6.3 Segmented Least Squares

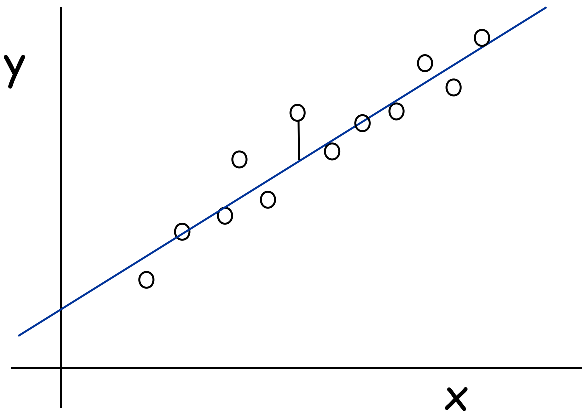
Segmented Least Squares

Least squares.

Foundational problem in statistic and numerical analysis.

Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

Q. How to find a line $y = ax + b$ that fits these points?



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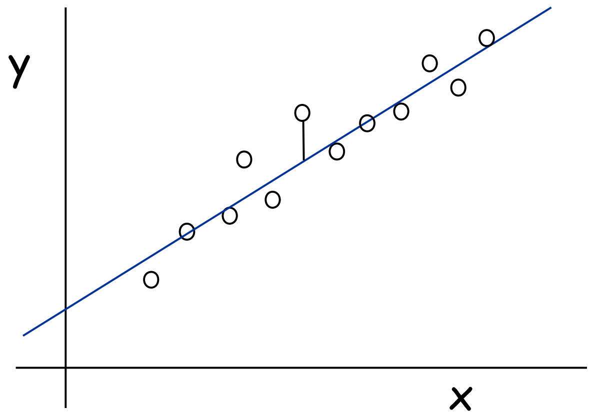
Q. How to find a line y = ax + b that fits these points?

Find a line y = ax + b that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$



Solution. Calculus $\Rightarrow$ min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented Least Squares

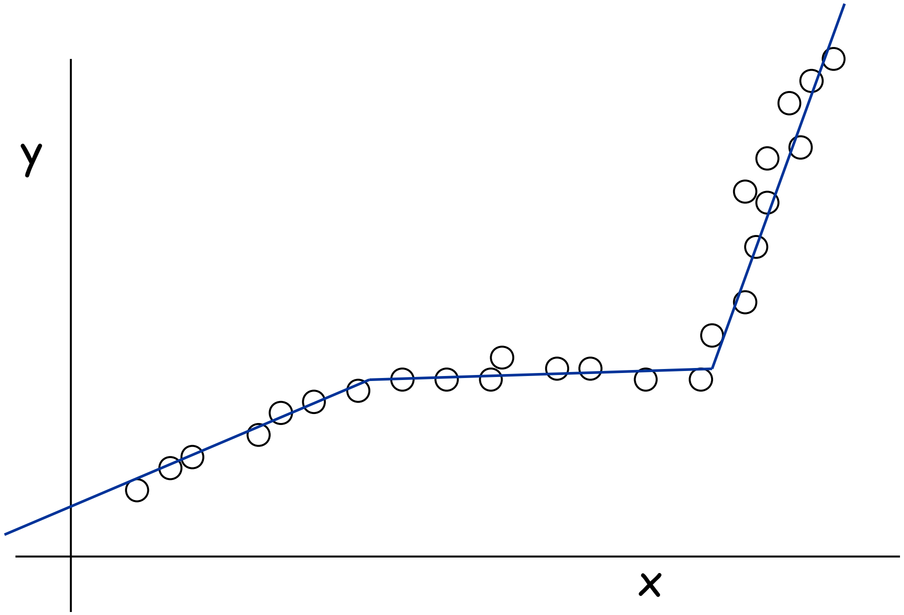
Segmented least squares.

Points lie roughly on a sequence of several line segments.
Given n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of line segments that minimizes $f(x)$.

Q. What's a reasonable choice for f(x) to balance accuracy and parsimony?

goodness of fit

number of lines



Segmented Least Squares

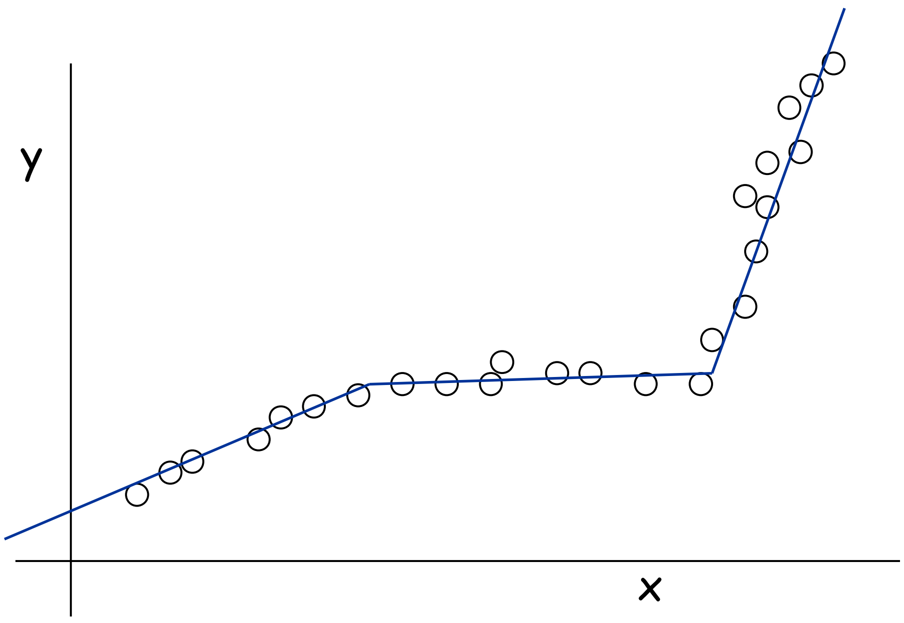
Segmented least squares.

Points lie roughly on a sequence of several line segments.

Given n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of line segments that minimizes:

- the sum of the sums of the squared errors e in each segment
- the number of lines L

Tradeoff function: $e + c\,L$, for some constant $c > 0$.



Dynamic Programming: Multiway Choice

Notation.

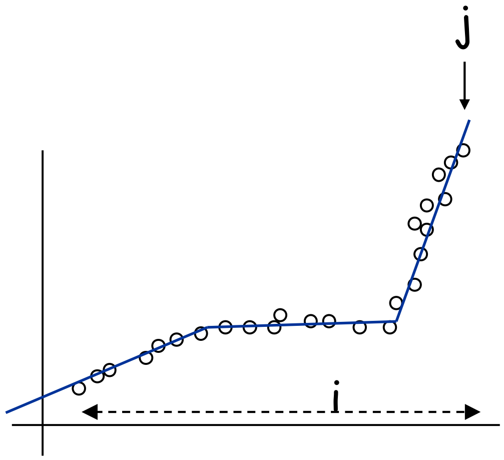
OPT(j) = minimum cost for points $p_1, p_2, \ldots, p_j$.

e(i, j) = minimum sum of squared errors for points $p_i, p_{i+1}, \ldots, p_j$.

Reason backward, computing OPT(j) using subproblems

Q. How can value of OPT(j) be expressed based on subproblems? (1 min)

Q. What are the options here?



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Reason backward, computing OPT(j) using subproblems

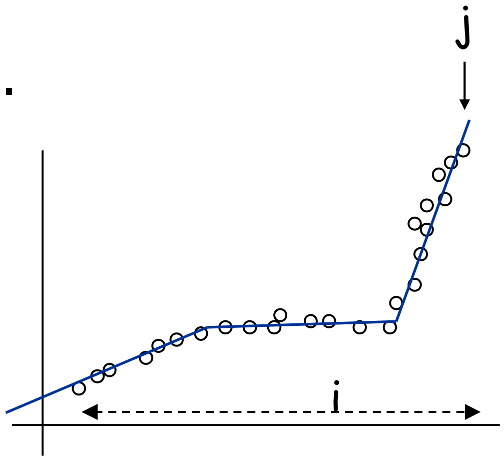
Q. How can value of OPT(j) be expressed based on subproblems? (1 min)

Q. What are the options here?

A. The start i of the last segment ($1 \le i \le j$).

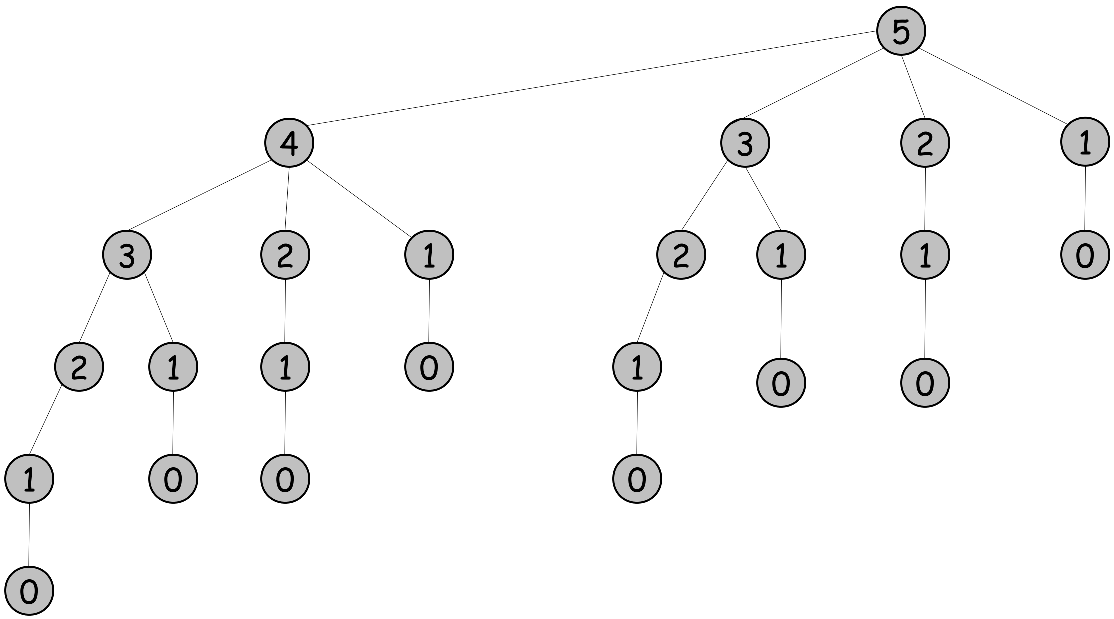
Last segment uses points $p_i, p_{i+1}, \ldots, p_j$ for some i.

Cost = e(i, j) + c + OPT(i-1).



Dynamic Programming: Multiway Choice

OPT(j) call graph



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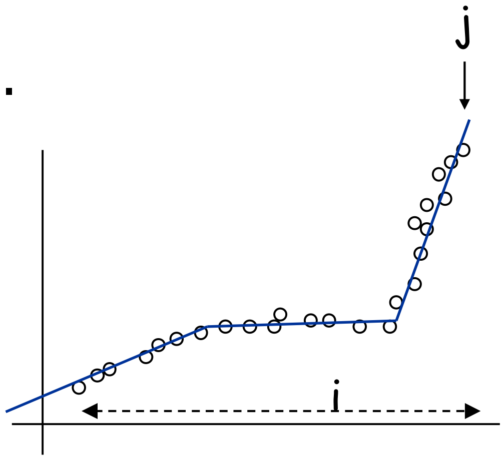

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min\limits_{1 \le i \le j} \left\{ \underbrace{e(i,j) + c + OPT(i-1)}_{\text{Cost of this choice}} \right\} & \text{otherwise} \end{cases}$$

Choose $i \in [1, j]$

Segmented Least Squares: Algorithm

```
INPUT: n, p1,…,pN, c

Segmented-Least-Squares() {
   M[0] = 0
   for j = 1 to n
      for i = 1 to j
         compute the least square error eij for
         the segment pi,…, pj

   for j = 1 to n
      M[j] = min 1 ≤ i ≤ j (eij + c + M[i-1])

   return M[n]
}
```

Q. What is the running time? (1 min)

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```

Q. What is the running time? (1 min)

A. $O(n^3)$. (but can be improved to $O(n^2)$ by pre-computing various statistics)

Bottleneck = initialization: computing e(i, j) for $O(n^2)$ pairs, $O(n)$ per pair using previous formula.

Example exam exercise

6.5 (Chinese) word segmentation problem

Given a string x of letters $x_1x_2 \ldots x_n$, give an efficient algorithm to split x into words (substrings) such that the sum of the quality of these words is maximized.

Example. "mogenzeslapen":

quality(mo, gen, ze, sla, pen) = 7
quality(mogen, ze, slapen) = 10

word	quality
mogen	4
enz	1
gen	2
sla	2
pen	2
slapen	5
ze	1
en	1

rules for this example:
- number of letters-1
- punish uncommon words