

Appendix B

Mathematical and numerical analysis of higher-order models

This appendix discusses the mathematical properties as well as a numerical solution approach for the Payne model, which consist of the conservation of vehicle equation

$$\frac{\partial k}{\partial t} + \frac{\partial (ku)}{\partial x} = 0 \quad (\text{B.1})$$

and the speed dynamics equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{V(k) - u}{\tau} - \frac{c^2(k)}{k} \frac{\partial k}{\partial x} \quad (\text{B.2})$$

where $V(k) = Q(k)/k$ denotes the equilibrium speed.

B.1 Riemann variables

To study the behaviour of the Payne model, this form of the model is not convenient: the conservation equation depends on the speed, while the speed dynamics depend on the density. To mathematically analyse the model, a different form of the model is preferred, which is referred to as the *characteristic form*. To derive this characteristic form, we first the equations in their *quasi-linear form*. Let the vector $w = (k, u)$; then the following equation is equivalent to Eqns. (B.1) and (B.2):

$$\frac{\partial}{\partial t} \begin{pmatrix} k \\ u \end{pmatrix} + \begin{pmatrix} u & k \\ \frac{c^2(k)}{k} & u \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} k \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{V(k)-u}{\tau} \end{pmatrix} \quad (\text{B.3})$$

$$\frac{\partial w}{\partial t} + A(w) \frac{\partial w}{\partial x} = B(w) \quad (\text{B.4})$$

The aim is to *diagonalise* the matrix $A(w)$. This can be achieved easily by determining the eigenvalues¹ and the eigenvectors² of $A(w)$. Some straightforward computations show that the eigenvalues $\lambda_{1,2}$ of the matrix $A(w)$ are equal to

$$\lambda_1 = u + c(k) \quad (\text{B.5})$$

and

$$\lambda_2 = u - c(k) \quad (\text{B.6})$$

¹The eigenvalues λ of the matrix A are defined by those values satisfying $\det(A - \lambda I) = 0$.

²An eigenvector e of the matrix A stemming from the eigenvalue λ is *any vector* that is in the null-space $\mathcal{N}(A - \lambda I)$ of the matrix $A - \lambda I$, i.e. e satisfies $(A - \lambda I)e = 0$.

The corresponding eigenvalues are $e_1 = \left(\frac{k}{2c(k)}, \frac{1}{2}\right)$ and $e_2 = \left(-\frac{k}{2c(k)}, \frac{1}{2}\right)$. Let us now define the matrix of eigenvectors S

$$S = \begin{pmatrix} \frac{k}{2c(k)} & -\frac{k}{2c(k)} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{with} \quad S^{-1} = \begin{pmatrix} \frac{c(k)}{k} & 1 \\ -\frac{c(k)}{k} & 1 \end{pmatrix} \quad (\text{B.7})$$

Using these matrices S and S^{-1} , we can determine the so-called *compability equations* by left-multiplying the eqn. (B.4), we get

$$S^{-1} \frac{\partial w}{\partial t} + S^{-1} A(w) \frac{\partial w}{\partial x} = S^{-1} B(w) \quad (\text{B.8})$$

Let us now consider the so-called Riemann variables z as follows. Consider any variation δw (either $\partial w / \partial t$ or $\partial w / \partial x$) in w and defined

$$\delta z = S^{-1} \delta w \quad (\text{B.9})$$

Inversely, one has

$$\delta w = S \delta z \quad (\text{B.10})$$

Note that these definition imply

$$S \frac{\partial z}{\partial t} = \frac{\partial w}{\partial t} \quad \text{and} \quad S \frac{\partial z}{\partial x} = \frac{\partial w}{\partial x} \quad (\text{B.11})$$

which means that the compability equation (B.8) can be written as follows:

$$S^{-1} S \frac{\partial z}{\partial t} + S^{-1} A(w) S \frac{\partial z}{\partial x} = \frac{\partial z}{\partial t} + \Lambda(w) \frac{\partial z}{\partial x} = \tilde{B}(z) \quad (\text{B.12})$$

where $\Lambda(w) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} u + c(k) & 0 \\ 0 & u - c(k) \end{pmatrix}$ and $\tilde{B}(z) = S^{-1} B(w)$.

If we consider the definition of the Riemann (or characteristic) variables, we see that

$$\delta z_1 = \delta v + \frac{c(k)}{k} \delta k \quad (\text{B.13})$$

and

$$\delta z_2 = \delta v - \frac{c(k)}{k} \delta k \quad (\text{B.14})$$

If we assume that the wave speed $c^2(k) = c_0^2$, when we can determine by simple integration that the Riemann variables are equal to

$$z_1 = \int \delta v + c_0 \frac{\delta k}{k} = v + c_0 \ln k \quad (\text{B.15})$$

$$z_2 = \int \delta v - c_0 \frac{\delta k}{k} = v - c_0 \ln k \quad (\text{B.16})$$

For the Payne model, where $c^2(k) = -\frac{1}{2\tau} \frac{dV}{dk}$, the Riemann variables depend on the precise shape of the fundamental diagram. For instance, for Greenshield's function, we have $c^2(k) = \frac{1}{2\tau} a$, where a is the slope of the linear speed density curve.

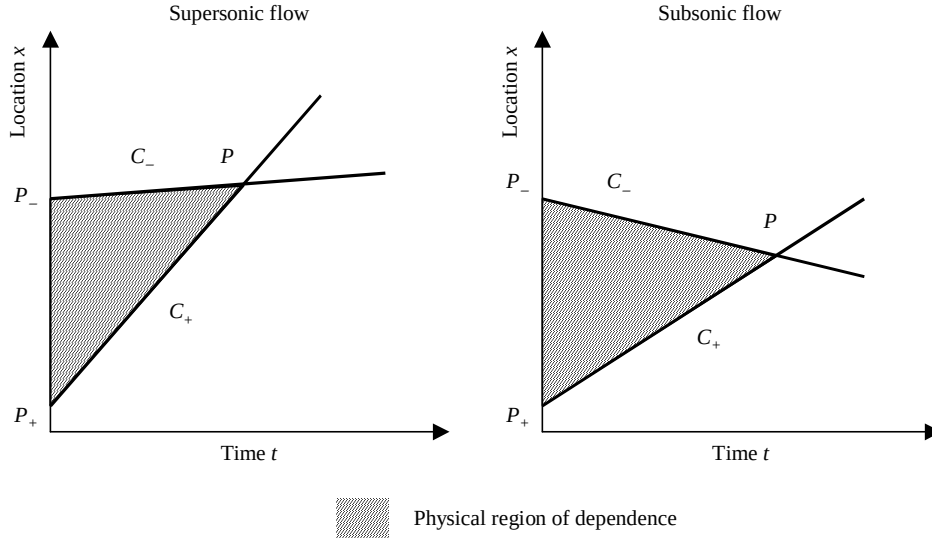


Figure B.1: Propagation of flow quantities.

B.2 Propagation of disturbances

The decoupling applied in the previous section, yielded the following decoupled partial differential equation for the Riemann variables $z_{1,2}$

$$\frac{\partial z_1}{\partial t} + \lambda_1 \frac{\partial z_1}{\partial x} = \tilde{B}_1(z) \quad (\text{B.17})$$

$$\frac{\partial z_2}{\partial t} + \lambda_2 \frac{\partial z_2}{\partial x} = \tilde{B}_2(z) \quad (\text{B.18})$$

These equations show that the quantities $z_{1,2}$ propagate along the corresponding characteristics with speed $\lambda_{1,2} = (u \pm c(k))$. Thus, $\delta z_1 = \delta v + \frac{c(k)}{k} \delta k$ propagates along the characteristic C_+ defined by $\frac{dx}{dt} = (u + c(k))$; $\delta z_2 = \delta v - \frac{c(k)}{k} \delta k$ propagates along the characteristic C_- defined by $\frac{dx}{dt} = (u - c(k))$.

Similar to applying the method of characteristics to the LWR model, the method of characteristics can be applied to the higher-order model discussed here. If we consider any point $P(x, t)$, we know that the density and speed at this point are determined from the quantities $z_{1,2}$ transported with speed $(u \pm c(k))$ along the characteristics, that is

$$(z_1)_P = (u + c_0 \ln k)_P = (u + c_0 \ln k)_{P_+} = (z_1)_{P_+} \quad (\text{B.19})$$

$$(z_2)_P = (u - c_0 \ln k)_P = (u - c_0 \ln k)_{P_-} = (z_2)_{P_-} \quad (\text{B.20})$$

The left side of Fig. B.1 is drawn for the case of supersonic flow, while for a subsonic flow, the C_- characteristic has a negative slope and one has the situation shown on the right side of the figure. Each point P in the (x, t) plane is reached by one characteristic of each type.

For the LWR model, we have seen that when $\frac{\partial u}{\partial x} < 0$ (speed decreases over space), that shocks will occur. This will generally also in the higher-order models: when $\frac{\partial(u+c)}{\partial x} < 0$ one can have the situation depicted in Fig. B.2, where the C_+ characteristic from P_{1+} intersects the C_+ characteristic from P_+ , and hence multi-valued quantities would occur in P_1 ; that is, one would have

$$(u + c_0 \ln k)_{P_1} = (u + c_0 \ln k)_{P_+} \quad (\text{B.21})$$

and

$$(u + c_0 \ln k)_{P_1} = (u + c_0 \ln k)_{P_{+1}} \quad (\text{B.22})$$

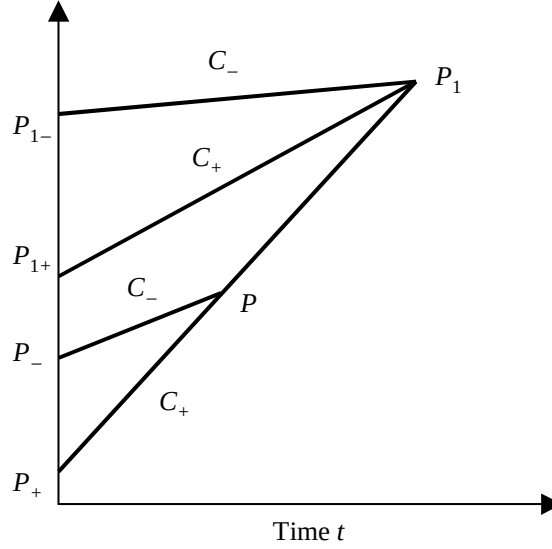


Figure B.2: Intersecting characteristics with $\frac{\partial}{\partial x}(u + c) < 0$.

where

$$(u + c_0 \ln k)_{P_+} \neq (u + c_0 \ln k)_{P_{+1}} \quad (\text{B.23})$$

This impossible situation leads to a discontinuity in the flow that we refer to as a shock wave.

As mentioned earlier, the Payne model has two families of characteristics, while the LWR model only has one. As a result, the Payne model has *two families of shock and expansion waves*: the first family is associated with the first characteristic C_+ , while the second is associated with the second characteristic C_- .

B.3 Instabilities and jam formation

It is clear that the Payne model allows homogeneous and stationary equilibrium solutions (k_e, u_e) , which satisfy $u_e = V(k_e)$. We will now examine the stability of this solution. This is achieved by considering small disturbances $\delta k(x, t), \delta u(x, t)$ to the solution. Let us furthermore assume that

$$k(x, t) = k_e(x, t) + \delta k(x, t) \text{ and } u(x, t) = u_e(x, t) + \delta u(x, t) \quad (\text{B.24})$$

is a solution of the Payne model. If we assume that $\delta k(x, t)$ and $\delta u(x, t)$ are small, we can linearise Eqns. (B.1) and (B.2) resulting in linear partial differential equations for $\delta k(x, t)$ and $\delta u(x, t)$

$$\frac{\partial(\delta k)}{\partial t} + u_e \frac{\partial(\delta k)}{\partial x} = k_e \frac{\partial(\delta u)}{\partial x} \quad (\text{B.25})$$

and

$$\frac{\partial(\delta u)}{\partial t} + u_e \frac{\partial(\delta u)}{\partial x} = -\frac{c^2(k_e)}{k_e} \frac{\partial(\delta k)}{\partial x} + \frac{1}{\tau} \left[\frac{dV}{dk} \delta k - \delta u \right] \quad (\text{B.26})$$

Using simple wave analysis, we can write the solutions $\delta k(x, t)$ and $\delta u(x, t)$ in terms of the following series

$$\delta k(x, t) = \int_s \sum_j k_j^*(s) \exp(i \cdot sx + [\lambda_j(k_e, s) - i \cdot \omega_j(k_e, s)] t) ds \quad (\text{B.27})$$

and

$$\delta u(x, t) = \int_s \sum_j u_j^*(s) \exp(i \cdot sx + [\lambda_j(k_e, s) - i \cdot \omega_j(k_e, s)] t) ds \quad (\text{B.28})$$

where $i = \sqrt{-1}$. In fact, we assume that the disturbances δk and δu can be decomposed into simple waves, traveling with a certain speed ω , while its amplitude increases / decreases according to λ_j . Let us now consider one simple wave solution

$$\delta k(x, t) = k^* \exp(i \cdot sx + [\lambda - i \cdot \omega]) \quad (\text{B.29})$$

$$\delta u(x, t) = u^* \exp(i \cdot sx + [\lambda - i \cdot \omega]) \quad (\text{B.30})$$

Note that s denotes the wave length, ω denotes the wave's frequency, and λ the wave's damping rate; k^* and u^* are amplitudes. If we substitute Eqns. (B.29) and (B.30) into Eqns. (B.25) and (B.26), we find the following set of algebraic equations

$$M \begin{pmatrix} k^* \\ u^* \end{pmatrix} = 0 \quad (\text{B.31})$$

with

$$M = \begin{pmatrix} -\tilde{\lambda} & -i \cdot s k_e \\ -\frac{i \cdot s}{k_e} c^2(k_e) + \frac{1}{\tau} \frac{dV}{dk} & -\tilde{\lambda} - \frac{1}{\tau} \end{pmatrix} \quad (\text{B.32})$$

where we have used

$$\tilde{\lambda} = \lambda - i \cdot \tilde{\omega} \quad \text{with} \quad \tilde{\omega} = \omega - sV(k_e) \quad (\text{B.33})$$

Eqn. (B.31) will only have a non-zero solution for certain values of $\tilde{\lambda}$, namely the eigenvalues³ of

$$\begin{pmatrix} 0 & -i \cdot s k_e \\ -\frac{i \cdot s}{k_e} c^2(k_e) + \frac{1}{\tau} \frac{dV}{dk} & -\frac{1}{\tau} \end{pmatrix} \quad (\text{B.34})$$

To compute these eigenvalues, we determine the characteristic polynomial $\zeta(\tilde{\lambda}) = \det(M)^4$, which equals

$$\zeta(\tilde{\lambda}) = \tilde{\lambda}^2 + \frac{1}{\tau} \tilde{\lambda} + i \cdot s k_e \left[-\frac{i \cdot s}{k_e} c^2(k_e) + \frac{1}{\tau} \frac{dV}{dk} \right] = 0 \quad (\text{B.35})$$

The two solutions to this equation are

$$\tilde{\lambda}_{1,2} = \tilde{\lambda}_{1,2}(k_e, s) = -\frac{1}{2\tau} \pm \sqrt{\frac{1}{4\tau^2} - (C_r + i \cdot C_i)} \quad (\text{B.36})$$

$$= -\frac{1}{2\tau} \pm \left[\sqrt{\mathcal{R} + i \cdot |\mathcal{I}|} \right] \quad (\text{B.37})$$

$$= -\frac{1}{2\tau} \pm \left[\sqrt{\frac{1}{2} \left(\sqrt{\mathcal{R}^2 + \mathcal{I}^2} + \mathcal{R} \right)} \pm i \cdot \sqrt{\frac{1}{2} \left(\sqrt{\mathcal{R}^2 + \mathcal{I}^2} - \mathcal{R} \right)} \right] \quad (\text{B.38})$$

where

$$C_r = s^2 c^2(k_e) \geq 0 \quad \text{and} \quad C_i = \frac{s k_e}{\tau} \frac{dV}{dk} \leq 0 \quad (\text{B.39})$$

and

$$\mathcal{R} = \frac{1}{4\tau^2} - C_r \quad \text{and} \quad \pm |\mathcal{I}| = -C_i = \frac{s k_e}{\tau} \left| \frac{dV}{dk} \right| \quad (\text{B.40})$$

In sum: when the wave damping factors $\tilde{\lambda}(k_e, s)$ are equal to Eqn. (B.38), we can find amplitudes k_* , u_* that satisfy Eqn. (B.31), which means that the simple waves (B.29) and (B.30) are solutions to the partial differential equations (B.25) and (B.26). This means that the simple waves provide a *non-zero contribution* to the solution given by Eqns. (B.3) and (B.4).

³Recall that by definition, the eigenvalues λ of a matrix M are defined by those values for which $\det(M - \lambda I) = 0$, which in turn implies that the matrix $M - \lambda I$ is singular and has a non-empty null-space.

⁴Recall that for any 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $\det(A) = ad - bc$.

Let us now consider the conditions under which the contributions of the simple waves (B.29) and (B.30) will cause the solution (B.3) and (B.4) to become unstable. This will be the case when either of the damping factors $\tilde{\lambda}_{1,2}$ has a real part which is larger than zero, which would mean that the perturbation would grow, rather than decrease. Since the real-part of $\tilde{\lambda}_1$ is per definition larger than the real-part of $\tilde{\lambda}_2$, we only have to consider $\tilde{\lambda}_1$. Let us now study under which circumstances $\text{Re}(\tilde{\lambda}_1) \geq 0$. We then have

$$\text{Re}(\tilde{\lambda}_1) \geq -\frac{1}{2\tau} + \sqrt{\frac{1}{2}(\sqrt{\mathcal{R}^2 + \mathcal{I}^2} + \mathcal{R})} = 0 \quad (\text{B.41})$$

which implies that

$$\left[\frac{1}{4\tau^2} - \frac{\mathcal{R}}{2} \right]^2 \leq \frac{1}{4}(\mathcal{R}^2 + \mathcal{I}^2) \quad (\text{B.42})$$

By simple substitution, we can conclude that the Payne model is unstable, under the following conditions

$$k_e \left| \frac{dV}{dk} \right| \geq c(k) \quad (\text{B.43})$$

B.4 Numerical solutions to the Payne model

Given that the progress regarding the mathematical analysis of the Payne model lacks far behind the LWR model, the schemes that are applicable to the Payne model are less far developed. In this section, we will extend the flux-splitting scheme developed for the LWR model. To this end, we first need to rewrite the Payne model - which is given in its so-called primitive form, i.e. describing the dynamics of density and speed - in its conservative form, that is describing the dynamics of the density $k(x, t)$ and the flow $q(x, t)$. When this is achieved, we apply a flux-splitting approach to the model in an equivalent way that was used to determine the flux-splitting scheme for the LWR model

B.4.1 Payne model in conservative form

The form is called conservative, since the density and the flow are conserved (i.e. change only by inflow minus outflow) in case only convective processes are present. Let us consider the speed dynamics equation (B.2) and multiply it by the density $k(x, t)$. Subsequently we multiply the conservation of vehicle equation (B.1) by the speed $u(x, t)$ and add it to the left hand side of the former result. We get the following result

$$k \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + u \left(\frac{\partial k}{\partial t} + \frac{\partial(ku)}{\partial x} \right) = k \frac{V(k) - u}{\tau} - k \frac{c^2(k)}{k} \frac{\partial k}{\partial x} \quad (\text{B.44})$$

which can be written as follows

$$k \frac{\partial u}{\partial t} + u \frac{\partial k}{\partial t} + ku \frac{\partial u}{\partial x} + u \frac{\partial(ku)}{\partial x} = \frac{\partial q}{\partial t} + \frac{\partial(uq)}{\partial x} \quad (\text{B.45})$$

$$= \frac{Q(k) - q}{\tau} - c^2(k) \frac{\partial k}{\partial x} \quad (\text{B.46})$$

Let us define the function $P(k)$ as follows

$$\frac{dP}{dk} = c^2(k) \quad (\text{B.47})$$

thus

$$c^2(k) \frac{\partial k}{\partial x} = \frac{\partial P(k)}{\partial x} \quad (\text{B.48})$$

Then, we speed dynamics equation can be transformed into the following equation describing the dynamics of the flow

$$\frac{\partial q}{\partial t} + \frac{\partial (uq + P(k))}{\partial x} = \frac{Q(k) - q}{\tau} \quad (\text{B.49})$$

In defining the vector $r = (k, q)$, we can rewrite the conservation of vehicle equation (B.1) and the flow dynamic equation (B.49) in the following form

$$\frac{\partial r}{\partial t} + \frac{\partial F(r)}{\partial x} = D(r) \quad (\text{B.50})$$

where

$$F(r) = F(k, q) = \begin{pmatrix} q \\ \frac{q^2}{k} + P(k) \end{pmatrix} \quad (\text{B.51})$$

and

$$D(r) = D(k, q) = \begin{pmatrix} 0 \\ \frac{Q(k) - q}{\tau} \end{pmatrix} \quad (\text{B.52})$$

B.4.2 Flux-splitting numerical solution approach

This section discusses one of the more refined approaches to numerically approximating solutions to the Payne model. While having its drawbacks, the approach is generally applicable and robust, and provides accurate solutions.

Similar to the flux-splitting approach applied to the LWR model, the considered roadway is split-up into cells i of length l_i . In cell i , we consider space-averaged *densities* and *flows*⁵

$$k_{i,j} = \int_{x_{i-1}}^{x_i} k(x, t_j) dx \quad \text{and} \quad q_{i,j} = \int_{x_{i-1}}^{x_i} q(x, t_j) dx \quad (\text{B.53})$$

The densities and flows change due to convection (inflow and outflow), as well as relaxation processes. The latter reflect traffic accelerating or decelerating to adapt to the equilibrium speed $V(k)$. To model the convective processes, the density-fluxes and the flow-fluxes at the cell interface (described by the vector F in the quasi-linear formulation (B.50)) need to be numerically approximated. Similar to the LWR model, the flux-splitting scheme will split the flux (density and flow flux) F into contributions of the sending cell i and the receiving cell $i + 1$, i.e.

$$F(r_{ij}, r_{i+1j}) = F^+(r_{i,j}) + F^-(r_{i+1,j}) \quad (\text{B.54})$$

subject to $F = F^+ + F^-$.

To determine appropriate definitions for F^+ and F^- , we need to see how the ‘information’ is transported in the flow. In the LWR model, we saw that the densities are transported along the characteristics. For the Payne model, we have seen that the flow properties (in terms of the Riemann variables or characteristic variables) are transported along the characteristics C_+ and C_- defined by (9.109) and (9.110). The characteristic C_+ , defined by $\frac{dx}{dt} = u + c(k) = u + \sqrt{\frac{dP}{dk}}$ always has a positive slope. As a result, the Riemann variable propagates along C_+ in the downstream direction. Thus, the part of the equation that approximates the dynamics of the Riemann variable z_1 depends on the cell i only. However, the characteristic C_- may have a negative slope, i.e. when $u - c(k) = u - \sqrt{\frac{dP}{dk}} < 0$. In this case, the waves carrying the Riemann variable z_2 move upstream (i.e. from cell $i + 1$ to cell i), which should be reflected in the definition of F^- .

⁵We will see that we will explicitly distinguish between space-averaged flows and flows at the cross-section x_i , defining the interface between two cells.

B.4.3 Quasi-linear formulation of Payne model

The approach to determine the correct description for F^+ and F^- is relatively straightforward. First, we rewrite the Payne model in its quasi-linear form

$$\frac{\partial r}{\partial t} + C(r) \frac{\partial r}{\partial x} = D(r) \quad (\text{B.55})$$

where

$$C(r) = \begin{pmatrix} 0 & 1 \\ \frac{dP}{dk} - u^2 & 2u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c^2(k) - u^2 & 2u \end{pmatrix} \quad (\text{B.56})$$

where $u = q/k$. Note that this form is equivalent to the quasi-linear form of Eqns. (B.3) and (B.4). We can use the same approach to diagonalise the matrix $C(r)$, namely by determining the eigenvalues $\lambda_{1,2}$ and the eigenvectors $e_{1,2}$ of the matrix $C(r)$. It turns out that

$$\lambda_{1,2} = u \pm c(k) \quad (\text{B.57})$$

and

$$e_{1,2} = \begin{pmatrix} 1 \\ u \pm c \end{pmatrix} \quad (\text{B.58})$$

Let R denote the matrix of eigenvectors (similar to the matrix S of eigenvectors of $A(w)$). As we have seen before, the Riemann variables can be defined by considering small variations δr and defining

$$\delta z = R^{-1} \delta r \quad (= S^{-1} \delta v) \quad (\text{B.59})$$

The Riemann variables z satisfy the so-called *compatibility equations*

$$\frac{\partial z}{\partial t} + \Lambda(w) \frac{\partial z}{\partial x} = \tilde{D}(z) \quad (\text{B.60})$$

where $\Lambda = R^{-1} C(r) R = S^{-1} A(w) S$, and where $\tilde{D}(z) = R^{-1} B(z)$. It is left to the reader to determine that

$$R = \begin{pmatrix} 1 & 1 \\ u+c & u-c \end{pmatrix} \quad \text{and} \quad R^{-1} = \begin{pmatrix} -\frac{u-c}{2c} & \frac{1}{2c} \\ \frac{u+c}{2c} & -\frac{1}{2c} \end{pmatrix} \quad (\text{B.61})$$

To determine the appropriate flux-splitting, the approach presented in section 9.3.4 is applied to each of the Riemann variables z_1 and z_2 separately. Then, with the definitions

$$\lambda_{1,2}^+ = \frac{\lambda_{1,2} + |\lambda_{1,2}|}{2} \quad (\text{B.62})$$

and

$$\lambda_{1,2}^- = \frac{\lambda_{1,2} - |\lambda_{1,2}|}{2} \quad (\text{B.63})$$

we can define the matrices

$$\Lambda^\pm = \begin{pmatrix} \lambda_1^\pm & 0 \\ 0 & \lambda_2^\pm \end{pmatrix} = \begin{pmatrix} \frac{(u+c) \pm |u+c|}{2} & 0 \\ 0 & \frac{(u-c) \pm |u-c|}{2} \end{pmatrix} \quad (\text{B.64})$$

Clearly, the eigenvalues of Λ^+ are all positive; Λ^- only has negative eigenvalues. That is, Λ^+ describes the contributions of all waves that propagate downstream, while Λ^- describes the contributions of the waves that move upstream. Note that, when $u \geq c$, $\Lambda^+ = \Lambda$ and $\Lambda^- = 0$, implying that only downstream propagating waves are present. When $u < c$

$$\Lambda^+ = \begin{pmatrix} u+c & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Lambda^- = \begin{pmatrix} 0 & 0 \\ 0 & u-c \end{pmatrix} \quad (\text{B.65})$$

Since $C(r) = R\Lambda R^{-1}$, it makes sense to define

$$C^+(r) = R\Lambda^+R^{-1} \quad \text{and} \quad C^-(r) = R\Lambda^-R^{-1} \quad (\text{B.66})$$

Then, $C^+(r)$ describes the contributions of the downstream moving waves, while $C^-(r)$ describes the contributions of the upstream moving waves. Note again that $C^-(r) \neq 0$, only if $u < c(k)$. That is, when $u > c(k)$, we have

$$C^+(r) = C(r) \quad \text{and} \quad C^-(r) = 0 \quad (\text{B.67})$$

For $u < c(k)$, we find

$$C^+(r) = \frac{1}{2c}(u+c) \begin{pmatrix} -(u-c) & 1 \\ -(u+c)(u-c) & (u+c) \end{pmatrix} \quad (\text{B.68})$$

$$C^-(r) = \frac{1}{2c}(u-c) \begin{pmatrix} (u+c) & -1 \\ (u-c)(u+c) & -(u-c) \end{pmatrix} \quad (\text{B.69})$$

Unlike for the LWR model, it is in general not possible to find flux functions $F^+(r)$ and $F^-(r)$ that satisfy $\frac{\partial}{\partial r}F^\pm(r) = C^\pm(r)$. Therefore, we adopt an approximation known as the Steger-Warming scheme, where

$$F^+(r) = C^+(r) \cdot r \quad \text{and} \quad F^-(r) = C^-(r) \cdot r \quad (\text{B.70})$$

It is left to the reader to determine that for $u > c(k)$

$$F^+(r) = \begin{pmatrix} q \\ \frac{q^2}{k} + P(k) \end{pmatrix} \quad \text{and} \quad F^-(r) = 0 \quad (\text{B.71})$$

while for $u < c(k)$, we have

$$F^+(r) = \frac{k}{2} \begin{pmatrix} u+c \\ (u+c)^2 \end{pmatrix} \quad \text{and} \quad F^-(r) = F^+(r) = \frac{k}{2} \begin{pmatrix} u-c \\ (u-c)^2 \end{pmatrix} \quad (\text{B.72})$$

