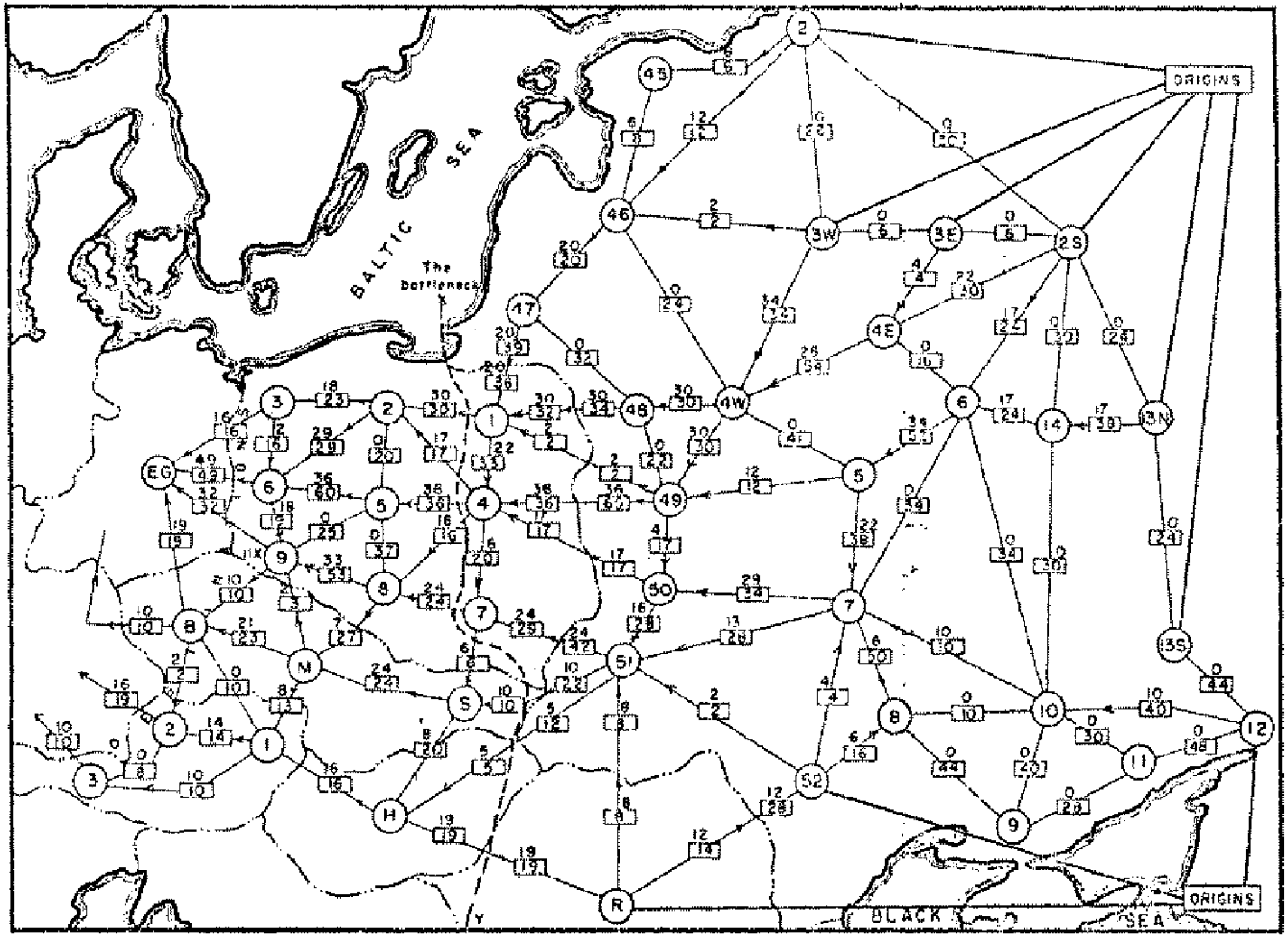
7.7 Extensions to Max Flow: circulations



Reference: *On the history of the transportation and maximum flow problems.* Alexander Schrijver in Math Programming, 91: 3, 2002. (See "External Links")

Circulation with Demands

Circulation with demands.

- Directed graph G = (V, E).
- Edge capacities c(e), e ∈ E.
- Node supply and demands d(v), v ∈ V.

↑ demand if d(v) > 0; supply if d(v) < 0; transshipment if d(v) = 0

Def. A circulation is a function f that satisfies:

- For each e ∈ E: $0 \leq f(e) \leq c(e)$ (capacity)
- For each v ∈ V: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

Circulation problem: given (V, E, c, d), does there exist a circulation?

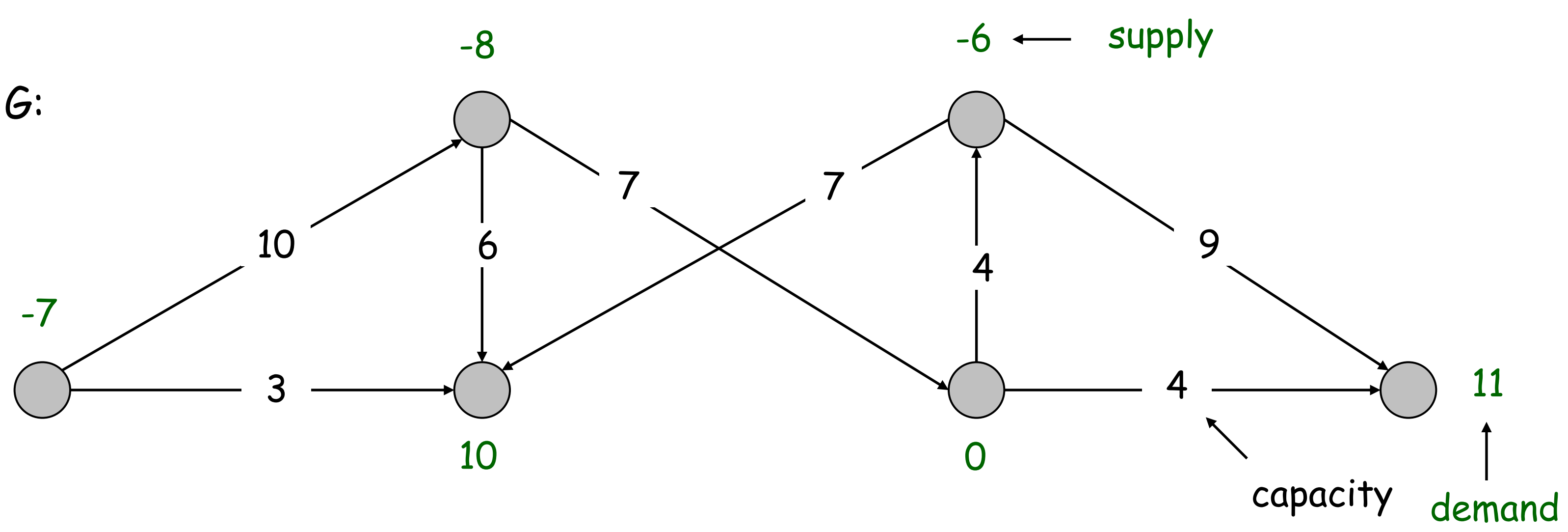
Circulation with Demands

Necessary condition: sum of supplies = sum of demands.

$$\sum_{v \,:\, d(v) > 0} d(v) \;=\; \sum_{v \,:\, d(v) < 0} -d(v) \;=:\; D$$

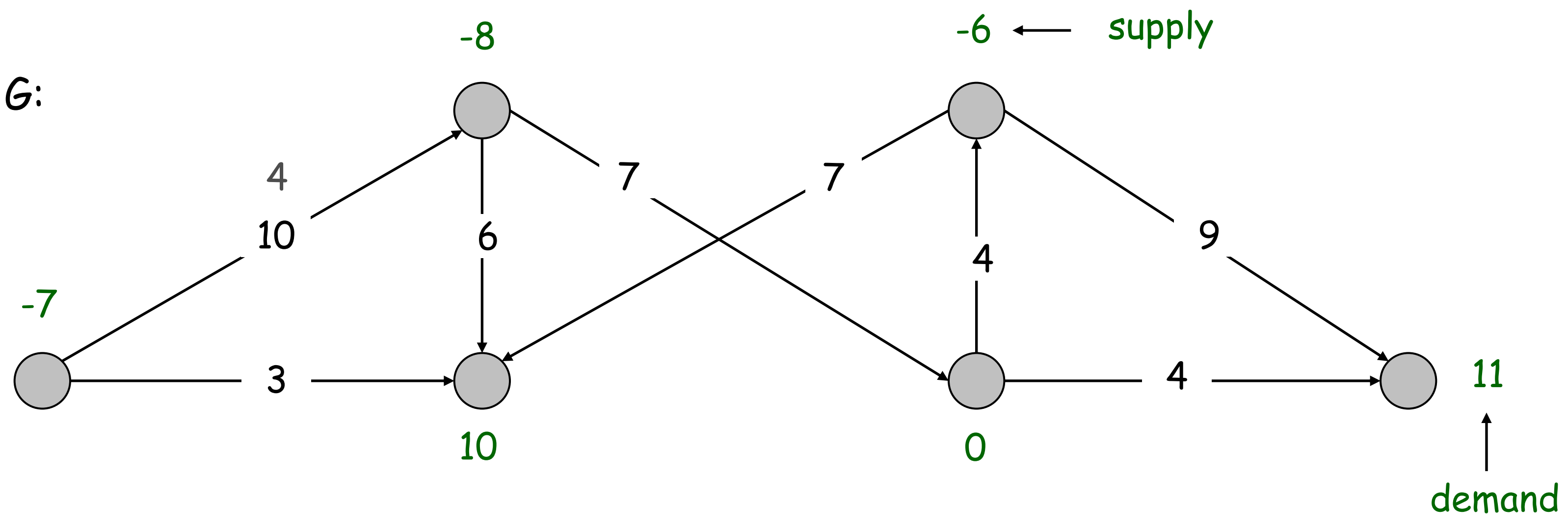
Pf. Sum conservation constraints for every demand node v.

Q. What is the value of D in the network below?



Circulation with Demands

Q. How can we formulate circulation with demands as a max flow problem?

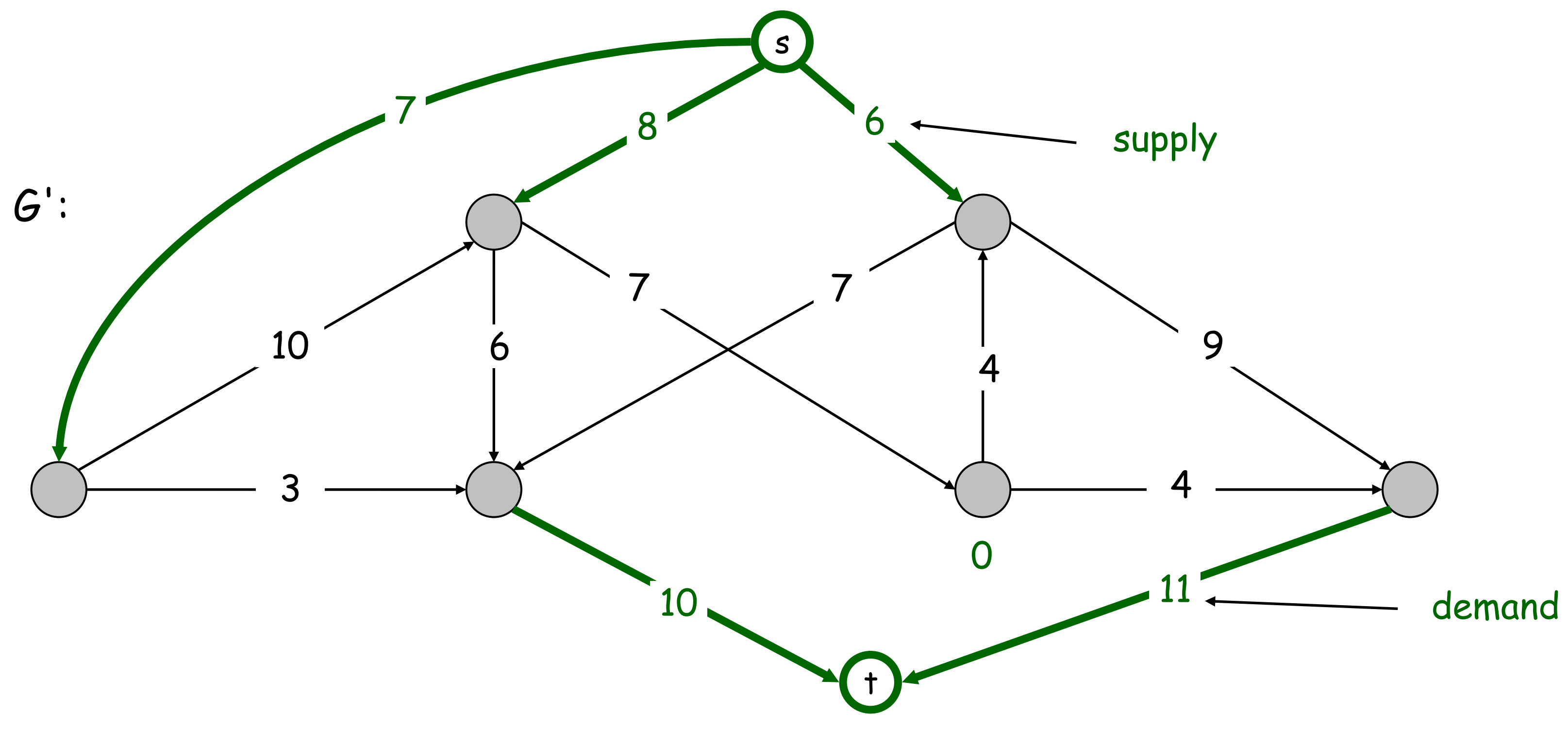


Circulation with Demands

Max flow formulation.

- Add new source s and sink t.
- For each v with d(v) < 0, add edge (s, v) with capacity -d(v).
- For each v with d(v) > 0, add edge (v, t) with capacity d(v).

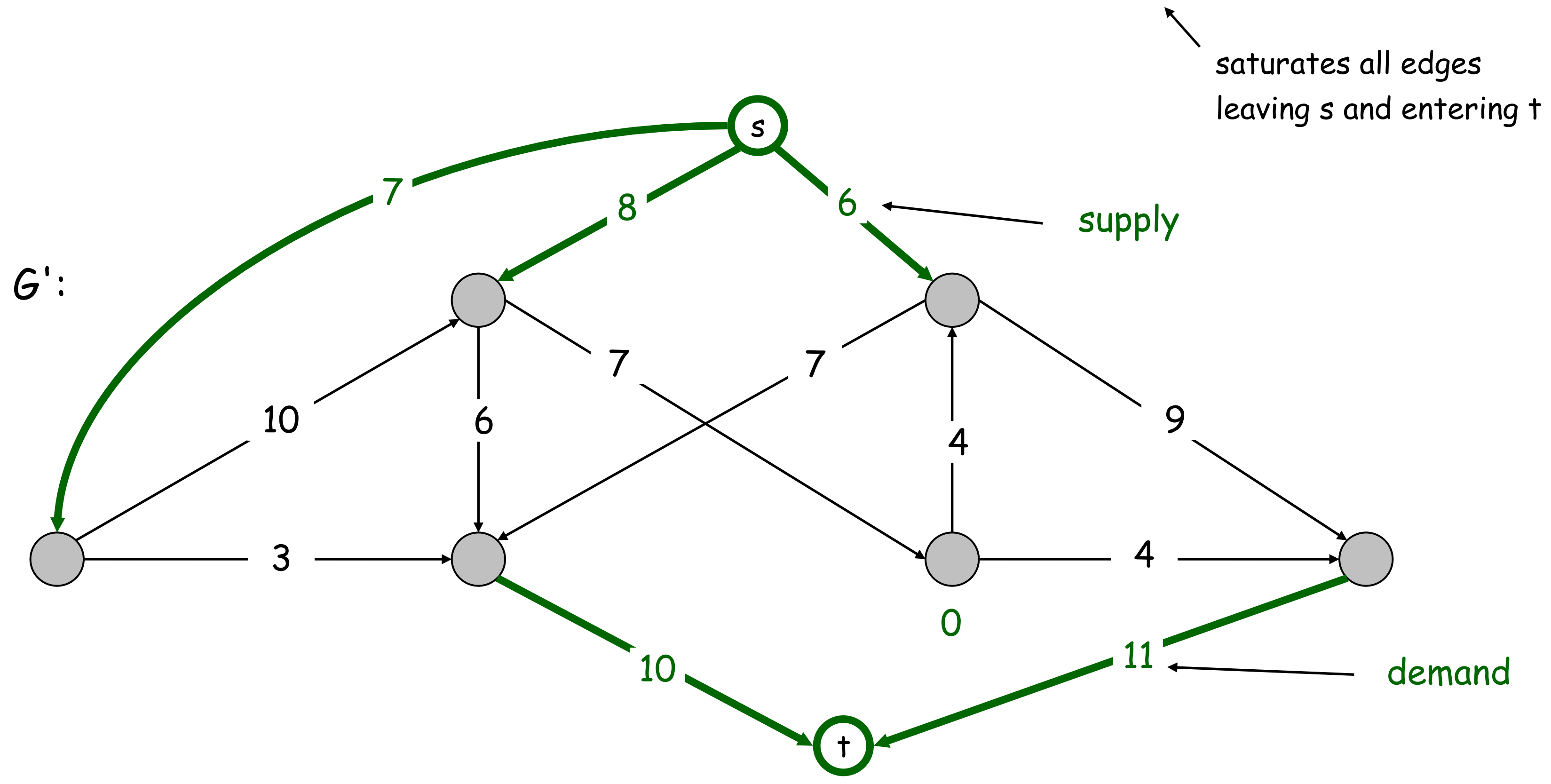
Q. When has G a circulation?



Circulation with Demands

Max flow formulation.

- Add new source s and sink t.
- For each v with d(v) < 0, add edge (s, v) with capacity -d(v).
- For each v with d(v) > 0, add edge (v, t) with capacity d(v).
- Claim: G has circulation iff G' has max flow of value D.



Circulation with Demands

Integrality theorem. If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued.

Pf. Follows from max flow formulation and integrality theorem for max flow.

Theorem. There exists a feasible circulation in G iff the max-flow in G' has value D.

Q. Given (V, E, c, d), when there is no circulation, how can we see why this is not possible?

Circulation with Demands

Integrality theorem. If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued.

Pf. Follows from max flow formulation and integrality theorem for max flow.

Theorem. There exists a feasible circulation in G iff the max-flow in G′ has value D.

Characterization. Given (V, E, c, d), there does not exists a circulation iff there exists a node partition (A, B) such that $\Sigma_{v \in B} d_v > cap(A, B)$

↑ demand by nodes in B exceeds supply of nodes in B plus max capacity of edges going from A to B

Pf idea. Look at min cut in G'.

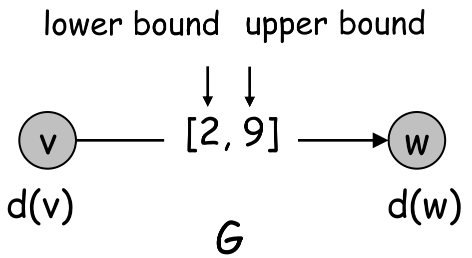
These results can be found on pages 381-382.

Circulation with Demands and Lower Bounds

Feasible circulation.
- Directed graph G = (V, E).
- Edge capacities c(e) and lower bounds ℓ (e), e $\in$ E.
- Node supply and demands d(v), v $\in$ V.

Def. A circulation is a function f that satisfies:
- For each e $\in$ E: $\ell(e) \le f(e) \le c(e)$ (capacity)
- For each v $\in$ V: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

Circulation problem with lower bounds. Given (V, E, ℓ, c, d), does there exists a a circulation?



Q. How to solve this? Do we need to design a new algorithm?

Circulation with Demands and Lower Bounds

Idea. Model lower bounds with demands.

- Send $\ell(e)$ units of flow along edge e.
- Update demands of both endpoints.

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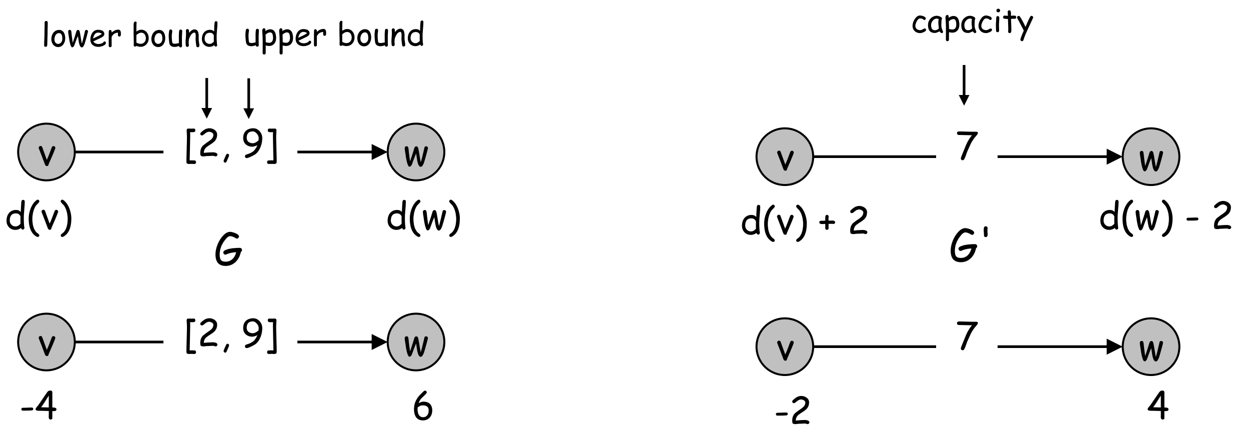

Q. If there is a circulation of 3 in G', what is the circulation in G? (1,3,5)

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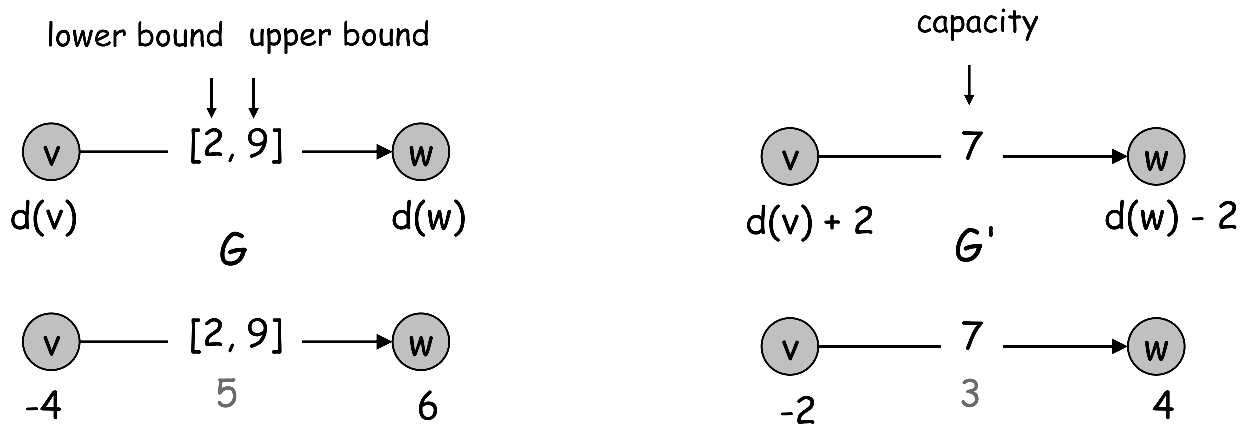

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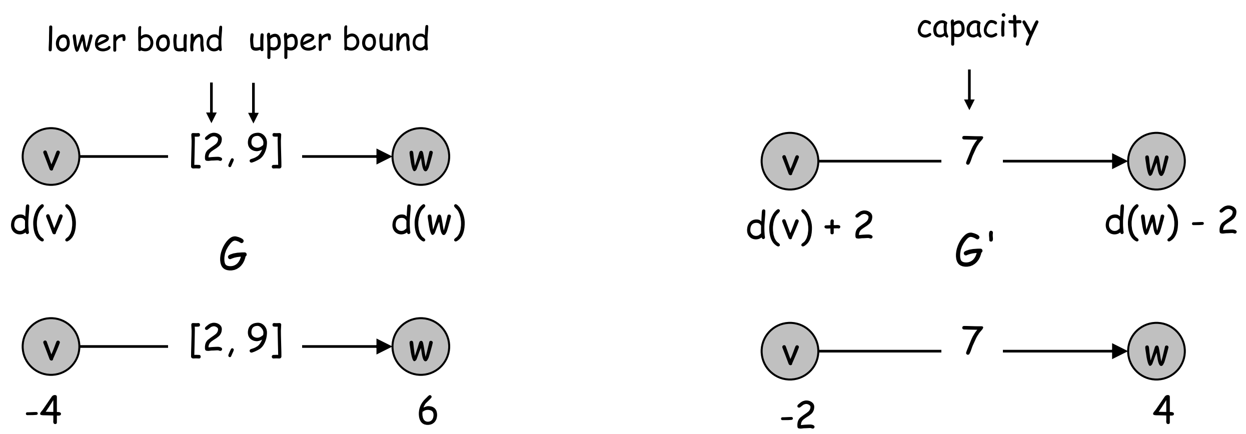

Theorem. There exists a circulation in G iff there exists a circulation in G'.

and if all demands, capacities, and lower bounds in G are integers, then there is a circulation in G that is integer-valued.

Pf sketch. f(e) is a circulation in G iff $f'(e) = f(e) - \ell(e)$ is a circulation in G'.