1.1 Stable Matching

Matching Residents to Hospitals

Goal. Given preferences of hospitals and medical school students, design a self-reinforcing admissions process. (Gale-Shapley ‘62)

Unstable pair: applicant x and hospital y are unstable if:

- x prefers y to its assigned hospital.
- y prefers x to one of its admitted students.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

favorite → 1st; least favorite → 3rd

	1st	2nd	3rd
Xander	Anna	Bertha	Clara
Youp	Bertha	Anna	Clara
Zeger	Anna	Bertha	Clara

Men's Preference Profile

favorite → 1st; least favorite → 3rd

	1st	2nd	3rd
Anna	Youp	Xander	Zeger
Bertha	Xander	Youp	Zeger
Clara	Xander	Youp	Zeger

Women's Preference Profile

TU Delft

Stable Matching Problem

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
- Unstable pair m-w could each improve by running away (eloping).

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

	1st (favorite)	2nd	3rd (least favorite)
Xander	Anna	Bertha	Clara
Youp	Bertha	Anna	Clara
Zeger	Anna	Bertha	Clara

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Anna	Youp	Xander	Zeger
Bertha	Xander	Youp	Zeger
Clara	Xander	Youp	Zeger

Women's Preference Profile

TU Delft

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

A1. No. Bertha and Xander will hook up.

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Xander	Anna	Bertha	Clara
Youp	Bertha	Anna	Clara
Zeger	Anna	Bertha	Clara

Men's Preference Profile

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Anna	Youp	Xander	Zeger
Bertha	Xander	Youp	Zeger
Clara	Xander	Youp	Zeger

Women's Preference Profile

TU Delft

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

A2. No. Anna and Xander will hook up.

	1st (favorite)	2nd	3rd (least favorite)
Xander	Anna	Bertha	Clara
Youp	Bertha	Anna	Clara
Zeger	Anna	Bertha	Clara

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Anna	Youp	Xander	Zeger
Bertha	Xander	Youp	Zeger
Clara	Xander	Youp	Zeger

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Xander	Anna	Bertha	Clara
Youp	Bertha	Anna	Clara
Zeger	Anna	Bertha	Clara

Men's Preference Profile

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Anna	Youp	Xander	Zeger
Bertha	Xander	Youp	Zeger
Clara	Xander	Youp	Zeger

Women's Preference Profile

TU Delft

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

A. Yes.

	1st (favorite)	2nd	3rd (least favorite)
Xander	Anna	Bertha	Clara
Youp	Bertha	Anna	Clara
Zeger	Anna	Bertha	Clara

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Anna	Youp	Xander	Zeger
Bertha	Xander	Youp	Zeger
Clara	Xander	Youp	Zeger

Women's Preference Profile

TU Delft

Stable Roommate Problem

Q. Do stable matchings always exist?

Stable Roommate Problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

Stable roommate problem.

- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D $\Rightarrow$ B-C unstable
A-C, B-D $\Rightarrow$ A-B unstable
A-D, B-C $\Rightarrow$ A-C unstable

Observation. Stable matchings do not always exist for stable roommate problem.

TU Delft

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Run-time

Q. How many proposals (iterations of while loop) are made at most?

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop.
Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wim	B	C	D	A	E
Xander	C	D	A	B	E
Youp	D	A	B	C	E
Zeger	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Anna	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clara	Y	Z	V	W	X
Diana	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

TU Delft

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf.

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

Q. How do we start a proof by contradiction?

TUDelft

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeger is not matched upon termination of algorithm (w.l.o.g. holds for anyone).

TU Delft

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeger is not matched upon termination of algorithm (w.l.o.g. holds for anyone).
- Then some woman, say Anna, is not matched upon termination (n men, n women).
- By Observation 2, Anna was never proposed to.

(Once a woman is matched, she never becomes unmatched)

- But Zeger proposes to everyone, since he ends up unmatched. (Obs.1)
- So he proposes also to Anna!

TU Delft

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeger is not matched upon termination of algorithm (w.l.o.g. holds for anyone).
- Then some woman, say Anna, is not matched upon termination (n men, n women).
- By Observation 2, Anna was never proposed to. (Once a woman is matched, she never becomes unmatched)
- But Zeger proposes to everyone, since he ends up unmatched. (Obs.1)
- So he proposes also to Anna!
- Contradiction!
- So Zeger *is* matched!

- No further assumptions on Zeger, so holds for all men. (∀-intro)
- n men and n women, so also all women are matched. ▪

TU Delft

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?

TU Delft