# Appendix C

# Gas-kinetic modelling

**Remark 68** *Gas-kinetic modelling of traffic flow is an advanced topic that can be skipped on a first reading.*

This section has discussed models that bear a strong resemblence to the models used to describe the dynamics of a fluid or a gas, or other continuous media. Therefore, these models are often referred to as *continuum models*. There is however, one specific type of continuum model that has not yet been discussed in this chapter, namely the so-called *gas-kinetic models*. These models offer a more refined and complicated description of traffic, and are deduced from theories applied in theoretical physics.

## C.1 Prigogine and Herman model

The starting point of the gas-kinetic models, is the so-called *phase-space density (PSD)*

$$\kappa\left(x,t,v\right) \tag{C.1}$$

The PSD is a function of the location $x$, the time $t$, and the speed $v$. Simply stating, it describes not only the mean number of vehicles per unit roadway length (such as the density), but also includes information on the speed distribution at that location. First, recall that by definition, the number

$$k(x,t)dx \tag{C.2}$$

denotes the probability that at time-instant $t$ in the small roadway segment $[x, x+dx)$ a vehicle is present. Equivalently, the number

$$\kappa\left(x,t,v\right)dxdv \tag{C.3}$$

denotes the probability that at time-instant $t$ in a small roadway segment $[x, x+dx)$ a vehicle is present *with a speed in the region* $[v, v+dv)$. Note that

$$\kappa\left(x,t,v\right) = f(x,t,v)\cdot k(x,t) \tag{C.4}$$

where $f\left(x,t,v\right)$ denotes the probability density function describing the speed distribution of the vehicles at $x$.

Prigogine and Herman were the first to use the notion of the PSD to derive a model describing the behaviour of traffic flow. They achieved this by assuming that the PSD changes according to the following processes:

1. *Convection* $\frac{\partial(\kappa v)}{\partial x}$. Vehicles with a speed $v$ flow into and out of the roadway segment $[x, x+dx)$, causing a change in the PSD $\kappa(x,t,v)$. This term is equivalent to the term $\frac{\partial(ku)}{\partial x}$ in the conservation of vehicle equation.

2. *Acceleration towards the desired speed* $\frac{V_0(x,t,v)-v}{\tau}$. Vehicles not driving at their desired speed will accelerate; $V_0(x,t,v)$ denotes the expected desired speed of vehicles driving with speed $v$; $\tau$ denotes the acceleration time.

3. *Deceleration when catching up with a slower vehicle* $(1-\pi)\,\kappa(x,t,v)\int(w-v)\,\kappa(x,t,w)$. A vehicle that encounters a slower vehicle will need to reduce its speed if it cannot immediately overtake.

Their deliberations yielded the following partial differential equation (PH-model):

$$\frac{\partial\kappa}{\partial t}+v\frac{\partial\kappa}{\partial x}=\frac{\partial}{\partial v}\left(\kappa\frac{V_0(x,t,v)-v}{\tau}\right)+(1-\pi)\,\kappa(x,t,v)\int(w-v)\,\kappa(x,t,w)\,dw \tag{C.5}$$

The most complex process here is probably the deceleration process reflected by the term $(1-\pi)\,\kappa(x,t,v)\int(w-v)\,\kappa(x,t,w)\,dw$. Let us briefly discuss how this term is determined from the following, simple behavioural assumptions:

1. The "slowing down event" has a probability of $(1-\pi)$, where $\pi$ denotes the so-called *immediate overtaking probabity.* This probability reflects the event that a fast car catching up with a slow car can immediatly overtake to another lane, without needing to reduce its speed.

2. The speed of the slow car is not affected by the encounter with the fast car, whether the latter is able to overtake or not.

3. The lengths of the vehicles can be neglected.

4. The "slowing down event" has an instantaneous duration.

5. Only two vehicle encounters are to be considered, multivehicle encounters are excluded.

To derive the deceleration term, we recall the result of Hoogendoorn and Bovy (2001). The argue that event-based (event = encounter) changes in the PSD can be divided into changes that decrease the PSD $\kappa(x,t,v)$ and those that increase the PSD $\kappa(x,t,v)$. With respect to the former, they show that the rate of decrease of the PSD due to encounters equals

$$\left(\frac{\partial\kappa(x,t,v)}{\partial t}\right)^{-}_{encounter}=\int_w\int_{v'}\sigma_w(v'|v)\kappa(x,t,v)\,\Pi_w(v)\,dv'dw \tag{C.6}$$

Here, $\Pi_w(v)$ denotes the *event-rate*, i.e. the number of encounters per unit time that a vehicle driving with speed $v$ has with another vehicle driving with speed $w$; $\sigma_w(v'|v)$ denotes the *transition probability* describing the probability that in case a vehicle with speed $v$ encounters another vehicle with speed $w$, the former vehicle changes its speed to $v'$.

The event-rate can be determined by recalling that the number of *active overtakings* that a vehicle driving with speed $v$ makes with slower vehicle driving with speed $w$ in a small period $[t,t+dt)$ equals (see section 2.10.1)

$$(v-w)\,\kappa(x,t,w)dt \tag{C.7}$$

Is it clear that when $v<w$ (a *passive encounter*), the vehicle driving with speed $v$ will not change its speed. This can be described by the transition probabilities via the following expression

$$\sigma_w\left(v'|v\right)=\delta\left(v'-v\right)\quad\text{for}\quad v<w \tag{C.8}$$

where $\delta(v)$ denotes the *delta-dirac function*[1]. When $v>w$, the vehicle can either immediately change lanes, without reducing its speed (implying $v'=v$), or it needs to decelerate to the speed

[1] Recall that the $\delta$-dirac function by definition satisfies $\int f(x)\,\delta(x-x_0)\,dx=f(x_0)$

$w$ of the preceding vehicle (i.e. $v' = w$). Since $\pi$ denotes the immedate overtaking probality, we get

$$\sigma_w\left(v'|v\right) = \pi\delta\left(v'-v\right) + (1-\pi)\,\delta\left(v'-w\right) \tag{C.9}$$

At the same time, encounters cause the PSD to increase. For instance, $\kappa\,(x,t,v)$ is increased when vehicles are slowed down *to the speed* $v$ when encountering another vehicle. These increases can be modelled via the following term

$$\left(\frac{\partial\kappa(x,t,v)}{\partial t}\right)^{+}_{encounter} = \int_w\int_{v'} \sigma_w\left(v|v'\right)\kappa\left(x,t,v'\right)\Pi_w\left(v'\right)dv'dw \tag{C.10}$$

The transition probabilities are again given by the Eqns. (??) and (??). It is left to the reader to verify that the gross-effect of encounters equals

$$\left(\frac{\partial\kappa(x,t,v)}{\partial t}\right)^{+}_{encounter} - \left(\frac{\partial\kappa(x,t,v)}{\partial t}\right)^{-}_{encounter}$$
$$= (1-\pi)\,\kappa\,(x,t,v)\int(w-v)\,\kappa\,(x,t,w)\,dw \tag{C.11}$$

## C.2 Paveri-Fontana model

The model of Prigogine and Herman has been critised and improved by Paveri-Fontana (1975), who extended the PSD $\kappa\,(x,t,v)$ by also including the distribution of the desired speeds. That is, he considered $\hat{\kappa}\,(x,t,v,v_0)$, defined according to

$$\hat{\kappa}\,(x,t,v,v_0) = f\,(v,v_0)\cdot k(x,t) \tag{C.12}$$

where $f(v,v_0)$ is the joint speed-desired speed probability density function. The equation derived by Paveri-Fontana reads (PF-model):

$$\frac{\partial\hat{\kappa}}{\partial t} + \frac{\partial}{\partial x}\left(\hat{\kappa}\frac{dx}{dt}\right) + \frac{\partial}{\partial v}\left(\hat{\kappa}\frac{dv}{dt}\right) + \frac{\partial}{\partial v_0}\left(\hat{\kappa}\frac{dv_0}{dt}\right) = \left(\frac{\partial\kappa}{\partial t}\right)_{encounter} \tag{C.13}$$

Clearly, we have

$$\frac{dx}{dt} = v \tag{C.14}$$

The acceleration-term $dv/dt$ reflects the acceleration of vehicles towards their desired speed $v_0$. Paveri-Fontana assumed the following exponential acceleration law

$$\frac{dv}{dt} = \frac{v_0 - v}{\tau} \tag{C.15}$$

Furthermore, it was assumed that the desired speed did not change over time, i.e. $dv_0/dt = 0$. The term reflecting the impacts of fast vehicles catching up with a slow vehicle equals

$$\left(\frac{\partial\kappa}{\partial t}\right)_{encounter} = (1-\pi)\int_{w>v}\int_{w_0}|w-v|\,\hat{\kappa}\,(x,t,v,w_0)\,\hat{\kappa}\,(x,t,w,v_0)\,dw_0dw \tag{C.16}$$
$$-(1-\pi)\int_{w<v}\int_{w_0}|w-v|\,\hat{\kappa}\,(x,t,w,w_0)\,\hat{\kappa}\,(x,t,v,v_0)\,dw_0dw \tag{C.17}$$

Note that the PH model can be derived from the PF model by integrating both the left-hand-side and the right-hand-side of equation (C.12) with respect to the desired speed $v_0$. We find

$$\frac{\partial\kappa}{\partial t} + \frac{\partial}{\partial x}(v\kappa) + \frac{\partial}{\partial v}\left(\kappa\frac{V_0\,(x,t,v)-v}{\tau}\right) = (1-\pi)\,\kappa\int(w-v)\,\kappa\,(x,t,w)\,dw \tag{C.18}$$

where

$$V_0\,(x,t,v) = \int_{v_0}\frac{\hat{\kappa}(x,t,v,v_0)}{\kappa\,(x,t,v)}dv_0 \tag{C.19}$$

denotes the expected desired speed of vehicles driving with speed $v$ at $(x,t)$.

## C.3 Applications of gas-kinetic models

Gas-kinetic models are much more complicated than macroscopis models. It is not surprising to see that the mathematical analysis of these models lacks far behind the analysis of the LWR and even the Payne mode. Although it is possible to derive appropriate numerical solution, these scheme tend to be rather complicated. It would thus seem that the applicability of the gas-kinetic models is rather limited.

Amongst the most important and interesting applications of gas-kinetic models is its use to derive macroscopic models. Let us briefly show how this is achieved using the Paveri-Fontana equation (C.13). The method in question is called the *method of moments*. In rough terms, the approach consists of multiplying the Paveri-Fontana equation with powers of $v$ and $v_0$, and subsequently integrating the result over $v$ and $v_0$. Let us illustrate this approach by multiplying with $v^l v_0^m$. Then, we find

$$\frac{\partial}{\partial t}\left(k\left\langle v^l\right\rangle\right)+\frac{\partial}{\partial x}\left(k\left\langle v^{l+1}\right\rangle\right)-\frac{l}{\tau}k\left(\left\langle v^{l-1}v_0\right\rangle-\left\langle v^l\right\rangle\right)=(1-\pi)\,k^2\left(\langle v\rangle\left\langle v^l\right\rangle-\left\langle v^{l+1}\right\rangle\right) \tag{C.20}$$

where

$$\left\langle v^l v_0^m\right\rangle=\int_v\int_{v_0} v^l v_0^m f\left(v,v_0\right)dv dv_0 \tag{C.21}$$

where $f\left(v,v_0\right)$ denotes the joint probability distribution of the speed $v$ and the desired speed $v_0$. Note that the following relations hold; for the mean speed $u\left(x,t\right)$ we have

$$u\left(x,t\right)=\langle v\rangle \tag{C.22}$$

For the speed variance $\theta\left(x,t\right)$ (denote the variance of speed around the mean speed $u\left(x,t\right)$), we have

$$\theta\left(x,t\right)=\left\langle\left(v-u\left(x,t\right)\right)^2\right\rangle \tag{C.23}$$

If we consider $l=m=0$, we find

$$\frac{\partial k}{\partial t}+\frac{\partial\left(k\,u\right)}{\partial x}=0 \tag{C.24}$$

which is obviously the conservation of vehicle equation.

For $l=1$ and $m=0$, after dividing Eq. (C.20) by $k$, we find

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{k}\frac{\partial}{\partial x}\left(k\theta\right)-\frac{1}{\tau}\left(V_0-u\right)=-\left(1-\pi\right)k^2\theta \tag{C.25}$$

where $V_0$ denotes the mean desired speed of the entire traffic flow. This expression can be written as follows

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{V-u}{\tau}-\frac{1}{k}\frac{\partial}{\partial x}\left(k\theta\right) \tag{C.26}$$

where $V=V\left(k,\theta\right)$ denotes the relaxation or equilibrium speed, defined by

$$V\left(k,\theta\right)=V_0-\tau\left(1-\pi\right)k\theta \tag{C.27}$$

Note that Eq. (C.27) is equivalent to Eq. (B.2), assuming that

$$c^2\left(k\right)=\frac{dP}{dk}=\frac{d}{dk}\left(k\theta\right) \tag{C.28}$$

The latter expression provides a different interpretation of the wave speed $c\left(k\right)$, namely that is it related to the *standard speed deviations*. For instance, assuming that $d\theta/dk=0$, we have $c=\sqrt{\theta}$. Besides this alternative interpretation of $c$, we have also derived an expression for the equilibrium speed Eq. (C.27). This expression shows among other things that the slope of the speed-density curve depends on the relaxation time $\tau$, the immediate lane-changing probability $\pi$, and the speed variance $\theta$. Also note that under the assumption that all these are constant values, the speed-density relation equals Greenshield's function.

It is clear that for the model to be applicable, the speed variance $\theta$ must either be explicitly determined (either as a function of the density, the speed, etc., or simply as a constant), *or* we can consider Eq. (C.20) for $l = 1$ and $m = 0$. After some complicated mathematics, the latter option yields the following, dynamic equation for the speed variance

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} = -2\theta \frac{\partial u}{\partial x} - \frac{1}{k} \frac{\partial}{\partial x} (kJ) + 2 \frac{\Theta - \theta}{\tau} \tag{C.29}$$

where $J$ denotes the *skewness of the speed distribution*, defined by

$$J = \left\langle (v - u(x,t))^3 \right\rangle \tag{C.30}$$

and where the *equilibrium speed variance* equals

$$\Theta = C - \frac{\tau}{2} (1 - \pi) kJ \tag{C.31}$$

where $C$ denotes the *covariance between the speed and the desired speed*, i.e.

$$C(x,t) = \langle (v - u(x,t)) (v_0 - V_0) \rangle \tag{C.32}$$

The problem now remains to appropriately choose the skewness of the speed distribution, and the covariance between the speed and the desired speed. Note that, under the assumption that the speed distributions are Gaussian, we have $J = 0$