

Appendix D

Higher-order model extensions and generalisations

This last section discusses some further model extensions and generalisations of the macroscopic flow models discussed in this section. The most important generalisation, is the multiclass extension of the model, due to [26] and [24].

D.1 Multiclass generalisation of PF-model

Here, let a denote the user-class. The classes can be distinguished based on a number of characteristics, such as vehicle-driver characteristics, socio-economic importance of the user-class, destination of the traffic, etc. Let $\hat{\kappa}_a(t, x, v, v_0)$ denote the PSD of class a , denoting for instance the mean number of trucks that is driving at x with speed v , while having desired speed v_0 . The PF-equation (C.13) clearly also holds for each class a , i.e.

$$\frac{\partial \hat{\kappa}_a}{\partial t} + \frac{\partial}{\partial x} \left(\hat{\kappa}_a \frac{dx}{dt} \right) + \frac{\partial}{\partial v} \left(\hat{\kappa}_a \frac{dv}{dt} \right) + \frac{\partial}{\partial v_0} \left(\hat{\kappa}_a \frac{dv_0}{dt} \right) = \left(\frac{\partial \kappa_a}{\partial t} \right)_{\text{encounter}} \quad (\text{D.1})$$

where $\frac{dx}{dt} = v$, and $\frac{dv}{dt} = \frac{v_0 - v}{\tau_a}$, where τ_a is the class-specific acceleration time. The latter reflects the differences in acceleration capabilities of the different vehicle classes. Similar to the PH-model and the PF-model, we can show that for the multiclass generalisation, we have

$$\left(\frac{\partial \hat{\kappa}_a}{\partial t} \right)_{\text{encounter}}^- = \sum_{a', b} \int \sigma_{(w, w_0, b)}(v', v'_0, a' | v, v_0, a) \Pi_{(w, w_0, b)}(v, v_0, a) dv' dv'_0 dw dw_0 \quad (\text{D.2})$$

and

$$\left(\frac{\partial \hat{\kappa}_a}{\partial t} \right)_{\text{encounter}}^+ = \sum_{a, b} \int \sigma_{(w, w_0, b)}(v, v_0, a | v', v'_0, a') \Pi_{(w, w_0, b)}(v', v'_0, a') dv' dv'_0 dw dw_0 \quad (\text{D.3})$$

Eq. (D.2) denotes the reductions in $\hat{\kappa}_a(t, x, v, v_0)$ due to interactions with vehicles of class b , driving with speed w having desired speed w_0 ; Eq. (D.3) denotes the increases in $\hat{\kappa}_a(t, x, v, v_0)$ caused by vehicles of class a' driving with speed v' having desired speed v'_0 changing their class / speed / desired speed to a , v and v_0 respectively after an interaction.

Thus, $\Pi_{(w, w_0, b)}(v, v_0, a)$ denotes the *rate* at which drivers of class a driving with speed v while having desired speed v_0 (referred to as vehicle (v, v_0, a)) *interact* with drivers of class b driving with speed w while having desired speed w_0 . Clearly, we have

$$\Pi_{(w, w_0, b)}(v, v_0, a) = |w - v| \hat{\kappa}_b(t, x, w, w_0) \quad (\text{D.4})$$

The *transition probabilities*

$$\sigma_{(w, w_0, b)}(v', v'_0, a' | v, v_0, a)$$

denote the probability that after an interaction / encounter of vehicle (v, v_0, a) with vehicle (w, w_0, b) , the speed / desired speed / class of vehicle (v, v_0, a) changes to (v', v'_0, a') .

Let us assume that the vehicle classes remain preserved upon interaction. Furthermore, assume that drivers will not change their desired speed when interacting with another vehicle. Then, using the same behavioural assumptions used to derive the PH-model, we can show that

$$\left(\frac{\partial \hat{\kappa}_a}{\partial t}\right)_{\text{encounter}} = \sum_b (1 - \pi_a) \int_{w>v} \int_{w_0} |w - v| \hat{\kappa}_b(x, t, v, w_0) \hat{\kappa}_a(x, t, w, v_0) dw_0 dw \quad (\text{D.5})$$

$$- \sum_b (1 - \pi_a) \int_{w<v} \int_{w_0} |w - v| \hat{\kappa}_b(x, t, w, w_0) \hat{\kappa}_a(x, t, v, v_0) dw_0 dw \quad (\text{D.6})$$

This expression shows how driver-vehicle combination of class a are decelerated by driver-vehicle combinations of their own class $b = a$ and by driver-vehicle combinations of other classes $b \neq a$.

D.2 Multiclass higher-order model

Multiclass PF-model Eq. (D.1) can be used in a similar fashion as the mixed-class PF-model Eq. (C.13) to derive a multiclass macroscopic model. The main benefit of using this approach is that the non-symmetrical interactions between fast and slow user-classes is contained within the model. In applying the method-of-moments to the multiclass PF-model, we find for $l = m = 0$

$$\frac{\partial k_a}{\partial t} + \frac{\partial (k_a u_a)}{\partial x} = 0 \quad (\text{D.7})$$

Here, $k_a(x, t)$ and $u_a(x, t)$ respectively denote the class-specific density and mean speed. This equation describes the multiclass conservation of vehicle equation. It shows that the class-specific density $k_a(x, t)$ is conserved. Clearly, conservation will only hold when driver-vehicle combinations do not change classes during the considered time period.

Considering $l = 1$ and $m = 0$, we can determine the speed-dynamic equations for class a

$$\frac{\partial u_a}{\partial t} + u_a \frac{\partial u_a}{\partial x} = \frac{V_a - u_a}{\tau_a} - \frac{1}{k_a} \frac{\partial}{\partial x} (k_a \theta_a) \quad (\text{D.8})$$

where V_a denotes the equilibrium speed of class a stemming from interactions with slower vehicles from classes $b = a$ (within-class interactions) and $b \neq a$ (between class interactions). Although the general form of the speed dynamics equation is conserved, for the multiclass model it is not possible to derive a close-form expression similar to Eq. (??). The reason for this is the asymmetry of the speed distributions of the different classes a and b . Hoogendoorn and Bovy [29] propose using the following approximation. First, let

$$S_{ab} = \theta_a + \theta_b - 2\xi \sqrt{\theta_a \theta_b} \quad (\text{D.9})$$

where θ_a denotes the speed variance of class a and $0 \leq \xi \leq 1$ denotes the correlation between the speed distributions of class a and b . Furthermore, let

$$\delta u_{ab} = \frac{u_a - u_b}{\sqrt{S_{ab}}} \quad (\text{D.10})$$

denote the normalised mean speed difference between class a and b . Then, Hoogendoorn and Bovy [29] propose

$$V_a = V_a^0 - (1 - \pi_a) \tau_a P_a \quad (\text{D.11})$$

where V_a^0 denotes the mean desired speed of class a and where

$$P_a(x, t) = \sum_b P_{ab}(x, t) \quad (\text{D.12})$$

with

$$P_{ab} = k_b S_{ab} [\delta u_{ab} f(\delta u_{ab}) + (1 + \delta u_{ab}^2) F_{ab}(\delta u_{ab})] \quad (\text{D.13})$$

describes the effects of interactions between vehicles on the average vehicle speed. Note that for $b = a$ we have

$$P_{aa} = k_a (1 - \xi) \theta_a \quad (\text{D.14})$$

For $\xi = 0$ (no correlation between speeds), we have $P_{aa} = k_a \theta_a$ (compare to Eq. (??)). This implies that the model discussed here is a true generalisation of the mixed-class model.

Hoogendoorn and Bovy [29] provide a further generalisation of their model by distinguishing roadway lanes. The resulting model can be used to explain and predict observed behavior on multilane facilities. It is beyond the scope of this course to explain the details of the resulting model.

