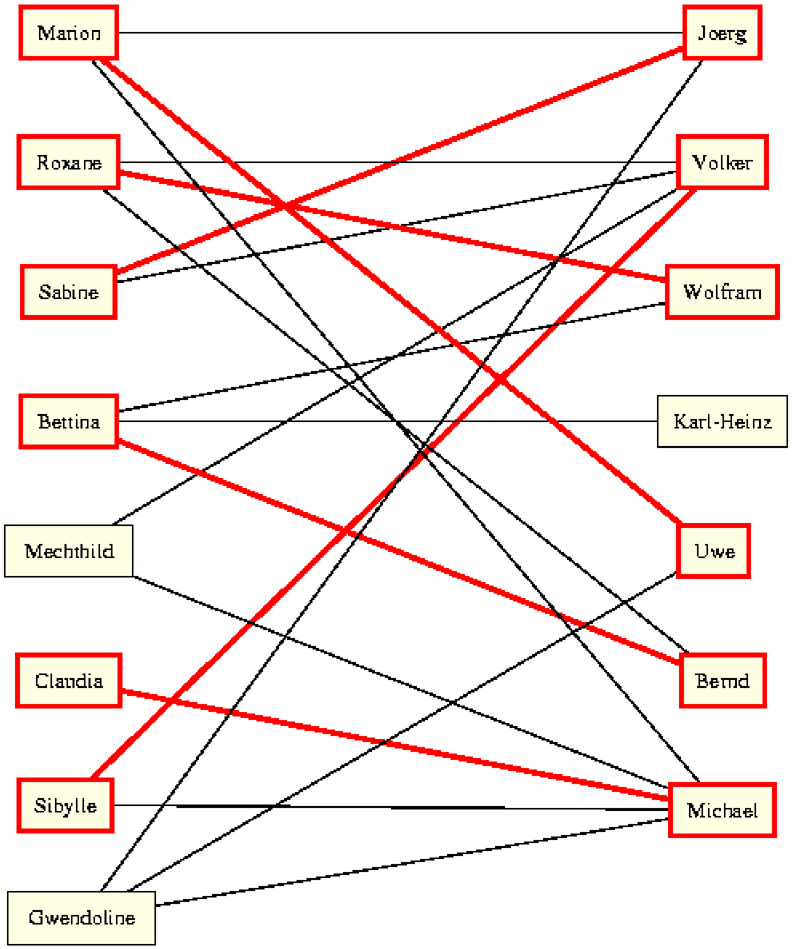
3.4 Testing Bipartiteness



Reference: http://www.leda-tutorial.org/en/unofficial/ch05s03s05.html

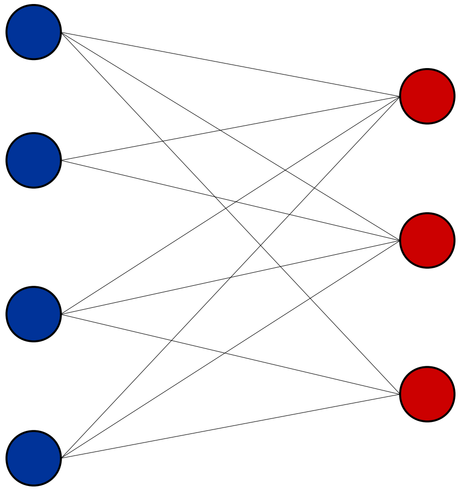
Bipartite Graphs

Def. An undirected graph G = (V, E) is bipartite if the nodes can be colored red or blue such that every edge has one red and one blue end.

Applications.

Matching: men = red, women = blue.

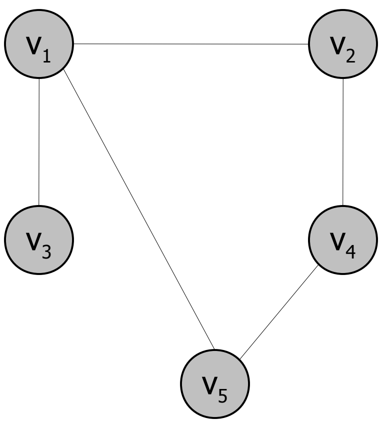
Scheduling: machines = red, jobs = blue.



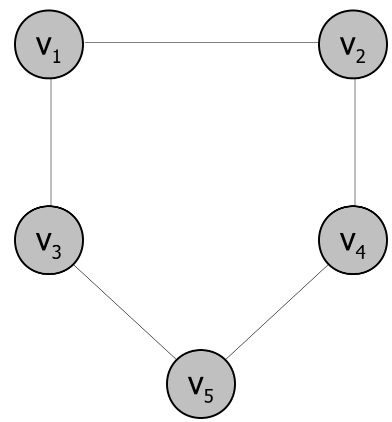
a bipartite graph

Testing Bipartiteness

Q. Which of the following graphs is/are bipartite?



G_1



G_2

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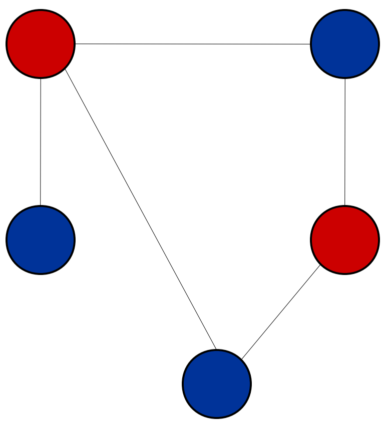
Testing Bipartiteness

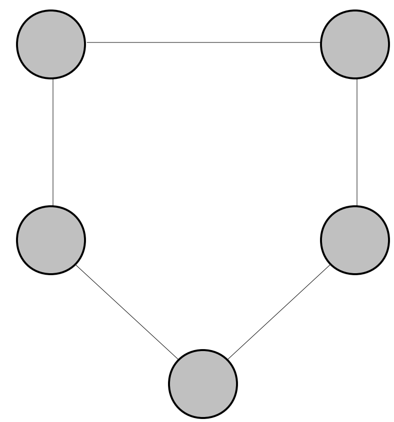
Q. Which of the following graphs is/are bipartite?

G_1



bipartite
(2-colorable)

G_2



not bipartite
(not 2-colorable)

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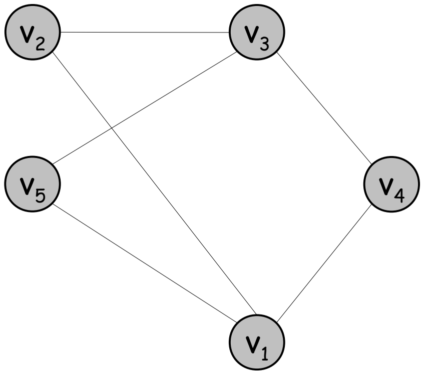
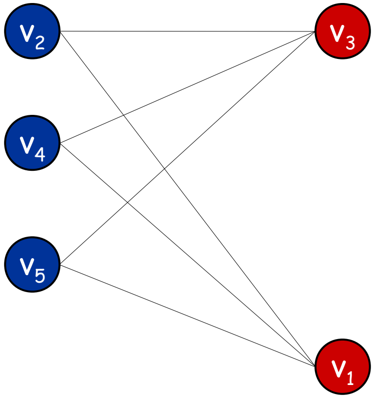
Testing Bipartiteness

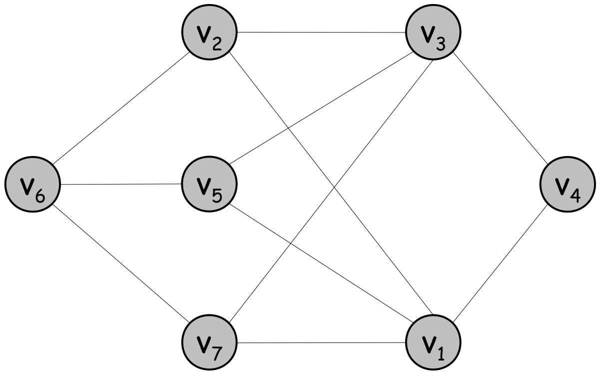


Q. Is graph G bipartite?



another drawing of G

Testing Bipartiteness



Q. Is graph G bipartite?

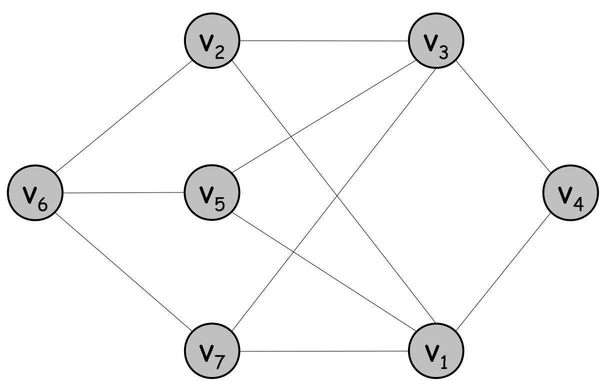
Testing Bipartiteness

Testing bipartiteness. Given a graph G, is it bipartite?

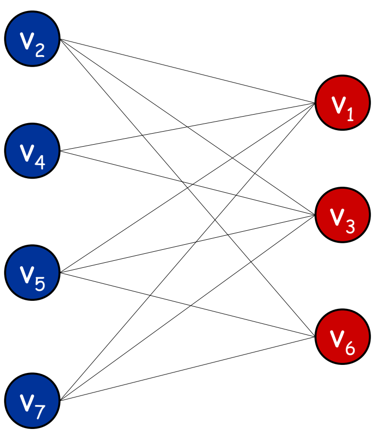
Many graph problems become:

- easier if the underlying graph is bipartite (matching)
- tractable if the underlying graph is bipartite (independent set)

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G



another drawing of G

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Testing Bipartiteness

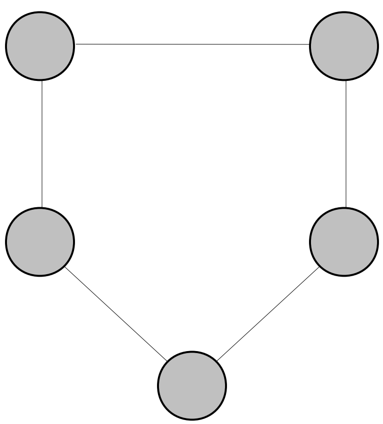
Testing bipartiteness. Given a graph G, is it bipartite?

Many graph problems become:

- easier if the underlying graph is bipartite (matching)
- tractable if the underlying graph is bipartite (independent set)

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.

Q. Why is the following graph not bipartite?



not bipartite
(not 2-colorable)

An Obstruction to Bipartiteness

Lemma 3.14. If graph G is bipartite, it cannot contain an odd length cycle.
(ie "G bipartite"$\rightarrow$ not "odd cycle")

Pf.

An Obstruction to Bipartiteness

Lemma 3.14. If graph G is bipartite, it cannot contain an odd length cycle.
(ie “G bipartite”$\rightarrow$ not ”odd cycle”)

Pf. (by proving the contrapositive: $(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$) (=contrapositie)
Suppose there is an odd cycle. $(\neg B)$
…
So G is not bipartite. $(\neg A)$ (So ”odd cycle” $\rightarrow$ not “G bipartite”)

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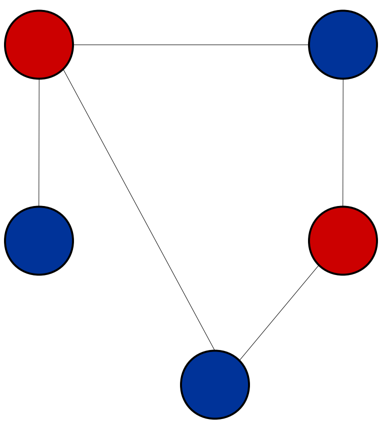
bipartite
(2-colorable)

not bipartite
(not 2-colorable)

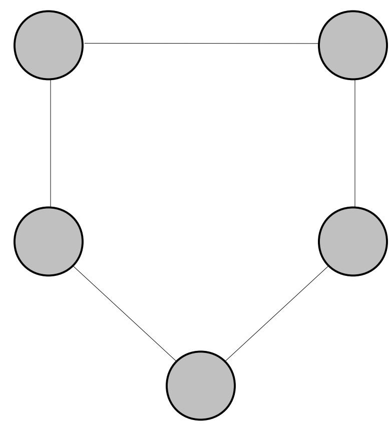
An Obstruction to Bipartiteness

Lemma 3.14. If graph G is bipartite, it cannot contain an odd length cycle.
(ie "G bipartite"$\rightarrow$ not "odd cycle")

Pf. (by proving the contrapositive: $(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$) (=contrapositie)
Suppose there is an odd cycle. ($\neg$B)
Then it is not possible to 2-color the odd cycle, let alone G.
So G is not bipartite. ($\neg$A) (So "odd cycle" $\rightarrow$ not "G bipartite") ▪



bipartite
(2-colorable)



not bipartite
(not 2-colorable)

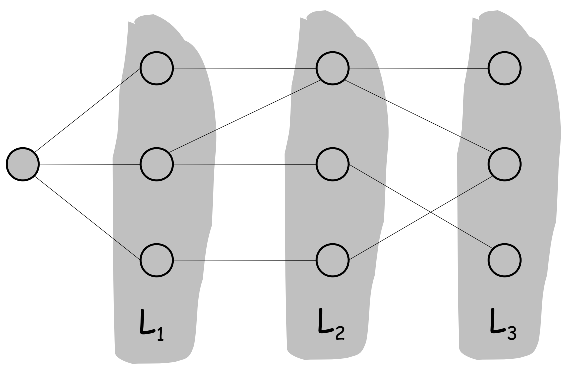
This proof can be found
on page 95.

Bipartite Graphs

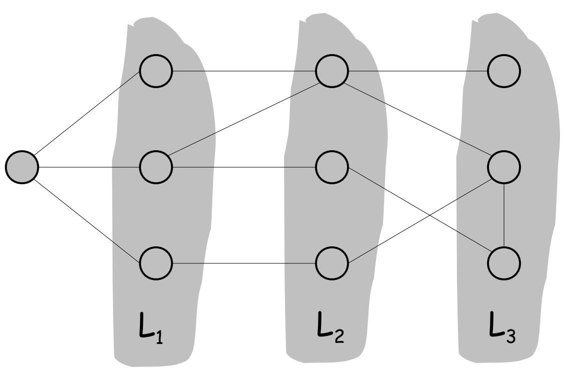
Lemma 3.15. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer; so G is bipartite.

(ii) An edge of G joins two nodes of the same layer; so G contains an odd-length cycle (and hence is not bipartite).



Case (i)



Case (ii)

Lemma 3.15. Let G be a connected graph, and let L_0, …, L_k be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer; so G is bipartite.

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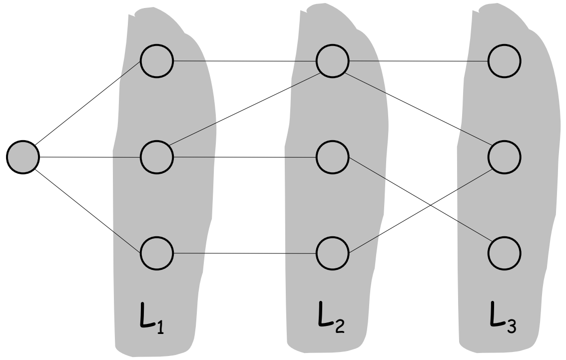
Pf. (i)

Suppose no edge joins two nodes in the same layer.

Q. How to prove that G is bipartite? How would you color the nodes?



Case (i)

Bipartite Graphs

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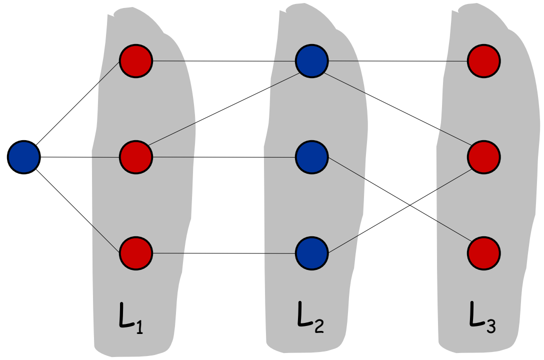
Pf. (i)

Suppose no edge joins two nodes in the same layer.

Bipartition: red = nodes on odd levels, blue = nodes on even levels.



Case (i)

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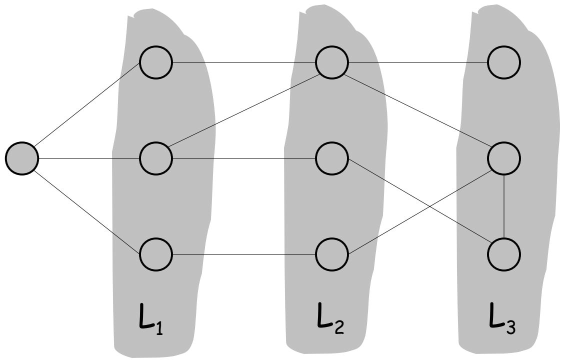
Pf. (ii)

Suppose (x, y) is an edge with x, y in same level L_j.

Q. Is there a cycle? Is it of odd length?



Bipartite Graphs

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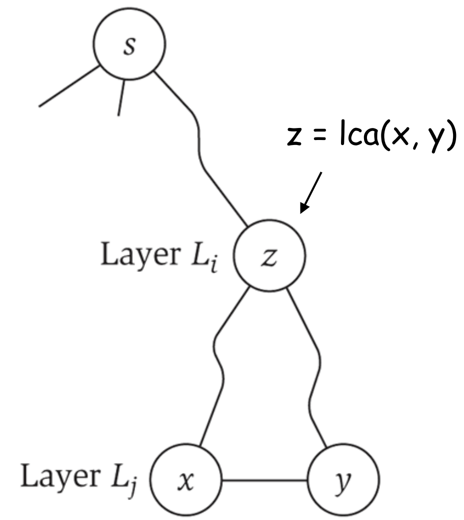
Pf. (ii)

Suppose (x, y) is an edge with x, y in same level L_j.

Let z = lca(x, y) = lowest common ancestor.



Bipartite Graphs

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Pf. (ii)

Suppose (x, y) is an edge with x, y in same level L_j.

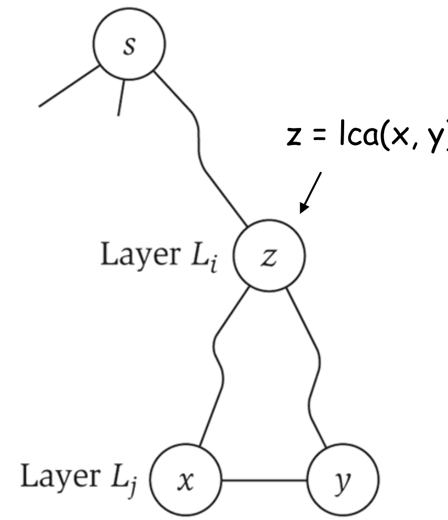
Let z = lca(x, y) = lowest common ancestor.

Let L_i be level containing z.

Consider cycle that takes edge from x to y,

then path from y to z, then path from z to x. Distinct.

Its length is $\underbrace{1}_{(x,\,y)} + \underbrace{(j-i)}_{\text{path from y to z}} + \underbrace{(j-i)}_{\text{path from z to x}}$, which is odd.



With Lemma 3.14, it is not bipartite. ▪

This proof can be found on pages 96-97.

Obstruction to Bipartiteness

Corollary. A graph G is bipartite iff it contains no odd length cycle.
(A↔B)

Q. How to proof?

Obstruction to Bipartiteness

Corollary. A graph G is bipartite iff it contains no odd length cycle.
(A↔B)

Using Lemma 3.14:
If graph G is bipartite, it cannot contain an odd length cycle.
(A→B)

Using Lemma 3.15:
If not (i) bipartite, then (ii) it has an odd length cycle.
Q. How does this help in proving the corollary?

Obstruction to Bipartiteness

Corollary. A graph G is bipartite iff it contains no odd length cycle.
($A \leftrightarrow B$)

Using Lemma 3.14:
If graph G is bipartite, it cannot contain an odd length cycle.
($A \rightarrow B$)

Using Lemma 3.15:
If not (i) bipartite, then (ii) it has an odd length cycle.
($B \rightarrow A$) $\leftrightarrow$ ($\neg A \rightarrow \neg B$) (=contrapositie)

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