Claim. No unstable pairs.Pf. (by contradiction)Q. How to start this proof?



Claim. No unstable pairs.

Pf. (by contradiction)

Suppose A-Z is an unstable pair: A and Z prefer each other to their partner in the Gale-Shapley matching S*.

Q. How could this have happened?



Claim. No unstable pairs.

Pf. (by contradiction)

Suppose A-Z is an unstable pair: A and Z prefer each other to their partner in the Gale-Shapley matching S*.

Q. How could this have happened?

Case 1: Z never proposed to A.

Case 2: Z proposed to A and A rejected/dumped Z

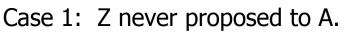


Claim. No unstable pairs.

Pf. (by contradiction)

Suppose A-Z is an unstable pair: A and Z prefer each other to their partner in the Gale-Shapley matching S^{*}.

/ order of preference

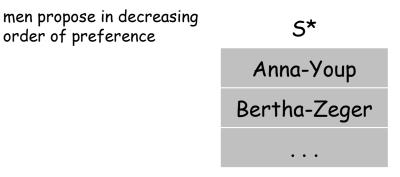


- \Rightarrow Z prefers his partner in S* to A.
- \Rightarrow A-Z is not an unstable pair.

Case 2: Z proposed to A.

- \Rightarrow A rejected Z (right away or later)
- \Rightarrow A prefers her partner in S* to Z. women only trade up
- \Rightarrow A-Z is not an unstable pair.

In either case A-Z is not an unstable pair, a contradiction.





Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1<sup>st</sup> woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Claim. Algorithm terminates after at most n² iterations of while loop.



Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (1.some man is free and hasn't proposed to every woman) {
    1.Choose such a man m
    w = 2. 1<sup>st</sup> woman on m's list to whom m has not yet proposed
    if (3.w is free)
        4.assign m and w to be engaged
    else if (5.w prefers m to her fiancé m')
        4.assign m and w to be engaged, and 1.m' to be free
    else
        2.w rejects m
}
```

Claim. Algorithm terminates after at most n² iterations of while loop.



Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Representing men and women.

```
Assume men are named 1, ..., n.
Assume women are named 1', ..., n'.
```

Engagements.

Maintain list of free men, e.g., in a queue. (1.)

Maintain two arrays wife[m], and husband[w].

- set entry to \circ if unmatched (3.)
- if m matched to w then wife[m]=w and husband[w]=m (4.)

Men proposing.

For each man, maintain list of women, ordered by preference. (2.) Maintain array count [m] for the number of proposals of man m. (2.)



Efficient Implementation

Women rejecting/accepting. (5.)

Q. How to implement efficiently: does woman w prefer man m to man m'? (1 min)

Anna	1st	2 nd	3 rd	4 th	5 th	6 th	7 th	8 th
Pref	8	3	7	1	4	5	6	2

Anna prefers man 3 to 6?



Efficient Implementation

Women rejecting/accepting. (5.)

Q. How to implement efficiently: does woman w prefer man m to man m'?
For each woman, create inverse of preference list of men.
Constant time access for each query after O(n) preprocessing.
Amortized constant time: worst-case O(1) on average

Anna	1 st	2 nd	3 rd	4 th	5 th	6 th	7 th	8 th
Pref	8	3	7	1	4	5	6	2
Anna	1	2	3	4	5	6	7	8
Inverse	4 th	8 th	2 nd	5 th	6 th	7 th	3 rd	1st

Anna prefers man 3 to 6 since inverse[3] < inverse[6]

7

2



Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

	1 st	2 nd	3 rd
Xander	А	В	С
Уоир	В	А	С
Zeger	А	В	С

	1 st	2 nd	3 rd
Anna	У	Х	Z
Bertha	Х	У	Z
Clara	Х	У	Ζ



Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

A-X, B-Y, C-Z. A-Y, B-X, C-Z.

	1 st	2 nd	3 rd
Xander	А	В	С
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Clara	Х	У	Ζ



Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Q. Does each man receive best valid partner based on the given preferences?



Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Q. Does each man receive best valid partner based on the given preferences?

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

No reason a priori to believe that man-optimal assignment is perfect, let alone stable.

Simultaneously best for each and every man.

No reason for lying about your preferences (incentive compatible).

Claim. GS matching S is man-optimal. Pf.



Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

Q. What does this mean?



Claim. GS matching S is man-optimal.Pf. by contradiction: suppose S is not man-optimal In execution: first moment some man Y is rejected by best valid partner A in S.

... (idea: create another stable matching S' where Y is not rejected to derive contradiction)

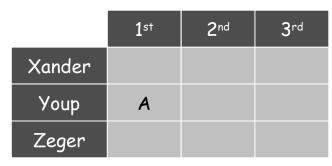


Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal In execution: first moment some man Y is rejected by best valid partner A in S.

When Y is rejected, A forms/stays engagement with a man, say Z, whom she prefers to Y.

... (idea: create another stable matching S' where Y is not rejected to derive contradiction)

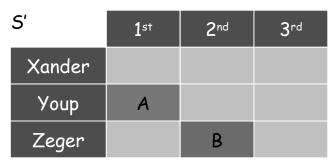


	1 st	2 nd	3 rd
Anna	Z	У	
Bertha			
Clara			



...-Youp Claim. GS matching S is man-optimal. Anna-Zeger **Pf.** by contradiction: suppose S is not man-optimal In execution: first moment some man Y is rejected by best . . . valid partner A in S. should exist: When Y is rejected, A forms/stays engagement with a man, S' say Z, whom she prefers to Y. Anna-Youp Stable S' with Y-A exists because Y-A is valid. Bertha-Zeger Let B be Z's partner in S'. Q. Given what happened in S, does Z prefer A or B? . . .

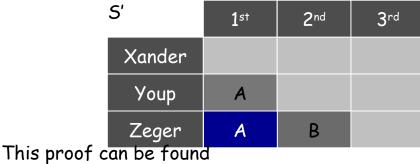
Contradiction! -



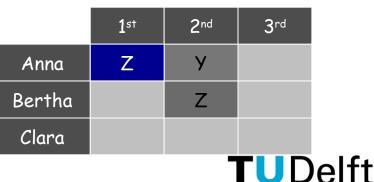
	1 st	2 nd	3 rd
Anna	Z	У	
Bertha		Z	
Clara			

S

S ...-Youp Claim. GS matching S is man-optimal. Anna-Zeger Pf. by contradiction: suppose S is not man-optimal In execution: first moment some man Y is rejected by best . . . valid partner A in S. should exist: When Y is rejected, A forms/stays engagement with a man, S' say Z, whom she prefers to Y. Anna-Youp Stable S' with Y-A exists because Y-A is valid. Bertha-Zeger Let B be Z's partner in S'. Z not rejected by any valid partner at the point when Y is rejected by A (in S). Thus, Z prefers A to B. But A prefers Z to Y. Thus A-Z is unstable in S'. since Y was first rejected by a valid partner Contradiction! • S 1st 2nd 3rd 1st 2nd 3rd



on pages 10-11.



Stable Matching Summary

```
Stable matching problem. Given preference profiles of n men and n women, find a stable matching.
no man and woman prefer to be with each other than assigned partner
Gale-Shapley algorithm. Finds a stable matching in O(n<sup>2</sup>) time.
Man-optimality. In version of GS where men propose, each man receives best valid partner.
```

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?



Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)Q. Which assumption to make?

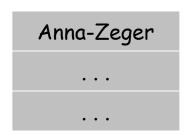


Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction) Suppose A-Z matched in S, but Z is not worst valid partner for A.

Idea: similar proof as man-optimal, and also use that fact!



S



Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)

Suppose A-Z matched in S, but Z is not worst valid partner for A. There exists stable matching S' in which A is paired with a man, say Y, whom she likes less than Z.

Let B be Z's partner in S'.



Q. Given what happened in S, does Z prefer A or B?

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)

Suppose A-Z matched in S, but Z is not worst valid partner for A. There exists stable matching S' in which A is paired with a man, say Y, whom she likes less than Z.

Let B be Z's partner in S'.

Z prefers A to B. \leftarrow man-optimality by GS in S

Thus, A-Z is an unstable pair in S'.

Contradiction: S' was stable! -

S' Anna-Youp Bertha-Zeger ... S

Anna-Zeger

. . .

This proof can be found on pages 11-12.

Extensions: Matching Residents to Hospitals

Ex: Men \approx hospitals, Women \approx med school residents.

- Variant 1. Some participants declare others as unacceptable.
- Variant 2. Unequal number of men and women.
- Variant 3. Limited polygamy.
- Variant 4. Also allow weak preferences. hospital X wants to hire 3 residents
- Variant 5. Online mechanism (new students / hospitals may arrive).
- Variant 6. Include contract details.

Def. Matching S unstable if there is a hospital h and resident r such that: h and r are acceptable to each other; and either r is unmatched, or r prefers h to her assigned hospital; and either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

Q. Does it help students to lie about their preferences if the hospitals "are the men"?

Extensions: Matching Residents to Hospitals

Q. Does it help students to lie about their preferences?

A. Yes (because they are the "women"), but:even for about 20,000 students/year in 1991-1996 only two years 2students worse off because they were the "women"

Alvin E. Roth & Elliott Peranson, 1999. "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design," *American Economic Review*, American Economic Association, vol. 89(4), pages 748-780.



Recent research

Weak preferences

Erdil, A., and H. Ergin (2008). "What's the Matter with Tie-Breaking? Improving Efficiency in School Choice." *American Economic Review* 98(3), 669-689.

Online matching

- Roth, A.E., Sonmez, T., and Unver, M.U. (2004). Kidney exchange. *Quarterly Journal of Economics*, 119(2), 457-488.
- Jalilzadeh, B., L.R. Planken, and M.M. de Weerdt (2009). Mechanism Design for the OnlineAllocation of Items without Monetary Payments. In O. Shehory and D. Sarne and E. David(Eds.). *Proc. of the workshop on Agent-Mediated Electronic Commerce*, 71-84.

Contract matching

- Hatfield, J.W., and Paul R. Milgrom (2005). Matching with Contracts. *The American Economic Review* 95(4), 913-935.
- Harrenstein, B.P., M.M. de Weerdt, and V. Conitzer (2009). A Qualitative Vickrey Auction. In J. Chuang and L. Fortnow and P. Pu (Eds.). *Proc. of the ACM Conference on Electronic Commerce*, 197-206.



Lessons Learned

Powerful ideas learned in course.

Isolate underlying structure of problem. Create useful and efficient algorithms.

Potentially deep social ramifications. [legal disclaimer]

- Historically, men propose to women. Why not vice versa?
- Men: propose early and often.
- Women: ask out the guys.
- Theory can be socially enriching and fun!
- CS students get the best partners!

