ME1633: Integration of Physics, Motion and Metrology

Assignment 5: Dynamics

6 December 2013

- You can do this assignment on your own or in groups, as long as you hand in your own solutions and it becomes clear that you understand your solutions. Formulate your solutions step-by-step, carefully pointing out the logical structure of your answer, but keep your answers brief.
- If in some question you happen to need an answer from a previous question that you don't know the answer to, assume an answer or at least explain the method you would use when you would have had that previous answer.
- *Only for the students who follow the PME track's course ME1633:* Your solution to this assignment must be submitted in the lecture or via Blackboard (in pdf format) *before* 29 January 2014, 23:59h.
- Your solution may be formulated in either *English* or *Dutch*.

Satellite shaker

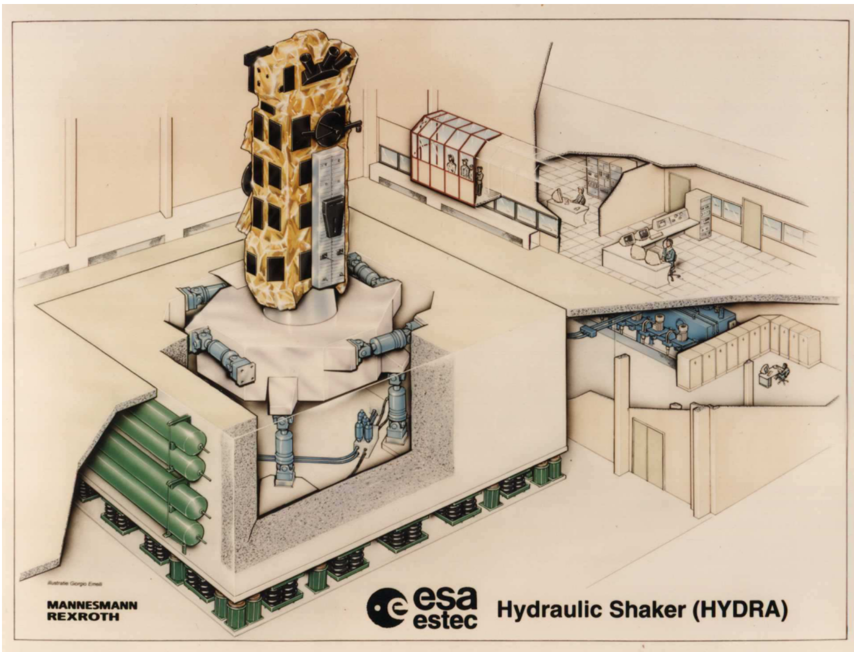
Satellites only experience light vibrations in space. During the launch, however, they experience large forces and are put heavily to the proof. To be sure that no damage will occur when launching the satellites, they are extensively tested. Amongst others, their dynamic behaviour is examined by measuring the response to vibrations applied by a shaker.

As modern satellites can easily weigh tens of tons, one can imagine that such a shaker will have extreme specifications. Figure 1a shows satellite shaker 'Hydra', which is located in the (huge) cleanroom facilities of the ESA in Noordwijk.[1] Satellites are mounted on the test table, which can be actuated in six degrees of freedom using a hydraulic system. The hydraulic cylinders are mounted in between the table and a very large seismic foundation body, whose purpose it is to reduce the forces transmitted to the ground. This allows people in neighbouring buildings to keep on working comfortably when the shaker is used. The seismic mass is supported by big springs that are connected to the ground.

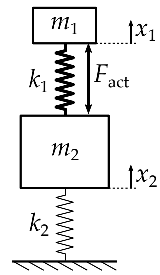
[1] http://www.european-test-services.net/services-mechanical-Hydra-Vibration.html

Table 1: Mass and stiffness parameters of the Hydra satellite shaker.

Parameter	Value
Mass of the shaker table (m_1)	$23.5 \cdot 10^3$ kg
Mass of the seismic body (m_2)	$1.4 \cdot 10^6$ kg
Stiffness of the hydraulic system (k_1)	10^9 N/m
Stiffness of the connection between the seismic body and the ground (k_2)	$7.5 \cdot 10^7$ N/m



(a) Drawing of the Hydra satellite shaker (source: ESA).



(b) Simplified one-dimensional model of a satellite shaker.

Figure 1: The satellite shaker system.

The satellite shaker is used for enforcing acceleration to the table with the satellite under test. In this assignment we aim at getting basic understanding of the system's dynamics and the possible control issues this dynamics imposes. We will only consider the vertical movement of the system and model the shaker as a simple, one-dimensional, two-body mass-spring system, see Figure 1b. m_1 represents the mass of the table and m_2 represents the mass of the seismic foundation body. Stiffness k_1 represents the stiffness of the hydraulic system, which works between the table and the foundation mass. The seismic body is coupled to the ground with a stiffness k_2. Finally, F_{act} is the actuation force that is applied by the hydraulic system. The mass and stiffness values are listed in Table 1. Unless explicitly stated otherwise, damping may be neglected.

Eigenmodes

The simplified one-dimensional model of the shaker possesses two bodies, with coordinates x_1 and x_2, so that two eigenmodes can be found. The modeshape, $\boldsymbol{\varphi}_1$, corresponding to the lowest eigenfrequency can be written as

$$\boldsymbol{\varphi}_1 = \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ 1-\alpha \end{array} \right\}, \qquad (1)$$

where $\alpha = 0.0012$. Clearly, both bodies move with almost equal amplitudes in the first eigenmode.

(8p) **1a.** Explain that this is the modeshape belonging to the *lowest* eigenfrequency of the system. Use a reasoning based on modal mass and the modal stiffness to answer this question.

Answer

- The formula of eigenfrequency has some form $\sqrt{\text{stiffness}/\text{mass}}$. To obtain a low eigenfrequency, the modal stiffness in the eigenmode has to be small.
- The seismic body is connected with high stiffness to the table, whereas the seismic body is connected with *relatively* low stiffness to the ground, so that $k_2 \ll k_1$.
- The modal stiffness would be low for a modeshape where the first spring deforms considerably and the second spring hardly. This is the case for equal displacement of the two bodies.
- In order to make equal displacements possible, the mass of the table has to be much smaller than the mass of the seismic body, which is the case ($m_1 \ll m_2$) . Otherwise a relatively high force has to be transmitted through the second spring, causing a relatively large deformation of that spring.
- (The coupling between the bodies would be *exact* when stiffness k_1 would be infinite or when the mass of the table would be zero.)

(6p) **1b.** To find a good estimate for the eigenfrequency corresponding to the modeshape of Equation (1) it is sufficient to use the following approximation of the modeshape:

$$\boldsymbol{\varphi}_1 = \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} \simeq \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}. \tag{2}$$

Find a simple approximate expression for the first eigenfrequency ω_1. For answering this, you should not need too extensive math.

Answer

- We saw that the two masses can be considered as almost coupled. Therefore we can calculate the first eigenfrequency as follows:

$$\omega_1 \simeq \sqrt{\frac{k_2}{m_1 + m_2}} = \sqrt{\frac{7.5 \cdot 10^7}{23.5 \cdot 10^3 + 1.4 \cdot 10^6}} = 7.3 \, \frac{\text{rad}}{\text{s}} \equiv 1.2 \text{ Hz}.$$

The modeshape corresponding to the highest eigenfrequency, $\boldsymbol{\varphi}_2$, is

$$\boldsymbol{\varphi}_2 = \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ -\beta \end{array} \right\}, \tag{3}$$

where β a *positive* number much smaller than 1 ($\beta \ll 1$).

(8p) **2a.** Explain that this is the modeshape belonging to the *highest* eigenfrequency of the system. Again, use a reasoning based on modal mass and modal stiffness to answer this question.

Answer

- As in Question 1a, we recall that the formula of eigenfrequency has some form $\sqrt{\text{stiffness}/\text{mass}}$. To obtain a high eigenfrequency, the modal mass of the eigenmode has to be small and the modal stiffness has to be large.
- As the stiffness of the second spring is much higher than the stiffness of the first ($k_2 \ll k_1$). The second spring should deform strongly to obtain a large modal stiffness.
- The mass of the table is much smaller than the mass of the seismic body. To obtain a small modal mass, the table should move much more than the seismic body, so that the seismic mass can be considered as practically at stand-still (at least with respect to the displacement of the table).
- (Mass would be *exact* at stand-still when stiffness k_2 would be infinite or when the mass of the seismic body would be infinite.)

(6p) **2b.** To find a good estimate for the eigenfrequency corresponding to the modeshape of Equation (3) it is sufficient to use the following approximate of the modeshape:

$$\boldsymbol{\varphi}_2 = \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} \simeq \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}. \tag{4}$$

Find a simple approximate expression for the second eigenfrequency ω_2. For answering this, you should not need too extensive math.

Answer

- *(3p)* We saw that the seismic body is almost at stand-still in the second eigenmode, so that practically only the table mass and the second spring and participate for the second eigenmode.
- *(3p)* Therefore, we can calculate the second eigenfrequency as follows:

$$\omega_2 \simeq \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{1 \cdot 10^9}{23.5 \cdot 10^3}} = 206 \ \frac{\text{rad}}{\text{s}} \equiv 33 \text{ Hz}.$$

Actually, the modeshapes in Equations (2) and (4) are approximations which are useful for approximating the eigenfrequencies of the system. For understanding the transfer functions from force to displacement or acceleration, however, we need to know the modeshapes more accurately.

(6p) **2c.** Calculate an estimate of β, using the assumption that the suspension stiffness between the seismic body and the ground has a negligible contribution to the modal stiffness of the second eigenmode,

i.e. $k_2 \simeq 0$.
Hint for this question: What is the movement of the system's centre of mass?

Answer

(2p)
- When assuming k_2 to be zero, no forces can be transmitted to the external world in this eigenmode. This means that the centre of mass of the two bodies need to be at standstill.

(2p)
- Call l_1 and l_2 respectively the distance between centre of mass and the first and second body. The following describes then the centre of mass:

$$\frac{l_2}{l_1} = \frac{m_1}{m_2}.$$

  To keep the centre of mass in place, the displacement x_1 and x_2 need also to obey

$$\frac{x_2}{-x_1} = \frac{m_1}{m_2}.$$

(2p)
- Now β is the x_2 corresponding to $x_1 = 1$, so that

$$\beta = \frac{-x_2}{x_1} = \frac{m_1}{m_2} = 0.017.$$

- (In line with the way of reasoning in Questions 1a and 2a, we would indeed expect a *different* sign between the displacement of the two bodies for the eigenmode with the highest eigenfrequeny. In that case the participation of the stiffness of the second spring is higher than when the sign would be *equal*, whereas the participation of the mass is equal in both situations.)

Answer (using the equations from the book)

- By assuming that $k_2 = 0$ we can use Equation (3.97) from the book[2]. For the second mode-shape one can write $C_2 x_2 = -C_1 x_1$, so that

$$\beta \simeq \frac{C_2}{C_1} = \frac{\frac{1}{m_1+m_2}\frac{m_1}{m_2}}{\frac{1}{m_1+m_2}} = \frac{m_1}{m_2} = 0.017.$$

- (Indeed the mode shape we find in this question approaches much better the *exact* solution, namely up to five decimal places.)

(4p) **2d.** Explain that an approximation as in Equation (4) could lead to big mistakes, when, for example, calculating the transfer from actuation forces to the force exerted to the ground.

[2] Equation (3.97) in the second edition, Equations (3.79-3.81) in the first edition.

Answer

- When we would for example excite the system at its second eigenfrequency, we would think that the second eigenmode would not contribute to the displacement of the seismic body. In that way we would not find the high forces that are in that case exerted on the ground ($F_{\text{ground}} = -k_2 x_2$).

Modal lever representation

As we know, the equations of motion of this dynamic system can be written in the matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \left\{ \begin{array}{c} \ddot{x}_1 \\ \ddot{x}_2 \end{array} \right\} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \left\{ \begin{array}{c} F_1 \\ F_2 \end{array} \right\}, \qquad (5)$$

where $x_1(t)$ and $x_2(t)$ are the physical coordinates of the system, in general referred to as 'nodal coordinates', and $F_1(t)$ and $F_2(t)$ are the forces on each of the bodies, which are for the satellite shaker

$$\left\{ \begin{array}{c} F_1 \\ F_2 \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\} F_{\text{act}}.$$

These equations of motion can be rewritten in terms of the so called 'modal coordinates', $q_1(t)$ and $q_2(t)$. The modal coordinates describe the amplitude of each of the modeshapes as a function of time. The equations can be rewritten in terms of modal coordinates as follows:

$$\begin{bmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{bmatrix} \left\{ \begin{array}{c} \ddot{q}_1 \\ \ddot{q}_2 \end{array} \right\} + \begin{bmatrix} \mathcal{K}_1 & 0 \\ 0 & \mathcal{K}_2 \end{bmatrix} \left\{ \begin{array}{c} q_1 \\ q_2 \end{array} \right\} = \left\{ \begin{array}{c} \mathcal{F}_1 \\ \mathcal{F}_2 \end{array} \right\},$$

where $\mathcal{M}_i$, $\mathcal{K}_i$ and $\mathcal{F}_i$ are respectively the modal mass, the modal stiffness and the modal force of the ith mode. Note that the modal mass and stiffness matrices are diagonal, the two equations are not coupled any more. The response of the total system is now simply the summation of the response of each of the individual modes:

$$\left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \boldsymbol{\varphi}_1 q_1 + \boldsymbol{\varphi}_2 q_2. \qquad (6)$$

(4p) **3a.** Explain the fact that each of the modal responses $q_i/\mathcal{F}_i$ can be regarded as the response of a single mass-spring system.

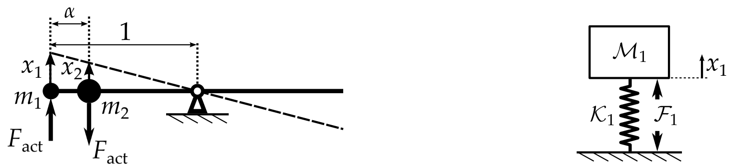
Answer

- The equations of motion in modal coordinates have exactly the well-known form of a single mass-spring system, i.e.

$$\mathcal{M}_1 \ddot{q}_1 + \mathcal{K}_1 q_1 = \mathcal{F}_1$$

and

$$\mathcal{M}_2 \ddot{q}_2 + \mathcal{K}_2 q_2 = \mathcal{F}_2.$$



(a) Modal lever representation. (b) Modal mass-spring system corresponding to coordinate x_1.

Figure 2: Representations of the first eigenmode.

Figure 2a shows the modal lever representation of the first eigenmode of the Hydra satellite shaker. Such a modal lever is a powerful tool for understanding the system's dynamics, for several reasons. First, a lever representation helps understanding that the scaling factor of a modeshape is arbitrary. Furthermore, it helps to more easily find the modal mass and modal force corresponding to a nodal coordinate. Figure 2b shows the modal mass-spring system corresponding to coordinate x_1 of the shaker's first eigenmode.

(8p) **3b.** Find, using the modal lever representation of Figure 2, the modal mass of the first modeshape ($\mathcal{M}_1$) and the modal stiffness ($\mathcal{K}_1$). Also, show that the modal force ($\mathcal{F}_1$) equals

$$\mathcal{F}_1 = \alpha F_{\text{act}}.$$

Hint for this question: The modal lever representation works not too intuitively for calculating modal stiffness; for this reason, use the modal mass in combination with the eigenfrequency to calculate $\mathcal{K}_1$.

Answer

(3p) • The modal mass is

$$\begin{aligned}\mathcal{M}_1 &= m_1 + \left(\frac{1-\alpha}{1}\right)^2 m_2 \\ &= 23.5 \cdot 10^3 + (0.9988)^2 \left(1.40 \cdot 10^6\right) = 1.42 \cdot 10^6 \text{ kg.}\end{aligned}$$

(3p) • The eigenfrequency of the equivalent mass-spring system is

$$\omega_1 = \sqrt{\frac{\mathcal{K}_1}{\mathcal{M}_1}},$$

so that the modal stiffness is

$$\begin{aligned}\mathcal{K}_1 &= \omega_1^2 \mathcal{M}_1 \\ &= (7.26)^2 \, 1.42 \cdot 10^6 = 7.48 \cdot 10^7 \, \frac{\text{N}}{\text{m}}.\end{aligned}$$

(2p) • The modal force is

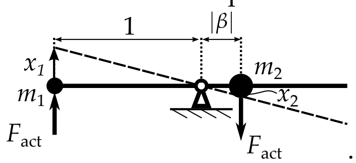
$$\begin{aligned}\mathcal{F}_1 &= F_{\text{act}} - \frac{1-\alpha}{1} F_{\text{act}} = \alpha F_{\text{act}}. \\ &= 0.0012 F_{\text{act}}\end{aligned}$$

QED.

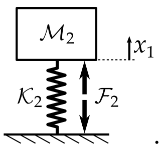
(12p) **3c.** Sketch the modal lever representation corresponding to the second eigenmode. Then, sketch the modal mass-spring system corresponding to coordinate x_1 with its modal mass $\mathcal{M}_2$, modal stiffness $\mathcal{K}_2$ and modal force $\mathcal{F}_2$.

Answer

(8p)
- The modal lever representation is:



(4p)
- The modal mass-spring system is:



(8p) **3d.** Calculate, using the modal lever representation of Question 3c, the modal mass ($\mathcal{M}_2$) and the modal stiffness ($\mathcal{K}_2$) for the second eigenmode. Also, calculate the modal force ($\mathcal{F}_2$) in terms of F_{act}.

Answer

(3p)
- The modal mass is

$$\begin{aligned}\mathcal{M}_2 &= m_1 + \left(\frac{\beta}{1}\right)^2 m_2 \\ &= 23.5 \cdot 10^3 + (0.0168)^2 \left(1.40 \cdot 10^6\right) = 23.9 \cdot 10^3 \text{ kg}.\end{aligned}$$

(3p)
- The modal stiffness is

$$\begin{aligned}\mathcal{K}_2 &= \omega_2^2 \mathcal{M}_2 \\ &= (208)^2 23.9 \cdot 10^3 = 1.034 \cdot 10^9 \ \frac{\text{N}}{\text{m}}.\end{aligned}$$

(2p)
- The modal force is

$$\begin{aligned}\mathcal{F}_2 &= F_{\text{act}} + \frac{\beta}{1} F_{\text{act}} = (1+\beta)\, F_{\text{act}} \\ &= 1.0168 F_{\text{act}}.\end{aligned}$$

(4p) **3e.** Could we call the modal masses and modal stiffnesses as calculated in 3b and 3d '*effective*'? Explain your answer.

Answer

- No, because the modal forces ($\mathcal{F}_1$ and $\mathcal{F}_2$) are not equal to F_{act}, so that the transfer function does not have the form as meant in Equation (3.98) of the book.

The system's response

Once the system's modal parameters (the modal masses, stiffnesses and forces) are known, it is possible to find the transfer of each of the eigenmodes and of the full system.

(18p) **4a.** Sketch *by hand, without using your computer,* the Bode plots of both the transfer q_1/F_{act} and the transfer q_2/F_{act}.

Answer

(10p) • Resonance occurs at the eigenfrequencies, respectively 1.2 and 33 Hz. Below these eiqenfrequencies, the spring lines dominate, resulting in a response of respectively

$$\frac{\mathcal{F}_1/F_{\text{act}}}{\mathcal{K}_1} = \frac{0.0012}{7.5 \cdot 10^7} = 1.7 \cdot 10^{-11}\ \frac{\text{m}}{\text{N}}$$

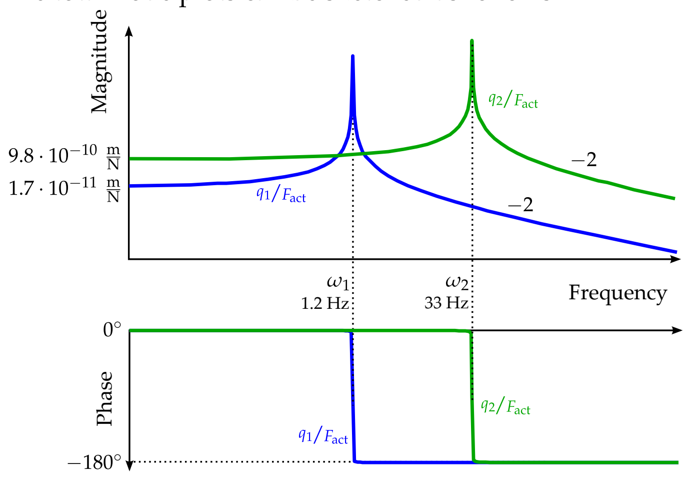
and

$$\frac{\mathcal{F}_2/F_{\text{act}}}{\mathcal{K}_2} = \frac{1.0168}{1.0 \cdot 10^9} = 9.8 \cdot 10^{-10}\ \frac{\text{m}}{\text{N}}.$$

Above the eigenfrequencies, the mass lines dominate. Mind the units of the transfer functions q_1/F_{act} and q_2/F_{act}. They depend on your choice of the units of the modal coordinates.

(8p) • For low frequencies, stiffness forces dominate, so that the amplitude of the mode is in phase with the actuator force. Displacement is a second order integral of acceleration. Therefore, the phase will start at $-180°$. Above the eigenfrequencies, inertia forces dominate, so that the phase drops to $-180°$.

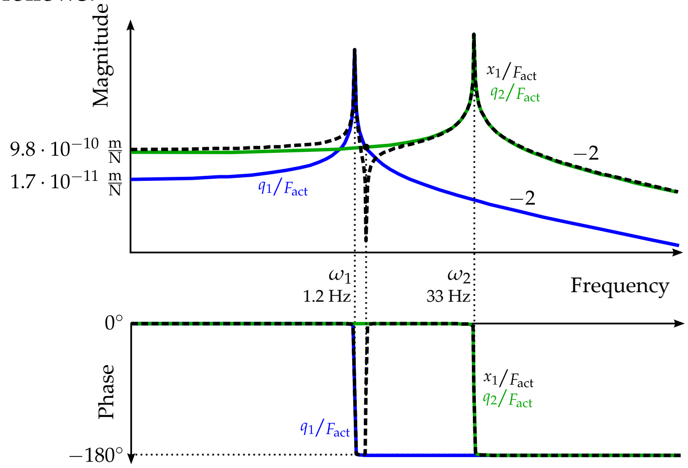
- The total Bode plots can be sketched as follows



(8p) **4b.** Now use the Bode plots found in Question 4a to sketch the Bode plot of the transfer from the actuator force to the displacement of the shaker table, x_1/F_{act}. Note that you can find this transfer by adding the transfers of each of the modeshape contributions. Mind the phase!

Answer

- The modeshapes were scaled such that $x_1 = 1q_1 + 1q_2$, which is useful, as the total transfer is now simply $x_1/F_{act} = q_1/F_{act} + q_2/F_{act}$. Mind the unit of the transfer function.
- At most frequencies the modeshape corresponding to the highest eigenfrequency dominates the transfer. At its eigenfrequency, however, the modeshape corresponding to the lower eigenfrequency dominates. Note the anti-resonance in the transfer, which occurs when both modes have equal amplitude but opposite phase for body 1. The Bode plot is as follows.



⟵ *Here ends the obligatory part of this assignment.*
The following part, bridging to the next subject (active control), is optional and will not be graded. ⟶

Changing the dynamics

Now that we found the system's transfer in terms of its modeshapes, we could use this knowledge to modify the dynamic behaviour. For this, both the shaker's physical parameters and the parameters of the active control system to be added could be tuned.

5a. The transfer from actuator force to displacement of the first body (the shaker table), x_1/F_{act}, possesses an anti-resonance. Which practical problem does this anti-resonance impose? Is it possible to overcome the problem with active control?

5b. Our system engineer wants to solve the anti-resonance problem by adding damping to the system. Explain, using the Bode plots of Question 4a, where he should add damping to the satellite shaker.

5c. Sketch the Bode plot of each of the modal transfers q_1/F_{act} and q_2/F_{act}, as you expect them to become after adding the damping of Question 5b, in one figure. Also sketch the resulting transfer to the shaker table (x_1/F_{act}). Neglect modal coupling due to damping.

Please indicate, for each question, the following:

This question was:

Difficult	$\longleftrightarrow$		Doable
○	○	○	○

I spent:

Hours