compute the max number of edge-disjoint s-t paths?

Theorem. Max number edge-disjoint s-t paths = max flow value.

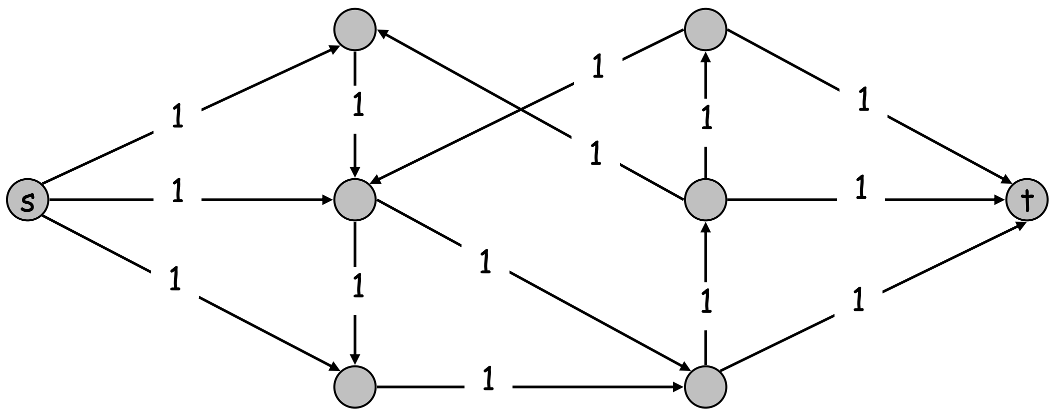
Pf. ≤

- Suppose there are max k edge-disjoint s-t paths $P_1, \ldots, P_k$.
- Set f(e) = 1 if e participates in some path P_i ; else set f(e) = 0.
- f is a valid flow.
- Since s-t paths are edge-disjoint, f is a flow of value k.
- f is less than or equal than the maximum flow. ▪

This proof can be found on pages 374-376.

Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



Theorem. Max number edge-disjoint s-t paths = max flow value.

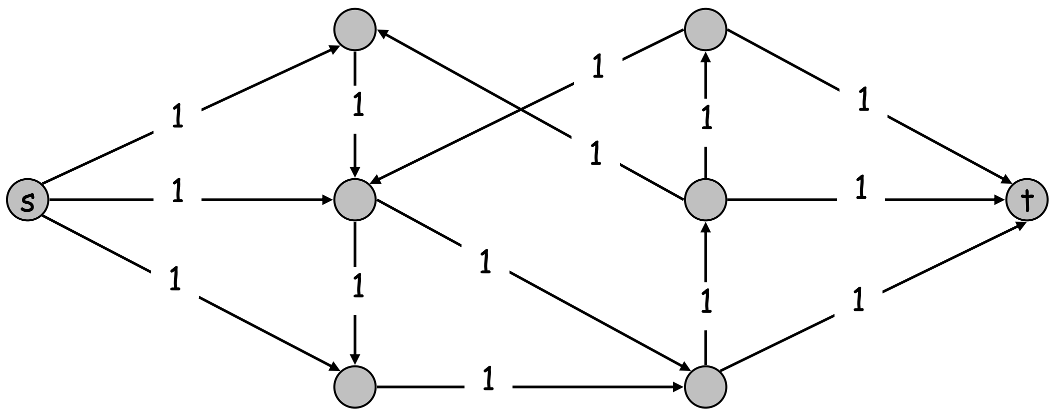
Pf. $\geq$

- Suppose max flow value is k.

- Produces k (not necessarily simple) edge-disjoint paths. ▪

Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



Theorem. Max number edge-disjoint s-t paths = max flow value.

Pf. $\geq$

- Suppose max flow value is k.
- Integrality theorem $\Rightarrow$ there exists 0-1 flow f of value k.
- Construct paths as follows: Consider edge (s, u) with f(s, u) = 1.
  - by conservation, there exists an edge (u, v) with f(u, v) = 1
  - continue until reach t, always choosing a new edge
- Produces k (not necessarily simple) edge-disjoint paths. ▪

can eliminate cycles to get simple paths if desired

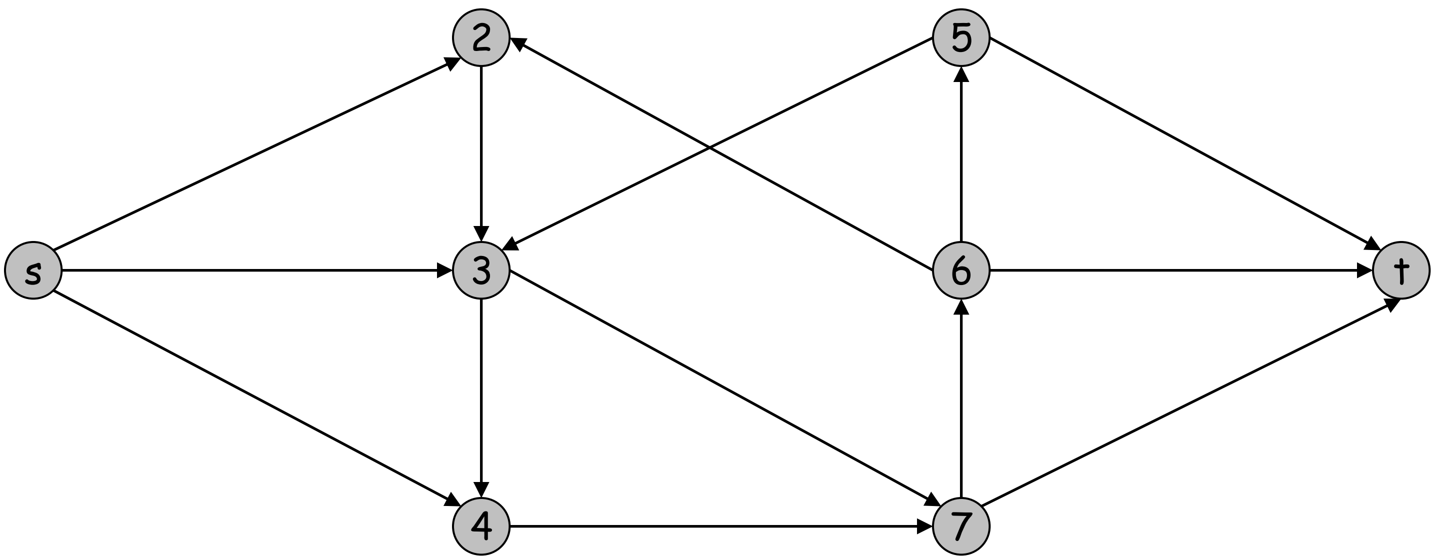
This proof can be found on pages 374-376.

Network Connectivity

Network connectivity problem. Given a digraph G = (V, E) and two nodes s and t, find min number of edges whose removal disconnects t from s.

Def. A set of edges $F \subseteq E$ **disconnects t from s** if all s-t paths use at least one edge in F.

Q. Which edges to remove here?



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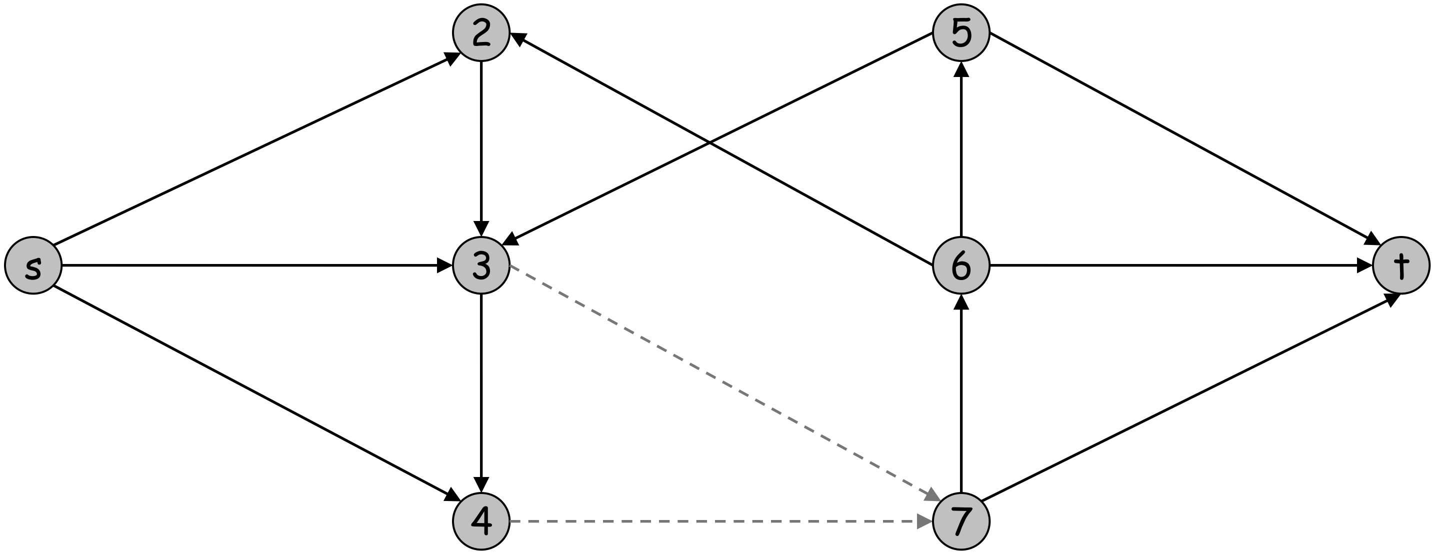
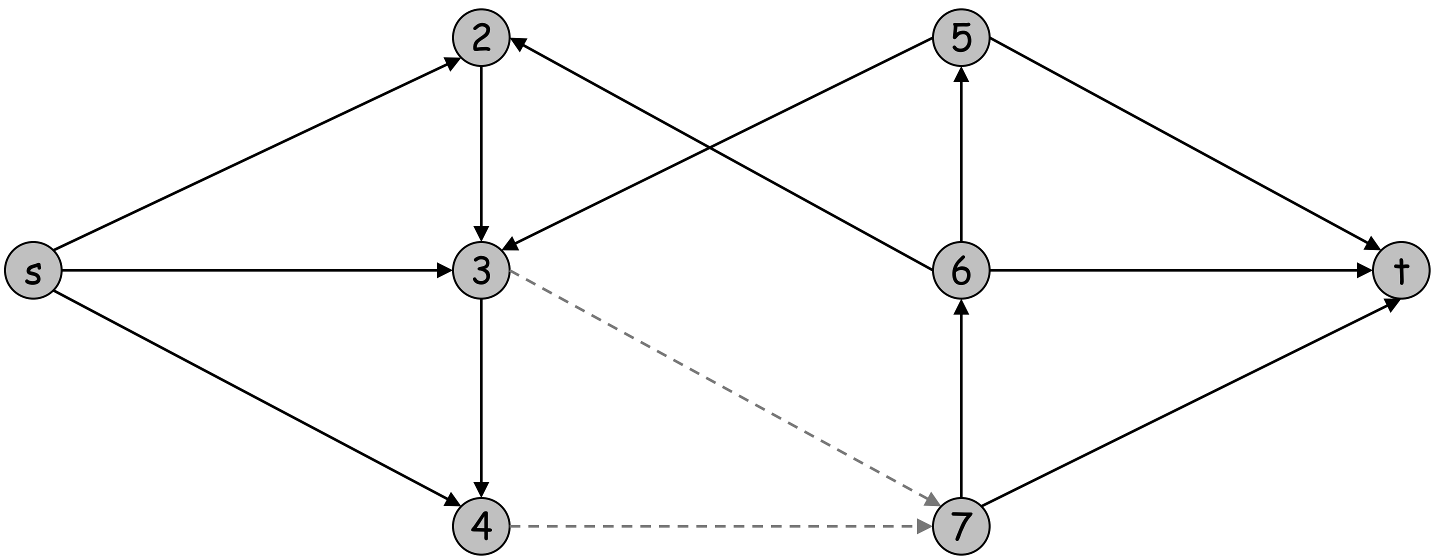


Network Connectivity

Network connectivity problem. Given a digraph G = (V, E) and two nodes s and t, find min number of edges whose removal disconnects t from s.

Def. A set of edges $F \subseteq E$ disconnects t from s if all s-t paths use at least one edge in F.

Q. How to find the minimal number of edges to disconnect t from s?



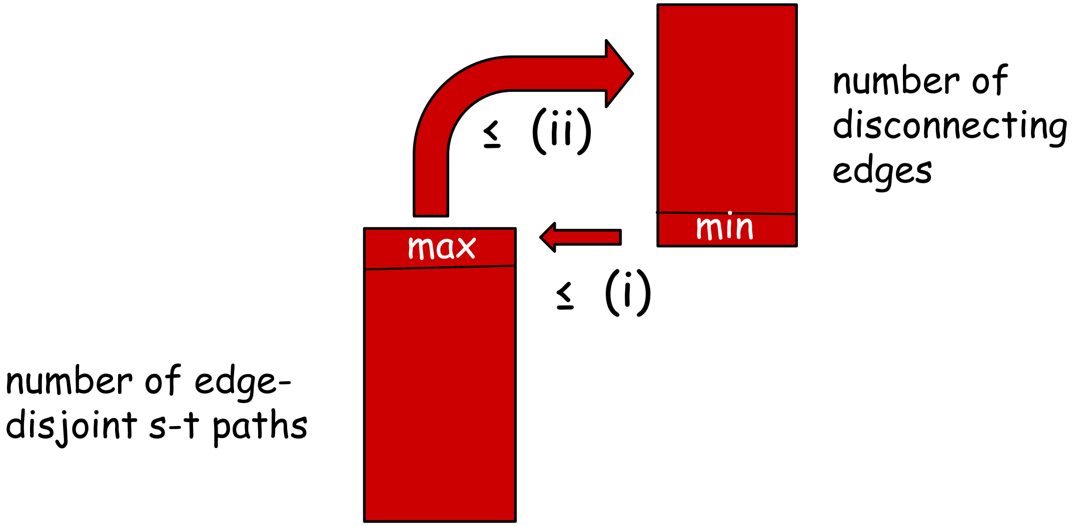
Disjoint Paths and Network Connectivity

Theorem. [Menger 1927] The max number of edge-disjoint s-t paths = the min number of edges whose removal disconnects t from s.

To prove:

(i) min # of disconnecting edges ≤ max # of edge disjoint s-t paths

(ii) (max) # of edge disjoint s-t paths ≤ (min) # of disconnecting edges



This proof can be found on page 377.

Disjoint Paths and Network Connectivity

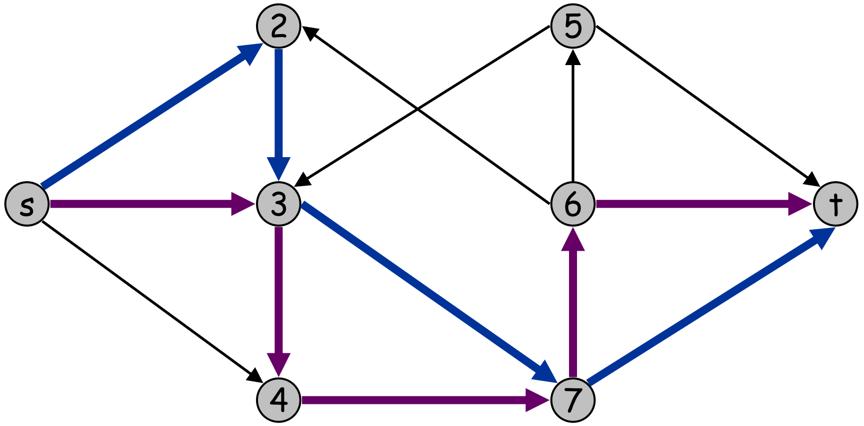
Theorem. [Menger 1927] The max number of edge-disjoint s-t paths = the min number of edges whose removal disconnects t from s.

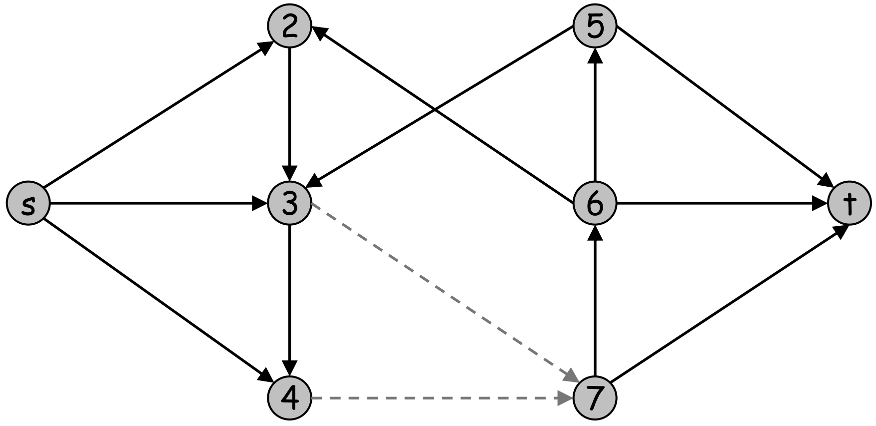
Pf. (i) min # of disconnecting edges ≤ max # of edge disjoint s-t paths

- Suppose max number of edge-disjoint paths is k.

Q. Which set of edges disconnects t from s?

- Let F be the set of edges …
- |F| = k and disconnects t from s.



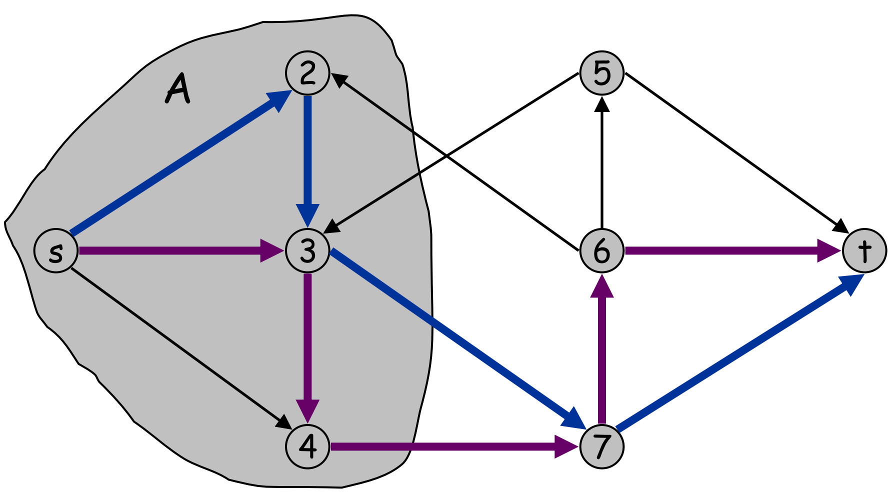


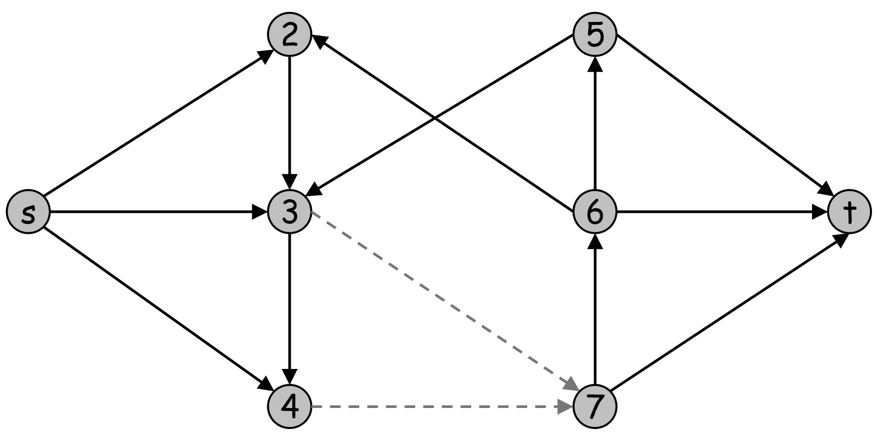
Disjoint Paths and Network Connectivity

Theorem. [Menger 1927] The max number of edge-disjoint s-t paths = the min number of edges whose removal disconnects t from s.

Pf. (i) min # of disconnecting edges ≤ max # of edge disjoint s-t paths

- Suppose max number of edge-disjoint paths is k.
- Then max flow value is k. (Last theorem on edge-disjoint paths.)
- Max-flow min-cut $\Rightarrow$ cut (A, B) of capacity k.
- Let F be the set of edges going from A to B.
- |F| = k and disconnects t from s. (Because each edge has capacity 1.)
- so min # of disconnecting edges ≤ k▪





This proof can be found on page 377.

Edge Disjoint Paths and Network Connectivity

Theorem. [Menger 1927] The max number of edge-disjoint s-t paths = the min number of edges whose removal disconnects t from s.

Pf. (ii) (max) # of edge disjoint s-t paths ≤ (min) # of disconnecting edges

- Let F be a set of disconnecting edges, and |F| = k.
- ...

- Hence, the number of edge-disjoint paths is at most k. ▪

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Edge Disjoint Paths and Network Connectivity

Theorem. [Menger 1927] The max number of edge-disjoint s-t paths = the min number of edges whose removal disconnects t from s.

Pf. (ii) (max) # of edge disjoint s-t paths ≤ (min) # of disconnecting edges

- Let F be a set of disconnecting edges, and |F| = k.
- All s-t paths use at least one edge of F.
- No overlap between paths is allowed.
- Hence, the number of edge-disjoint paths is at most k. ▪

This proof can be found on page 377.

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