Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

Q. How to start this proof?

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

Suppose A-Z is an unstable pair: A and Z prefer each other to their partner in the Gale-Shapley matching S*.

Q. How could this have happened?

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

Suppose A-Z is an unstable pair: A and Z prefer each other to their partner in the Gale-Shapley matching S*.

Q. How could this have happened?

Case 1: Z never proposed to A.

Case 2: Z proposed to A and A rejected/dumped Z

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

Suppose A-Z is an unstable pair: A and Z prefer each other to their partner in the Gale-Shapley matching S*.

Case 1: Z never proposed to A. ↙ men propose in decreasing order of preference

⇒ Z prefers his partner in S* to A.

⇒ A-Z is not an unstable pair.

S*
Anna-Youp
Bertha-Zeger
. . .

Case 2: Z proposed to A.

⇒ A rejected Z (right away or later)

⇒ A prefers her partner in S* to Z. ← women only trade up

⇒ A-Z is not an unstable pair.

In either case A-Z is not an unstable pair, a contradiction. ▪

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Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Claim. Algorithm terminates after at most n^2 iterations of while loop.

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Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (1.some man is free and hasn't proposed to every woman) {
    1.Choose such a man m
    w = 2. 1st woman on m's list to whom m has not yet proposed
    if (3.w is free)
        4.assign m and w to be engaged
    else if (5.w prefers m to her fiancé m')
        4.assign m and w to be engaged, and 1.m' to be free
    else
        2.w rejects m
}
```

Claim. Algorithm terminates after at most n^2 iterations of while loop.

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Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Representing men and women.

- Assume men are named 1, …, n.
- Assume women are named 1', …, n'.

Engagements.

- Maintain list of free men, e.g., in a queue. (1.)
- Maintain two arrays `wife[m]`, and `husband[w]`.
  - set entry to `0` if unmatched (3.)
  - if m matched to w then `wife[m]=w` and `husband[w]=m` (4.)

Men proposing.

- For each man, maintain list of women, ordered by preference. (2.)
- Maintain array `count[m]` for the number of proposals of man `m`. (2.)

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Efficient Implementation

Women rejecting/accepting. (5.)

Q. How to implement efficiently: does woman `w` prefer man `m` to man `m'`? (1 min)

Anna	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Anna prefers man 3 to 6?

Efficient Implementation

Women rejecting/accepting. (5.)

Q. How to implement efficiently: does woman `w` prefer man `m` to man `m'`?

For each woman, create inverse of preference list of men.

Constant time access for each query after O(n) preprocessing.

Amortized constant time: worst-case O(1) *on average*

Anna	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Anna	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Anna prefers man 3 to 6
since `inverse[3]` < `inverse[6]`

2 7

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

	1st	2nd	3rd
Xander	A	B	C
Youp	B	A	C
Zeger	A	B	C

	1st	2nd	3rd
Anna	Y	X	Z
Bertha	X	Y	Z
Clara	X	Y	Z

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Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

A-X, B-Y, C-Z.

A-Y, B-X, C-Z.

	1st	2nd	3rd
Xander	A	B	C
Youp	B	A	C
Zeger	A	B	C

	1st	2nd	3rd
Anna	Y	X	Z
Bertha	X	Y	Z
Clara	X	Y	Z

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Understanding the Solution

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An instance with two stable matchings.

A-X, B-Y, C-Z.

A-Y, B-X, C-Z.

	1st	2nd	3rd
Xander	A	B	C
Youp	B	A	C
Zeger	A	B	C

	1st	2nd	3rd
Anna	Y	X	Z
Bertha	X	Y	Z
Clara	X	Y	Z

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Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Q. Does each man receive best valid partner based on the given preferences?

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Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Q. Does each man receive best valid partner based on the given preferences?

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.
- No reason for lying about your preferences (incentive compatible).

Man Optimality

Claim. GS matching S is man-optimal.

Pf.

Man Optimality

Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

Q. What does this mean?

Contradiction! ▪

Man Optimality

Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

In execution: first moment some man Y is rejected by best valid partner A in S.

… (idea: create another stable matching S' where Y is not rejected to derive contradiction)

Contradiction! ▪

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Man Optimality

S

…-Youp
Anna-Zeger
. . .

Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

In execution: first moment some man Y is rejected by best valid partner A in S.

When Y is rejected, A forms/stays engagement with a man, say Z, whom she prefers to Y.

… (idea: create another stable matching S′ where Y is not rejected to derive contradiction)

Contradiction! ▪

	1st	2nd	3rd
Xander			
Youp	A		
Zeger			

	1st	2nd	3rd
Anna	Z	Y	
Bertha			
Clara			

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Man Optimality

Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

In execution: first moment some man Y is rejected by best valid partner A in S.

When Y is rejected, A forms/stays engagement with a man, say Z, whom she prefers to Y.

Stable S′ with Y-A exists because Y-A is valid.

Let B be Z's partner in S′.

Q. Given what happened in S, does Z prefer A or B?

Contradiction! ▪

S

...-Youp
Anna-Zeger
. . .

should exist:
S′

Anna-Youp
Bertha-Zeger
. . .

S′

	1st	2nd	3rd
Xander			
Youp	A		
Zeger		B	

	1st	2nd	3rd
Anna	Z	Y	
Bertha		Z	
Clara			

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Man Optimality

Claim. GS matching S is man-optimal.

Pf. by contradiction: suppose S is not man-optimal

In execution: first moment some man Y is rejected by best valid partner A in S.

When Y is rejected, A forms/stays engagement with a man, say Z, whom she prefers to Y.

Stable S′ with Y-A exists because Y-A is valid.

Let B be Z's partner in S′.

Z not rejected by any valid partner at the point when Y is rejected by A (in S). Thus, Z prefers A to B.

But A prefers Z to Y. Thus A-Z is unstable in S′.

Contradiction! ▪

since Y was first rejected by a valid partner

S

...-Youp
Anna-Zeger
. . .

should exist:
S′

Anna-Youp
Bertha-Zeger
. . .

S′

	1st	2nd	3rd
Xander			
Youp	A		
Zeger	A	B	

	1st	2nd	3rd
Anna	Z	Y	
Bertha		Z	
Clara			

This proof can be found on pages 10-11.

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Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.

no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)

Q. Which assumption to make?

Contradiction! ▪

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)

Suppose A-Z matched in S, but Z is not worst valid partner for A.

Idea: similar proof as man-optimal, and also use that fact!

S
Anna-Zeger
. . .
. . .

Contradiction! ▪

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)

Suppose A-Z matched in S, but Z is not worst valid partner for A.

There exists stable matching S′ in which A is paired with a man, say Y, whom she likes less than Z.

Let B be Z's partner in S′.

Q. Given what happened in S, does Z prefer A or B?

Contradiction! ▪

S

Anna-Zeger
. . .
. . .

S′

Anna-Youp
Bertha-Zeger
. . .

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Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S.

Pf. (by contradiction)

Suppose A-Z matched in S, but Z is not worst valid partner for A.

There exists stable matching S' in which A is paired with a man, say Y, whom she likes less than Z.

Let B be Z's partner in S'.

Z prefers A to B. ← man-optimality by GS in S

Thus, A-Z is an unstable pair in S'.

Contradiction: S' was stable! ▪

S

Anna-Zeger
. . .
. . .

S'

Anna-Youp
Bertha-Zeger
. . .

This proof can be found on pages 11-12.

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Extensions: Matching Residents to Hospitals

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.
Variant 2. Unequal number of men and women.
Variant 3. Limited polygamy.
Variant 4. Also allow weak preferences.
Variant 5. Online mechanism (new students / hospitals may arrive).
Variant 6. Include contract details.

resident A unwilling to work in Cleveland

hospital X wants to hire 3 residents

Def. Matching S unstable if there is a hospital h and resident r such that:
h and r are acceptable to each other; and
either r is unmatched, or r prefers h to her assigned hospital; and
either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

Q. Does it help students to lie about their preferences if the hospitals "are the men"?

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Extensions: Matching Residents to Hospitals

Q. Does it help students to lie about their preferences?

A. Yes (because they are the "women"), but:
even for about 20,000 students/year in 1991-1996 only two years 2 students worse off because they were the "women"

Alvin E. Roth & Elliott Peranson, 1999. "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design," *American Economic Review*, American Economic Association, vol. 89(4), pages 748-780.

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Recent research

Weak preferences

Erdil, A., and H. Ergin (2008). "What's the Matter with Tie-Breaking? Improving Efficiency in School Choice." *American Economic Review* 98(3), 669-689.

Online matching

Roth, A.E., Sonmez, T., and Unver, M.U. (2004). Kidney exchange. *Quarterly Journal of Economics*, 119(2), 457-488.

Jalilzadeh, B., L.R. Planken, and M.M. de Weerdt (2009). Mechanism Design for the Online Allocation of Items without Monetary Payments. In O. Shehory and D. Sarne and E. David (Eds.). *Proc. of the workshop on Agent-Mediated Electronic Commerce*, 71-84.

Contract matching

Hatfield, J.W., and Paul R. Milgrom (2005). Matching with Contracts. *The American Economic Review* 95(4), 913-935.

Harrenstein, B.P., M.M. de Weerdt, and V. Conitzer (2009). A Qualitative Vickrey Auction. In J. Chuang and L. Fortnow and P. Pu (Eds.). *Proc. of the ACM Conference on Electronic Commerce*, 197-206.

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Lessons Learned

Powerful ideas learned in course.

Isolate underlying structure of problem.

Create useful and efficient algorithms.

Potentially deep social ramifications. [legal disclaimer]

- Historically, men propose to women. Why not vice versa?
- Men: propose early and often.
- Women: ask out the guys.
- Theory can be socially enriching and fun!
- CS students get the best partners!