7.11 Project Selection

- Project Selection Problem
- Translation to Network Flow
- Correctness Proof of Translation

Project Selection

can be positive or negative ↓

Projects with prerequisites.

- Set P of possible projects. Project v has associated revenue p_v.
  - some projects generate money: create interactive e-commerce interface, redesign web page
  - others cost money: upgrade computers, get site license
- Set of prerequisites E. If $(v, w) \in E$, can't do project v and unless also do project w. (Arrow from v to w.)
- A subset of projects $A \subseteq P$ is feasible if the prerequisite of every project in A also belongs to A.

Project selection. Choose a feasible subset of projects to maximize revenue.

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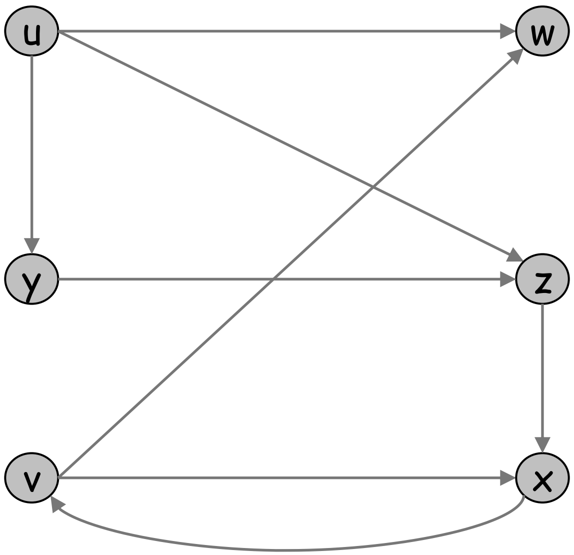
Project Selection: Min Cut Formulation

Min cut formulation

- such that (A, B) is min cut iff A – { s } is optimal set of projects.

Q. How to ensure that A – { s } is feasible?



formulation: cut (A,B):
A-{s}: selected projects
B: unselected projects

Project Selection: Min Cut Formulation

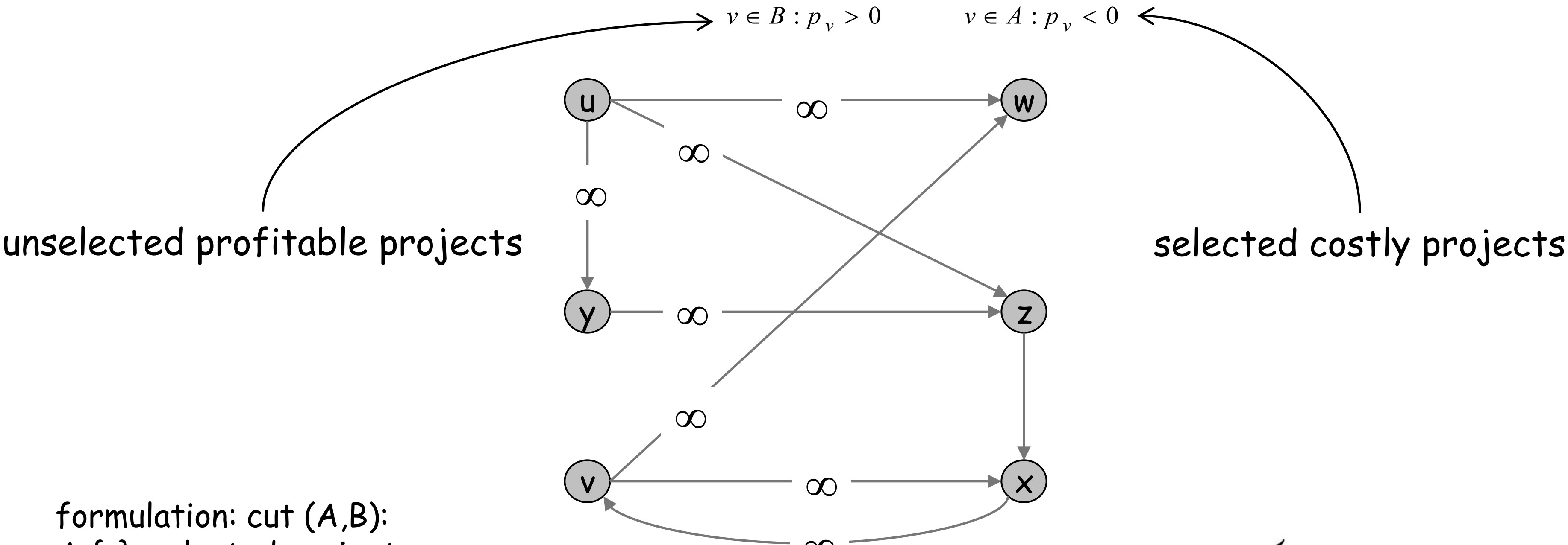
Min cut formulation

- such that (A, B) is min cut iff A – { s } is optimal set of projects.
- infinite capacity edges ensure A – { s } is feasible.

Q. How to ensure that capacity of min cut relates to profit?

A. minimize selected costly and unselected profitable projects:

$$cap(A, B) = \sum_{v \in B \, : \, p_v > 0} p_v + \sum_{v \in A \, : \, p_v < 0} (-p_v)$$



formulation: cut (A,B):
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Project Selection: Min Cut Formulation

Min cut formulation.

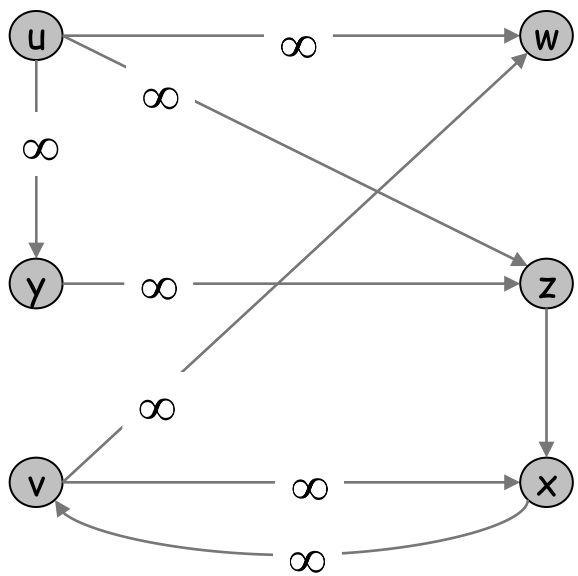
- Assign capacity ∞ to all prerequisite edges.

Q. How to assign other capacities to ensure that

$$cap(A,B) = \sum_{v \in B \,:\, p_v > 0} p_v + \sum_{v \in A \,:\, p_v < 0} (-p_v)$$

unselected profitable projects

selected costly projects



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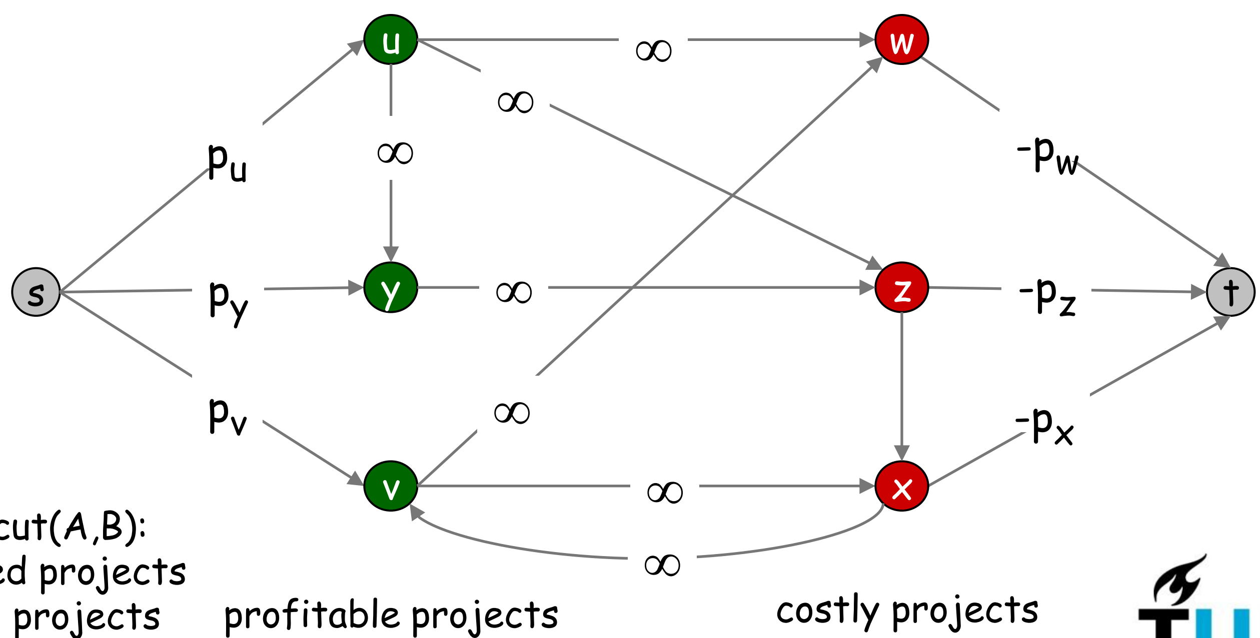
Project Selection: Min Cut Formulation

Min cut formulation.

- Assign capacity ∞ to all prerequisite edges.
- Add edge (s, v) with capacity p_v if $p_v > 0$.
- Add edge (v, t) with capacity $-p_v$ if $p_v < 0$.
- For notational convenience, define $p_s = p_t = 0$.



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Claim. If (A, B) is min cut then A – { s } is optimal set of projects.

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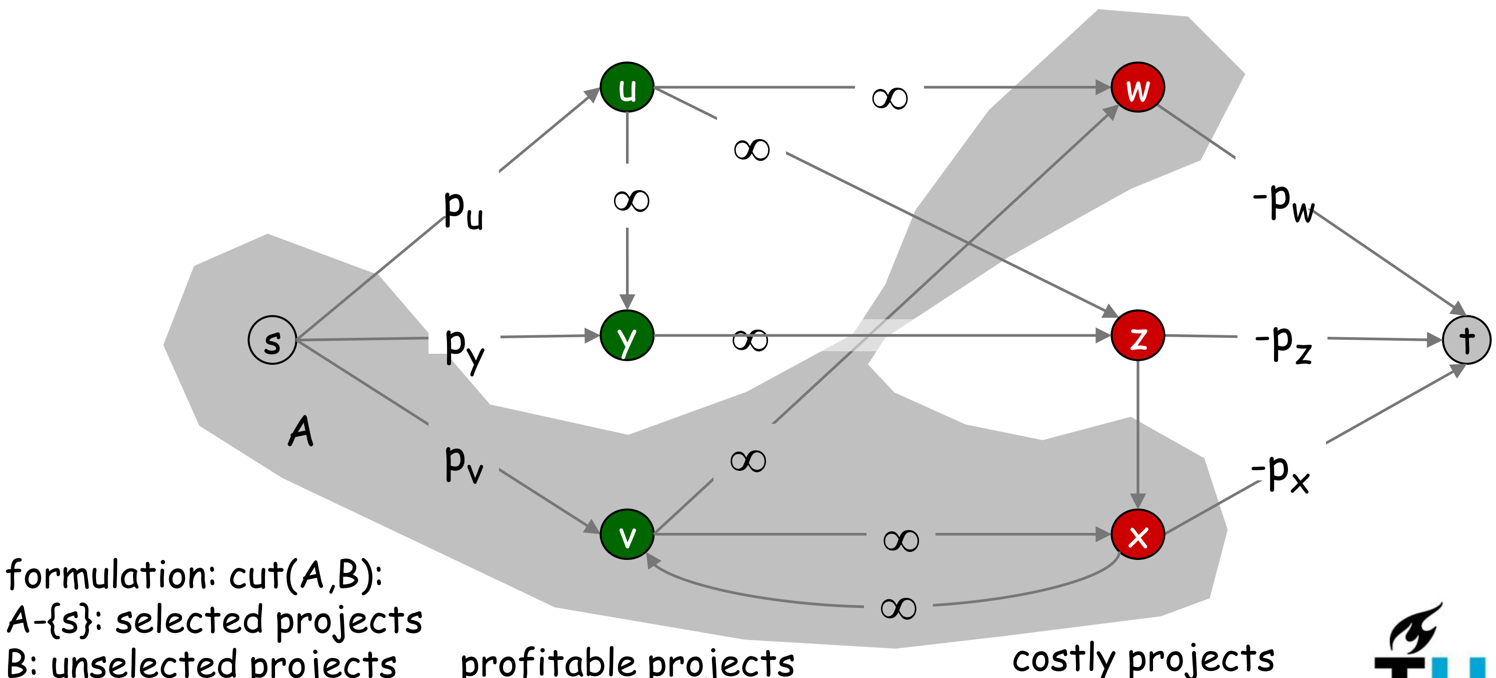

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Pf. (i) Let a min cut (A, B) be given.

- By contradiction:
  - Suppose A-{s} is infeasible.

  - Contradiction, so A-{s} must be feasible.

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This proof can be found on page 399.

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- By contradiction:
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  - Then a project $v \in A$ is selected, while $w \in B$ isn't and $(v,w) \in E$.
  - Then $$cap(A, B) = \sum_{e \text{ out of } A} c(e) = \infty,$$
  - because $c(v,w)=\infty$ and all capacities are positive.
  - (A, B) can thus not be min cut, because e.g. cap({s}, V-{s}) is finite.
  - Contradiction, so A-{s} must be feasible. ▪

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- The set A represents the projects to do.
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$= \ldots$

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distinguish profitable from unprofitable

$$= \left(\sum_{v \in V:\, p_v > 0} p_v - \sum_{v \in B:\, p_v > 0} p_v \right) + \sum_{v \in A:\, p_v < 0} p_v$$

profit of selected profitable projects is total profit – unselected profitable projects

$$= \sum_{v \in V:\, p_v > 0} p_v - \left(\sum_{v \in B:\, p_v > 0} p_v + \sum_{v \in A:\, p_v < 0} -p_v \right)$$

- Maximizing this revenue is equal to minimizing $\sum_{v \in B:\, p_v > 0} p_v + \sum_{v \in A:\, p_v < 0} -p_v = cap(A,B)$
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Open Pit Mining

Open-pit mining. (studied since early 1960s)

- Blocks of earth are extracted from surface to retrieve ore.
- Each block v has net value p_v = value of ore - processing cost.
- Can't remove block v before w or x.