1.1 Stable Matching

Matching Residents to Hospitals

Goal. Given preferences of hospitals and medical school students, design a self-reinforcing admissions process. (Gale-Shapley '62)

Unstable pair: applicant x and hospital y are unstable if:

- x prefers y to its assigned hospital.
- y prefers x to one of its admitted students.

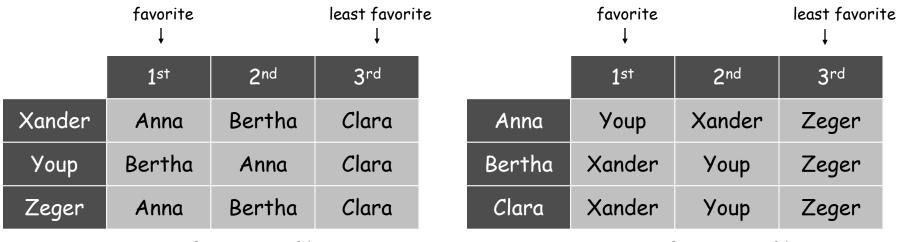
Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.



Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.



Men's Preference Profile



Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
- Unstable pair m-w could each improve by running away (eloping).

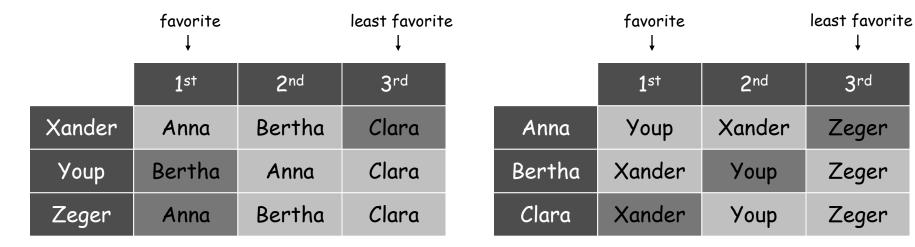
Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.



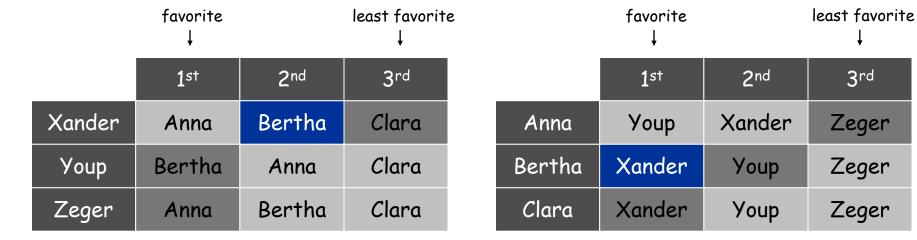
Q. Is assignment X-C, Y-B, Z-A stable?

Men's Preference Profile





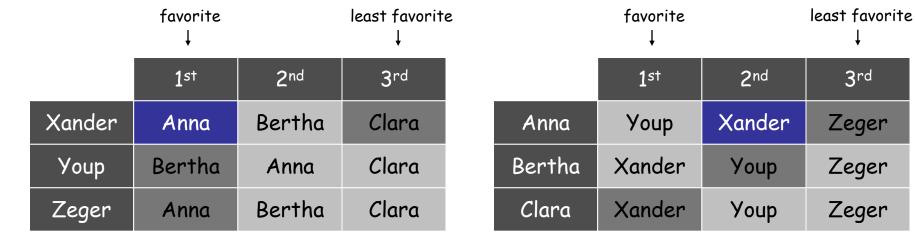
- Q. Is assignment X-C, Y-B, Z-A stable?
- A1. No. Bertha and Xander will hook up.



Men's Preference Profile



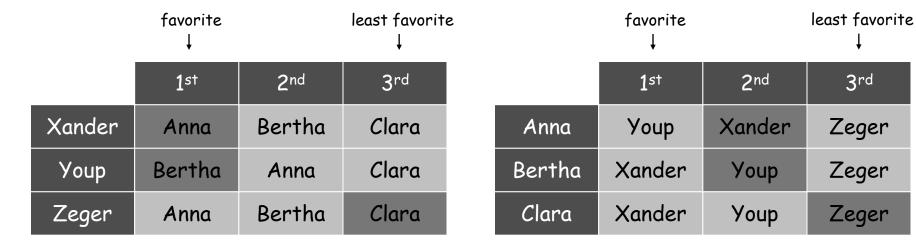
- Q. Is assignment X-C, Y-B, Z-A stable?
- A2. No. Anna and Xander will hook up.



Men's Preference Profile



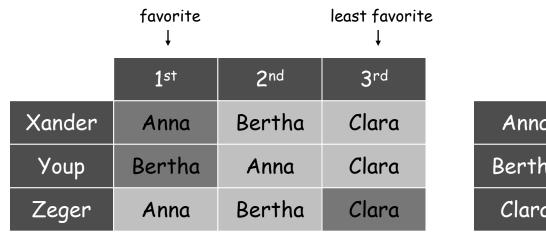
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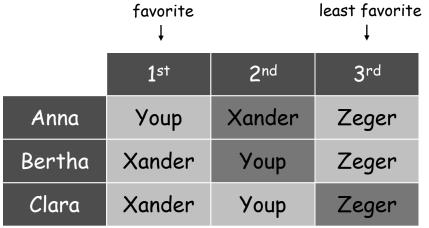
Men's Preference Profile



Q. Is assignment X-A, Y-B, Z-C stable?A. Yes.



Men's Preference Profile





Stable Roommate Problem

Q. Do stable matchings always exist?



Stable Roommate Problem

- Q. Do stable matchings always exist?
- A. Not obvious a priori.

Stable roommate problem.

- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1 st	2 nd	3 rd
Adam	В	С	D
Bob	С	А	D
Chris	А	В	D
Doofus	А	В	С

Observation. Stable matchings do not always exist for stable roommate problem.



Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
   Choose such a man m
   w = 1<sup>st</sup> woman on m's list to whom m has not yet proposed
   if (w is free)
        assign m and w to be engaged
   else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
   else
        w rejects m
}
```



Run-time

Q. How many proposals (iterations of while loop) are made at most?

Initialize each person to be free.

}

```
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Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop. Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals.

	1st	2 nd	3 rd	4 th	5 th		1 st	2 nd	3 rd	4 th	5 th
Victor	A	В	С	D	E	Anna	W	Х	У	Z	V
Wim	В	С	D	А	E	Bertha	Х	У	Z	V	W
Xander	С	D	А	В	E	Clara	У	Z	V	W	Х
Youp	D	А	В	С	Е	Diana	Z	V	W	Х	У
Zeger	A	В	С	D	E	Erika	V	W	Х	У	Z

n(n-1) + 1 proposals required



Claim. All men and women get matched. Pf.



Claim. All men and women get matched.

Pf. (by contradiction)

Q. How do we start a proof by contradiction?



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- Pf. (by contradiction)
 - Suppose, for sake of contradiction, that Zeger is not matched upon termination of algorithm (w.l.o.g. holds for anyone).



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 - Then some woman, say Anna, is not matched upon termination (n men, n women).
 - By Observation 2, Anna was never proposed to.

(Once a woman is matched, she never becomes unmatched)

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- But Zeger proposes to everyone, since he ends up unmatched. (Obs.1)
- So he proposes also to Anna!
- Contradiction!
- So Zeger is matched!
- No further assumptions on Zeger, so holds for all men. (∀-intro)
- n men and n women, so also all women are matched.



Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

- Q. How to implement GS algorithm efficiently?
- Q. If there are multiple stable matchings, which one does GS find?

