4.1 Interval Partitioning

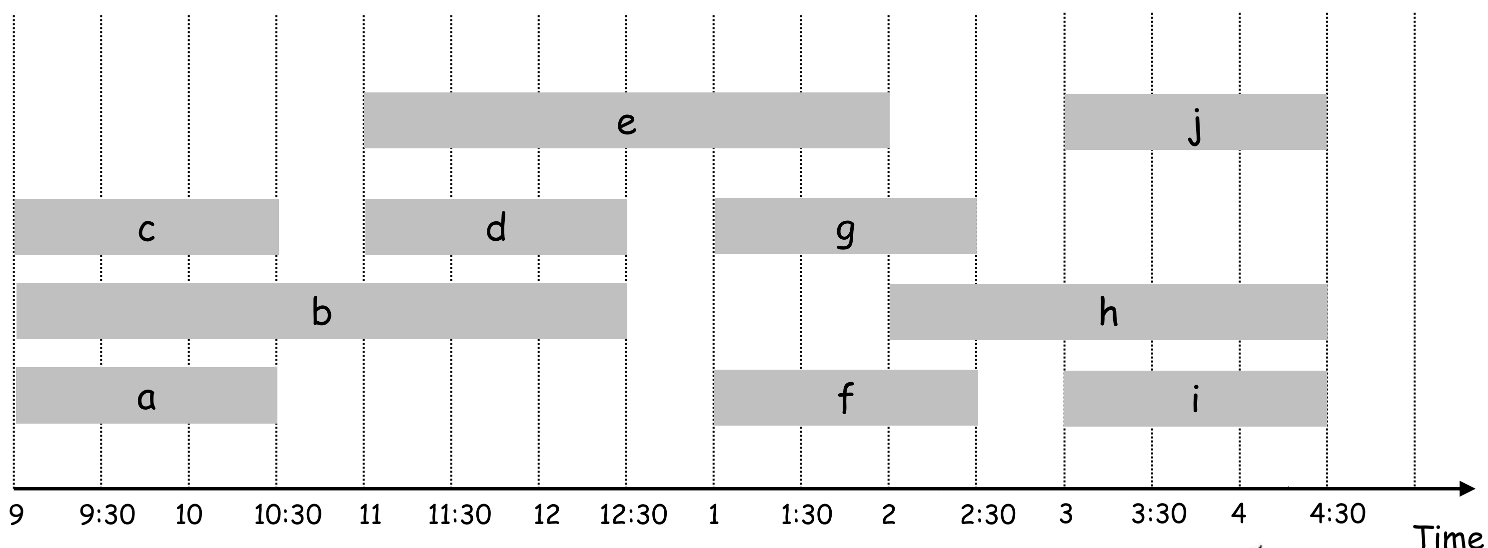
Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.

Q. Is there a schedule that requires less classrooms?



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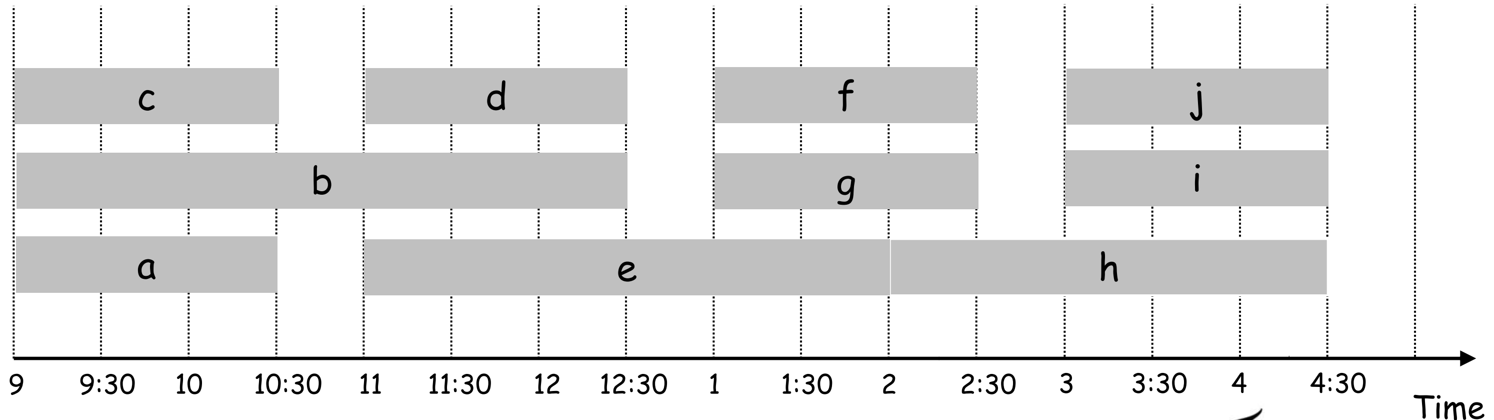
- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



Interval Partitioning: Lower Bound on Optimal Solution

Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of example = 3 $\Rightarrow$ schedule is optimal, because 3 rooms.

↑ a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?
If so, how to obtain such a schedule? If not, why not?
(1 min for both)

Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn.
d ← 0   ← number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Q. What is the worst-case running time?

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Q. How to implement finding a free classroom?

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Implementation. [Q. How to implement finding a free room?]

- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

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Implementation. O(n log n).

- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue (operations of O(log n)).

Interval Partitioning: Greedy Analysis

Key observation. Number of classrooms needed $\geq$ depth.

Observation 2. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm for interval partitioning is optimal.

Pf. (using a structural bound)

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Q. Why did classroom d open?

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To prove: Greedy allocates d classrooms $\Rightarrow$ we need at least d.

- Classroom d opened because lecture j incompatible with d-1 others.

Q. How many incompatibilities at time $s_j + \varepsilon$? (with $\varepsilon>0$ very small)

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Pf. (using a structural bound)
To prove: Greedy allocates d classrooms $\Rightarrow$ we need at least d.

- Classroom d opened because lecture j incompatible with d-1 others.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j+\varepsilon$ (with $\varepsilon \geq 0$ very small).

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- Using the key observation (ie number of classrooms needed $\geq$ depth) $\Rightarrow$ we know that all schedules use $\geq$ d classrooms. ▪

This proof can be found on on pages 123-125.