

Appendix A

Green's theorem and its applications

In section 2.6 we have introduced the cumulative vehicle count (also local stream function) $\tilde{N}(x, t)$ can be thought of as labelling the vehicles on the roadway that pass cross-section x until time instant t . Fig. A.1 illustrates this in case there is passing (vehicle 7 passes vehicle 8, causing their indices to interchange).

Theorem 67 (*Green's Theorem*) - *The line integral of the projection of the gradient along any curve joining two points is independent of the curve, and that the result equals the difference in $\tilde{N}$ at the extremes of the curve.*

In our case, we consider a curve C in the (x, t) plane between two points (x_1, t_1) and (x_2, t_2) . According to Green's theorem, we have

$$\tilde{N}(x_2, t_2) - \tilde{N}(x_1, t_1) = \int_C \frac{\partial \tilde{N}}{\partial t} dt + \frac{\partial \tilde{N}}{\partial x} dx = \int_C q dt - k dx \quad \text{for all curves } C \quad (\text{A.1})$$

The validity of Eq. (A.1) is understood easily by parameterizing the curve C , i.e.

$$C = \{(x(s), t(s)) \mid \text{with } s_1 \leq s \leq s_2 \text{ s.t. } (x(s_i), t(s_i)) = (x_i, t_i)\} \quad (\text{A.2})$$

Along the curve, we have

$$\frac{d\tilde{N}}{ds} = \frac{\partial \tilde{N}}{\partial t} \frac{dt}{ds} + \frac{\partial \tilde{N}}{\partial x} \frac{dx}{ds} \quad (\text{A.3})$$

and thus, we find that

$$\tilde{N}(s_2) - \tilde{N}(s_1) = \int_{s_1}^{s_2} \frac{d\tilde{N}}{ds} ds = \int_C \frac{\partial \tilde{N}}{\partial t} dt + \frac{\partial \tilde{N}}{\partial x} dx \quad (\text{A.4})$$

By considering a *closed curve* C , we can determine the following form of the conservation equation:

$$\int_C q dt - k dx = 0 \quad \text{for all curves } C \quad (\text{A.5})$$

The main benefit of Eq. (A.5) rather than the p.d.e. is that the equation is also valid if q and k are discontinuous.

Eqns. (A.1) and (A.5) have many applications. Consider for instance an observer travelling a distance L in a period of length T . Consider the number of cars that the observer passes or that pass the observer, assuming that the traffic conditions are stationary with flow q and density k . To determine this number, we can apply eqn. (A.1) by choosing a curve from (x_1, t_1) to $(x_1 + L, t_1 + T)$. For the sake of simplicity, we choose a curve that travels from (x_1, t_1) to $(x_1, t_1 + T)$ and then to $(x_1 + L, t_1 + T)$. We then have

$$n_{obs} = \int_C q dt - k dx = \int_{t_1}^{t_1+T} q dt - \int_{x_1}^{x_1+L} k dx = qT - kL \quad (\text{A.6})$$

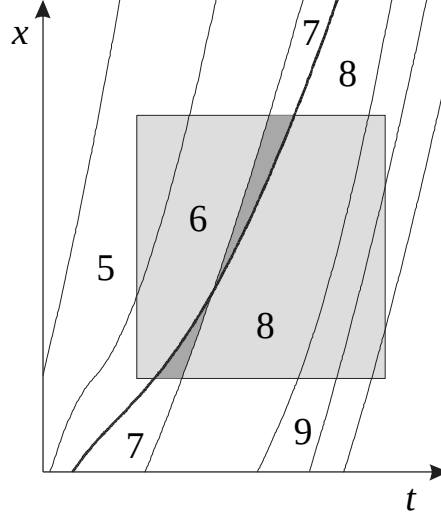


Figure A.1: Values of the cumulative flow function (adopted from [14])

which thus equals the number of vehicles passed by the moving observer. By rearranging the terms in this expression, we can show easily that the relative flow q_{obs} measured by the moving observer equals

$$q_{obs} = \frac{n_{obs}}{T} = q - k \frac{L}{T} = q - kv_{obs} \quad (\text{A.7})$$

where v_{obs} is the speed of the moving observer.

A.1 Alternative analytical solution scheme to the kinematic wave model

The approach discussed in this section is still rather qualitative, and actually determining an analytical solution will be quite painstaking. This section recalls the procedure from [14] to construct analytical solutions of the kinematic wave model. The approach is based on first determining the vehicle trajectories and determining the cumulative flow function $\tilde{N}(x, t)$ from these trajectories.

The thin lines of Fig. A.2 depict a family of characteristics (emanating from the line $t = 0$) corresponding to a hypothetical infinite roadway problem. We again assume that the traffic state $k(x, 0)$ is known at $t = 0$. Let $k(x_0(z), t_0(z)) = k_0(z)$. In the previous section we have already discussed the equations of the z -characteristic (i.e. the characteristic emanating from the $(x_0(z), t_0(z))$). We showed that the wave had slope $c(k) = Q'(k)$.

Since the intensity and the density are constant on the characteristic, we can use eq. (1.6) to evaluate the change in vehicle number seen by an observer travelling along the characteristic, i.e. with velocity $v_o = \tilde{c}(z)$, during a time interval Δt .

$$\Delta \tilde{N} = (q - kv_o) \Delta t = [Q(k_0(z)) - k_0(z)\tilde{c}(z)] = \Delta(z) \Delta t \quad (\text{A.8})$$

where the last equality simply introduces $-\Delta(z)$ as shorthand notation for the factor in brackets; this symbol represents the *known* fixed rate at which the vehicle number increases with time along the z -characteristic.

If the boundary data are specified in terms of density, $k_0(z)$, we first need to find the vehicle number $\tilde{N}_0(z)$ along the boundary for all z . This can be done by rewriting the conservation law eq. (1.3) in parametric form, using for C the portion of the boundary from $z = 0$ to $z = z$

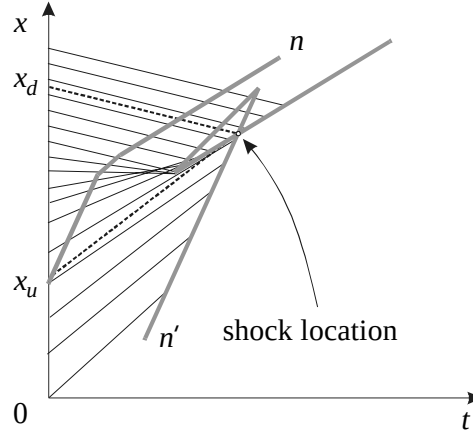


Figure A.2: Map of characteristic displaying the trajectories of vehicles n and n' ($n' > n$)

(end of the region of interest). If we define $N_0(0) = 0$, then the result is

$$\tilde{N}_0(z) - \tilde{N}_0(0) = \int_0^z \left[Q(k_0(\zeta)) \frac{dt}{d\zeta} - k_0(\zeta) \frac{dx}{d\zeta} \right] d\zeta \quad (\text{A.9})$$

which reduces to

$$\tilde{N}_0(z) = \int_0^z -k_0(\zeta) d\zeta \quad (\text{A.10})$$

for the initial value problem (because the boundary satisfies $dt/d\zeta = 0$, and $dx/d\zeta = 1$ in this case).

In any case, once $\tilde{N}_0(z)$ and $\Delta(z)$ have been determined, one can express the vehicle number at time s on the z -characteristic, denoted by $S(s; z)$ as follows

$$S(s; z) = \tilde{N}_0(z) + \Delta(z) [s - t_0(z)] \quad (\text{A.11})$$

Then, on setting $S(s; z)$ equal to an arbitrary n and solving it for s , one obtains the time at which vehicle n crosses the z -characteristic

$$s = t_0(z) + [n - \tilde{N}_0(z)] / \Delta(z) \quad (\text{A.12})$$

This equation can be evaluated for every characteristic shown in a sketch such as Fig. A.2. If the point on each characteristic corresponds to the calculated time is darkened, then the smooth darkened line connecting the points will be the trajectory of the n -th vehicle. Thus, we can view characteristics, not just as waves that tell every car what spacing and speed to adopt, but also when and where they should be.

If the n -th vehicle traverses a region in the (x, t) plane where the characteristics do not intersect as occurs for vehicle n in Fig. A.2, then the result of the procedure identifies the actual vehicle trajectory. As n is increased however, characteristics may begin to intersect and it may no longer be clear to which signal a vehicle should respond as it traverses such a region. In this case the purely mathematical recipe [14] described returns a parametric trajectory that intersects all the characteristics but also exhibits some loops, as shown by curve n' of Fig. A.2. Although such a curve is mathematically valid, it cannot be a real vehicle trajectory and something must be done to fix the problem.

Note that if we remove the loop from Fig. A.2 then the remaining portion of the curve is a valid and physically meaningful vehicle trajectory in that the vehicle speed (and the flow and density along its trajectory) is at all times equal to that demanded by the pertinent characteristic. Thus, points on the physically relevant trajectory still satisfy the conditions of the

kinematic wave model. To obtain this result, we had to avoid multi-signals, i.e. crossings of the characteristics. This was achieved by ignoring all signals that made the loop (i.e. those issued between x_u and x_d). The solution then includes a sudden vehicular deceleration at the point P of contact with the loop.

Note as well that the characteristics issued at x_u and x_d cannot effect the following vehicles, since they cross at point P , and must therefore terminate there. In the complete solution with all the vehicle trajectories, the collection of all points such as P (where trajectories bend and characteristics terminate) defined curves which have been called *shocks* in the previous sections.

Without further proof, we note that when $Q(k)$ is concave, the family of *active (truncated) characteristics* covers the whole region of the (x, t) plane without crossing shocks or one another, i.e. *there is one and only one characteristic passing through every point that is not on a shock*. This fundamental property can be expressed in two equivalent ways:

1. Every point in the solution can be connected in a unique way to the boundary through a truncated characteristic that does not cross any shocks or other truncated characteristics.
2. 'Entropy' condition: truncated characteristics run into a shock.

From (1) and (2) it follows that shocks can merge into a combined shock but cannot diverge. Furthermore, once a shock has formed, e.g., at time $t_s > 0$ the solution must contain a shock at any future time $t_f > t_s$.

Daganzo [14] notes that the collection of all the real (loopless) trajectories identifies the physically relevant solution to our problem in its entirety in that they give a 'topographical map' of $\tilde{N}(x, t)$. For more details on the approach, as well as yet other alternative approaches to determine analytical solutions, we refer to [14].

In practise, the approach can be implemented very easily in a spreadsheet. This requires the following steps:

1. Determine the x -locations from which the z -characteristics emanate at $t = 0$, i.e. $z_i = i\Delta z$, for $i = 0, \dots, m$ where $\Delta z = L/m$. m denotes the number of characteristics used to determine the solutions.
2. For all i , determine the initial density $k_0(z_i)$ at $x_i = z_i$ and $t = 0$.
3. For each characteristic, determine the rate $\Delta(z_i)$ at which the vehicle count increase along the z_i -characteristic. This is done using the equation $\Delta(z_i) = Q(k_0(z_i)) - c(k_0(z_i)) \cdot k_0(z_i)$.
4. Determine the number of vehicles at $t = 0$ for each characteristic z_i by numerically solving $\tilde{N}_0(z_i) = -\int_0^{z_i} k_0(\zeta) d\zeta$.
5. The trajectory of vehicle n is given by the points (x_i, t_i) where $t_i = \left[n - \tilde{N}_0(z_i) \right] / \Delta(z_i)$ and $x_i = z_i + c(k_i) t_i$

The remaining problem is to remove the loops. How to solve this problem is left to the reader. Fig. A.3 shows application of the procedure to describe the acceleration of vehicles from a queue of $200m$. The results resulting from the approach of Daganzo are similar to the approach described in the previous section.

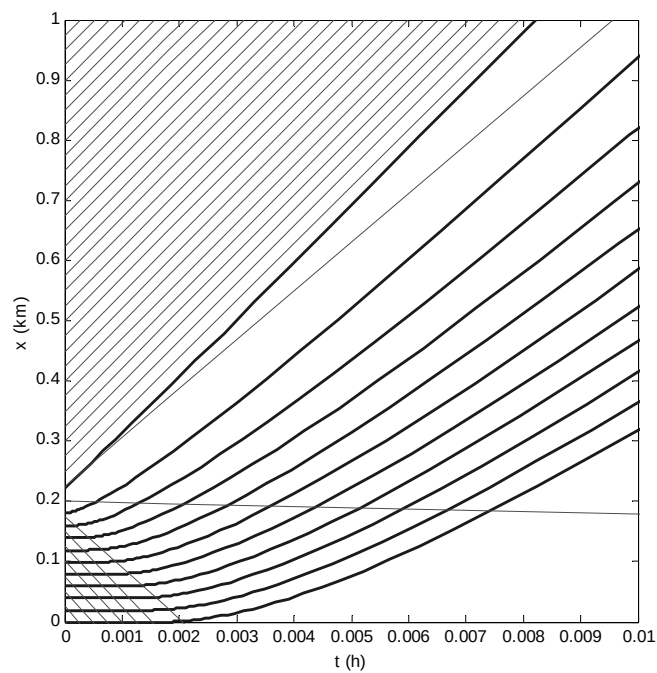


Figure A.3: Vehicles driving away from standing queue using Greenshields intensity-density curve. Thick lines reflect trajectories; thin lines are kinematic waves. Notice how the procedure predicts that the instantaneously accelerating first vehicle sends out an acceleration fan, causing the following vehicles to smoothly accelerate from the queue.

