Bending Deflection – Differential Equation Method

AE1108-II: Aerospace Mechanics of Materials

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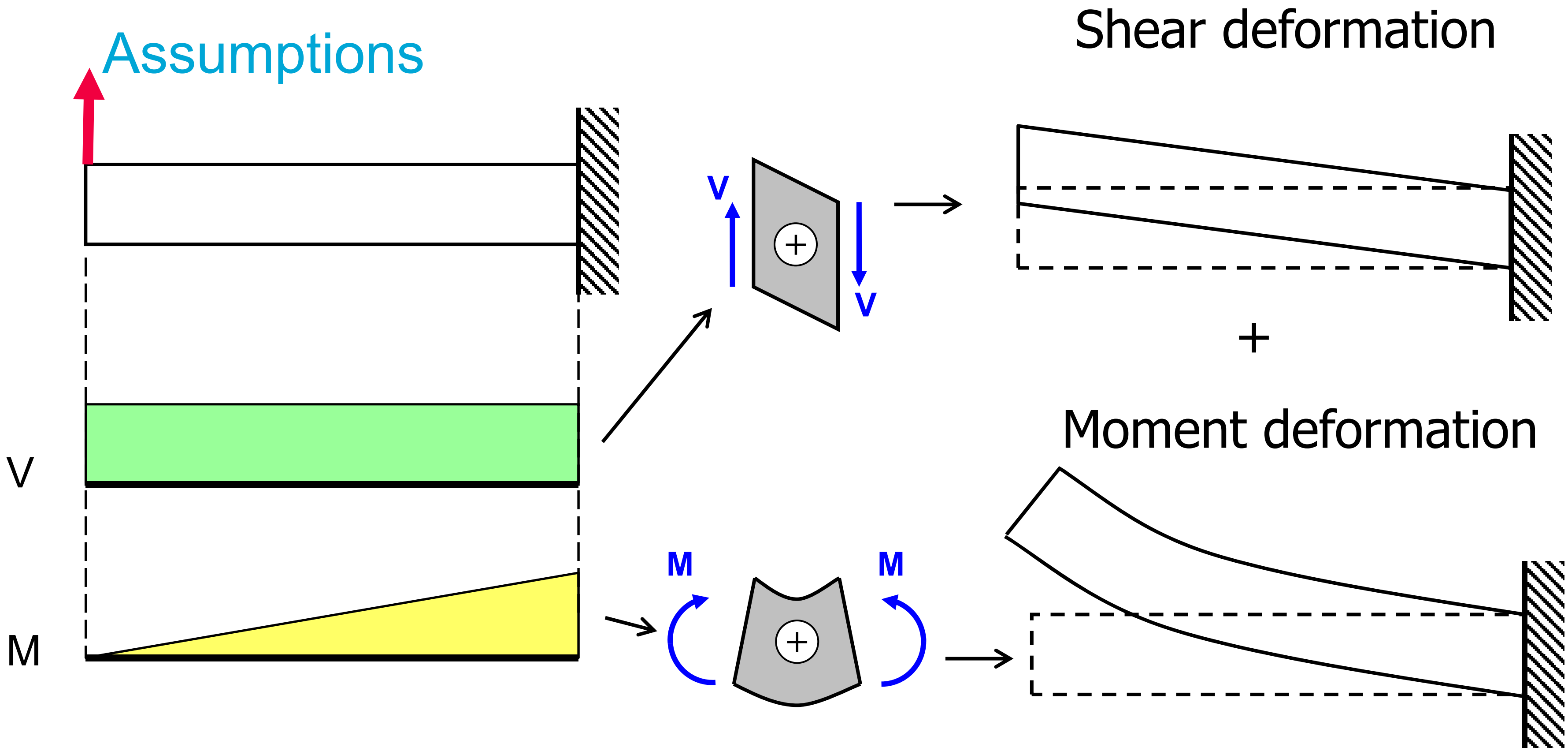
Recap

- So far, for symmetric beams, we have:
  - Looked at internal shear force and bending moment distributions
  - Determined normal stress distribution due to bending moments
  - Determined shear stress distribution due to shear force
- Need to determine deflections and slopes of beams under load
  - Important in many design applications
  - Essential in the analysis of statically indeterminate beams

Deformation of a Beam



Negligible (for long beams)

Bending Deformation = ~~Shear Deformation~~ + Moment Deformation

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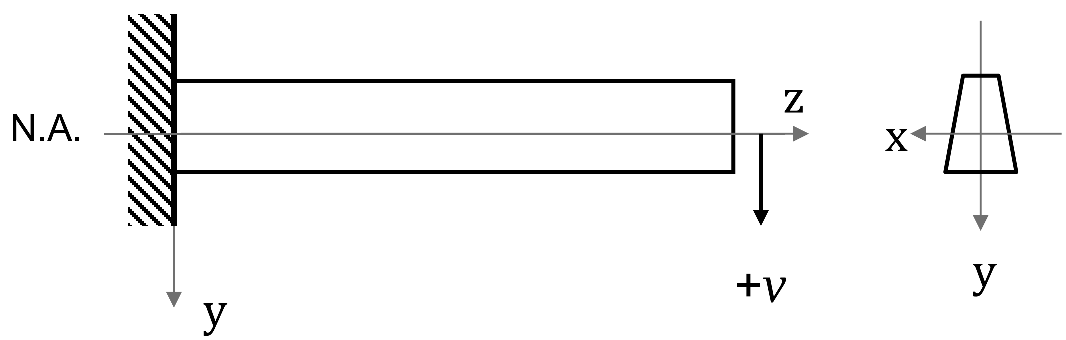
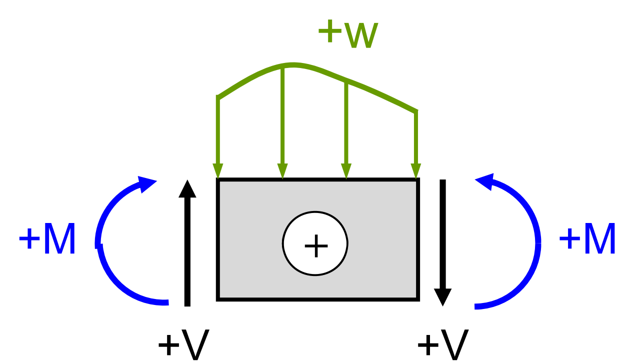
Deformation of a Beam

Assumptions

- For long beams (length much greater than beam depth), shear deformation is negligible
  - This is the case for most engineering structures
  - Will consider moment deformation only in this course
- Recap of sign convention





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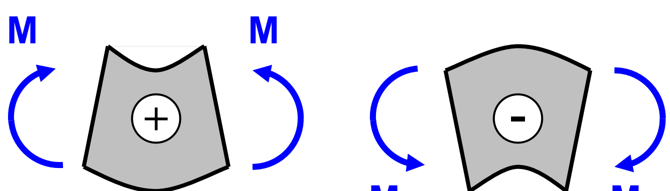
Deformation of a Beam

Visualizing Bending Deformation

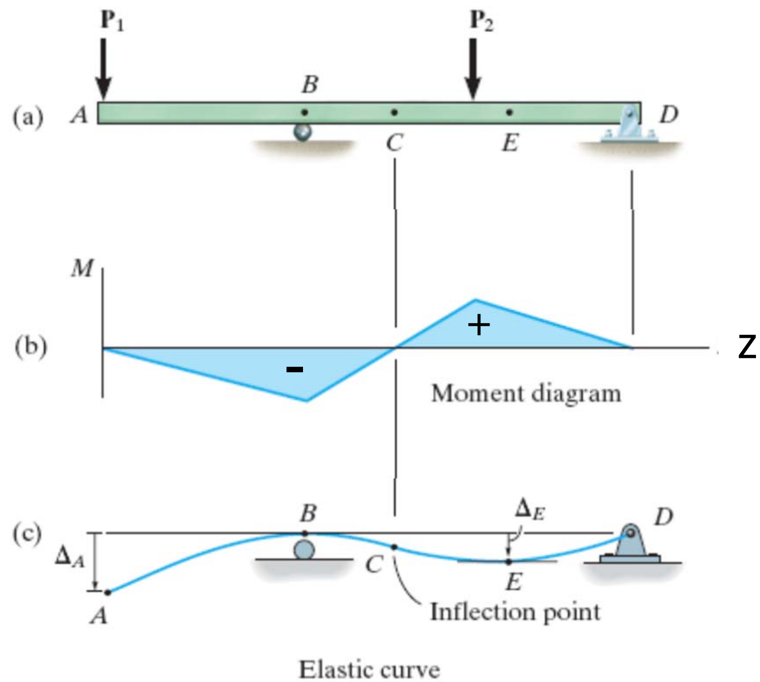
Elastic curve: plot of the deflection of the neutral axis of a beam

How does this beam deform?

We can gain insight into the deformation by looking at the bending moment diagram

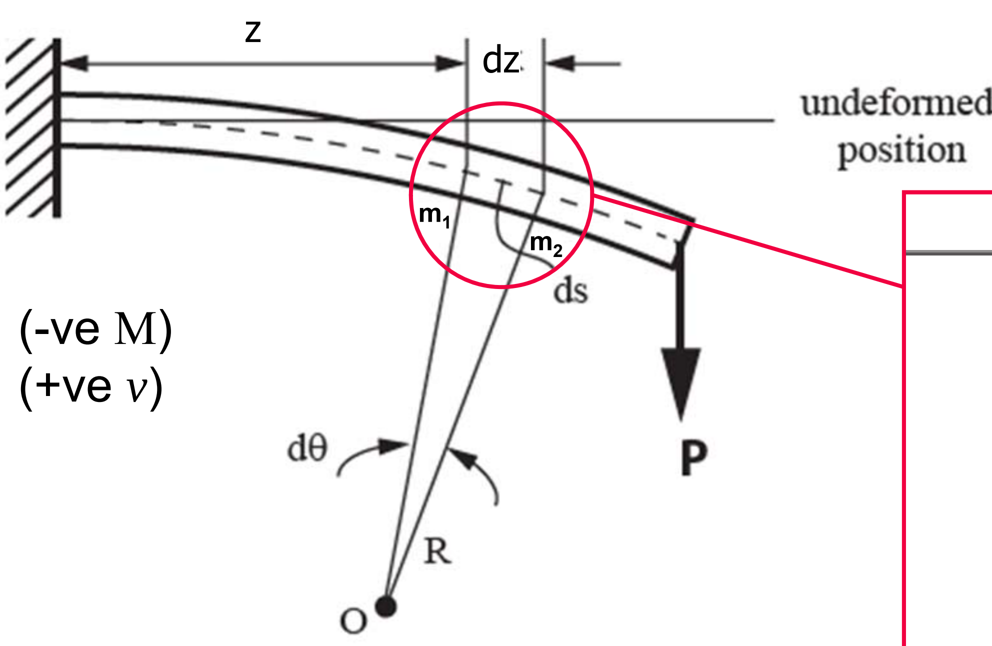


And by considering boundary conditions at supports

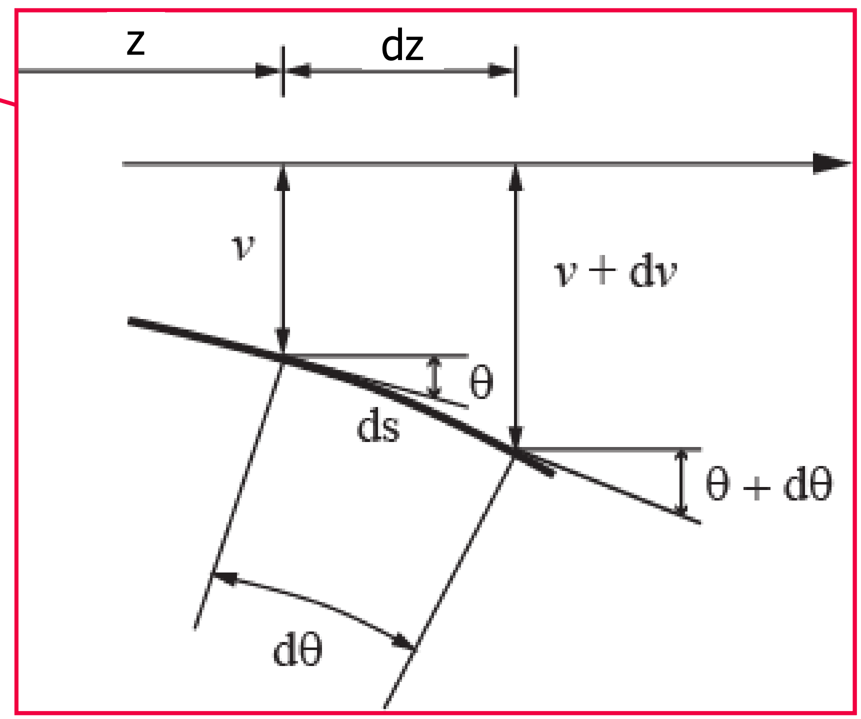


Qualitatively can determine elastic curve!

Moment-Curvature Relationship



(-ve M)
(+ve v)



$$v(\mathrm{z}) = \text{vertical deflection at z}$$

$$\theta(\mathrm{z}) = \text{slope at z} = \left(\frac{dv}{dz}\right)$$

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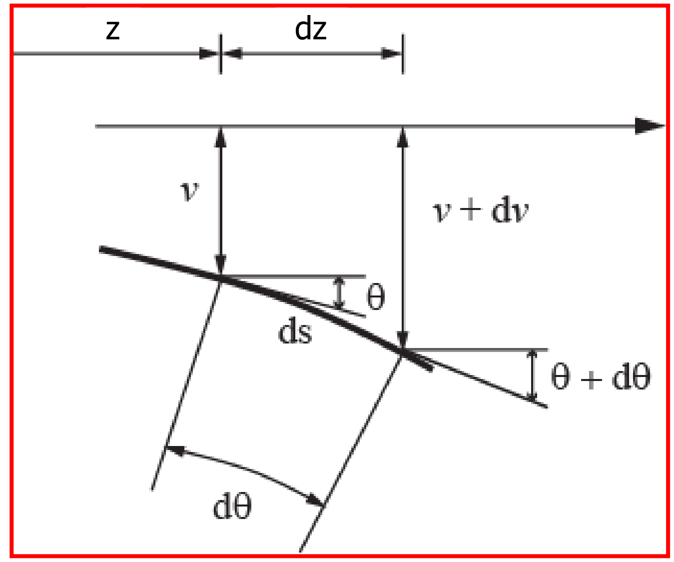
Moment-Curvature Relationship



For small $d\theta$: $$ds = R \cdot d\theta$$

or $$\kappa = \frac{1}{R} = \frac{d\theta}{ds}$$

curvature

For small θ: $$ds = \frac{dz}{\cos\theta} \approx dz$$

$\cos\theta \to 1$
when θ is small

$$\therefore \frac{1}{R} = \frac{d\theta}{dz} = \frac{d\left(dv/dz\right)}{dz} = \frac{d^2v}{dz^2}$$

Recall
$$\frac{M}{I} = \frac{E}{R}$$

$$\boxed{M = -EI\frac{d^2v}{dz^2} = -EIv''}$$

(negative sign a result of sign convention)

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Deflection by Method of Integration

$$M = -EI\frac{d^2v}{dz^2}$$

$\int$

$$\frac{dv}{dz} = -\frac{1}{EI}\int M \cdot dz = \theta$$

$\int$

$$v = -\frac{1}{EI}\iint M \cdot dz$$

Lets consider a prismatic beam
(ie: EI = constant)

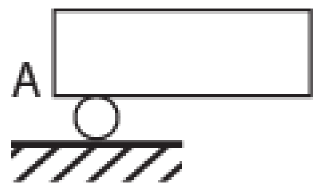
Indefinite integrals result in constants of integration that can be determined from boundary conditions of the problem

Constant of integration

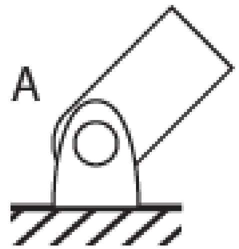
ie: $\int z \cdot dz = \frac{1}{2}z^2 + C$

Determining Constants of Integration

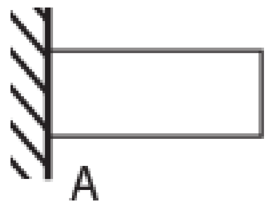
Support Conditions



$v = 0$



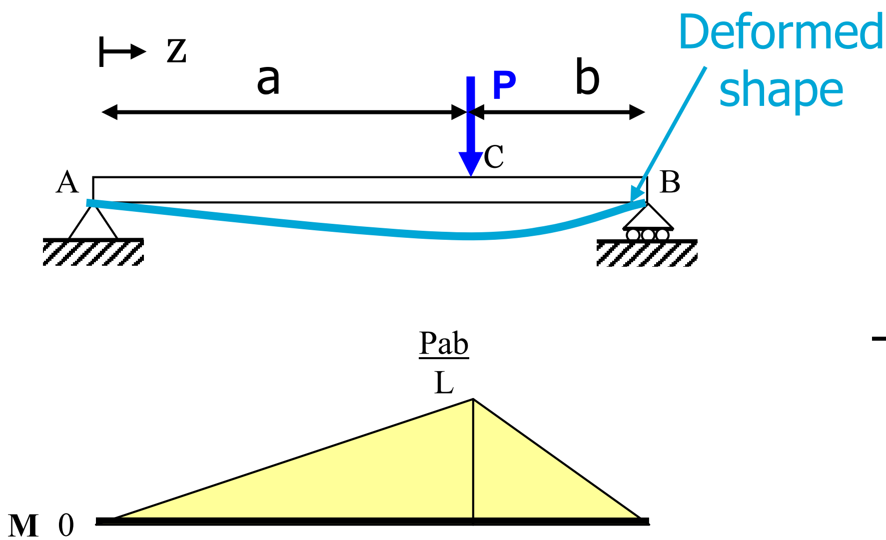
$v = 0$



$v = 0$
$\theta = 0$

Determining Constants of Integration

Continuity Conditions



Discontinuity at z = a

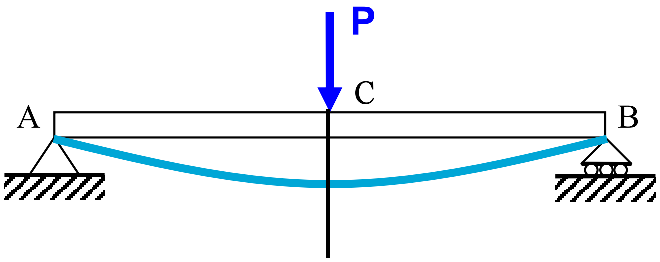
$z \leq a$	$z \geq a$
$M_{AC} = \dfrac{Pb}{L} z$	$M_{CB} = \dfrac{Pa}{L}(L - z)$

$$v_{AC}(a) = v_{CB}(a)$$

$$\theta_{AC}(a) = \theta_{CB}(a)$$

Determining Constants of Integration

Symmetry Conditions



- Symmetry implies reflection of deformation across symmetry plane
  - v is equal
  - θ is opposite
- Continuity implies equal deformation at symmetry plane
  - v is equal
  - θ is equal

$$\therefore \theta_C = 0$$

Procedure for Analysis

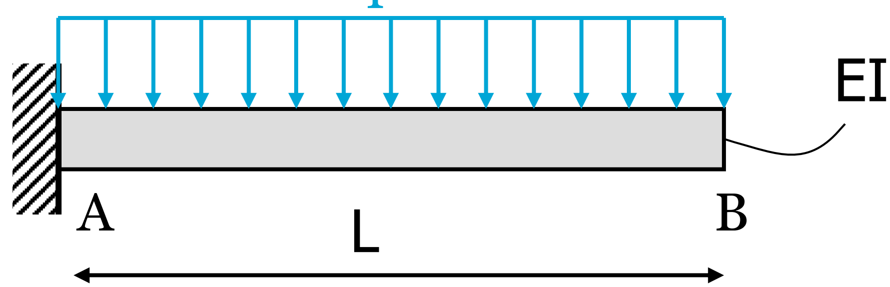
Deflection by Integration

- Draw a FBD including reaction forces
- Determine V and M relations for the beam
- Integrate Moment-displacement differential equation
- Select appropriate support, symmetry, and continuity conditions to solve for constants of integration
- Calculate desired deflection (v) and slopes (θ)

Example 1a

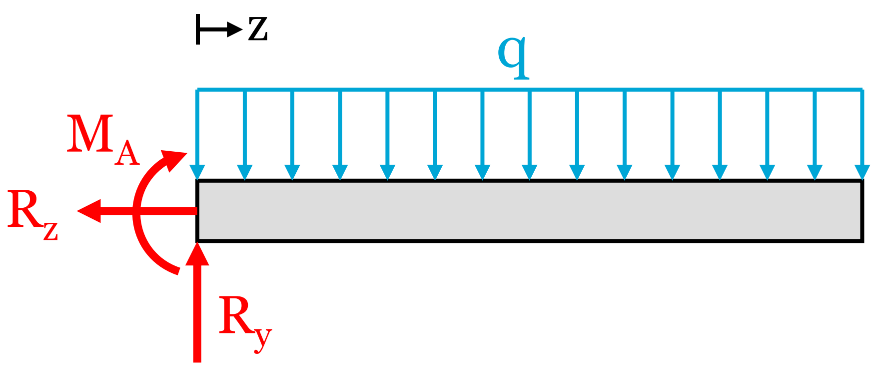
Problem Statement

Determine the deflection and slope at point B in a prismatic beam due to the distributed load q



Solution

1) FBD & Equilibrium



$$\sum \overset{\leftarrow +}{F} = 0 = R_z$$

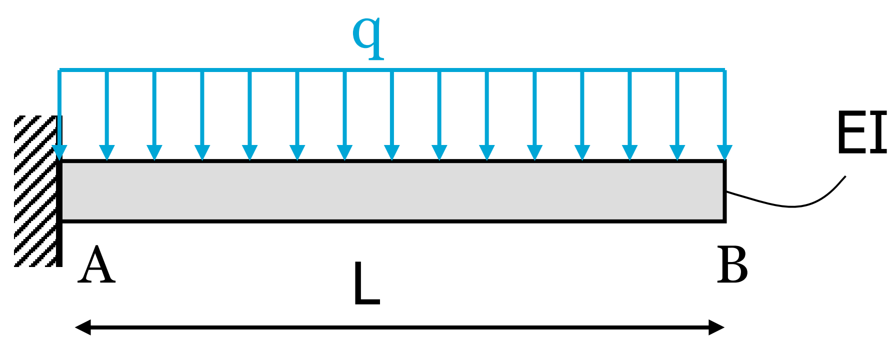
$$\sum \overset{\uparrow +}{F} = 0 = R_y - qL \Rightarrow R_y = qL$$

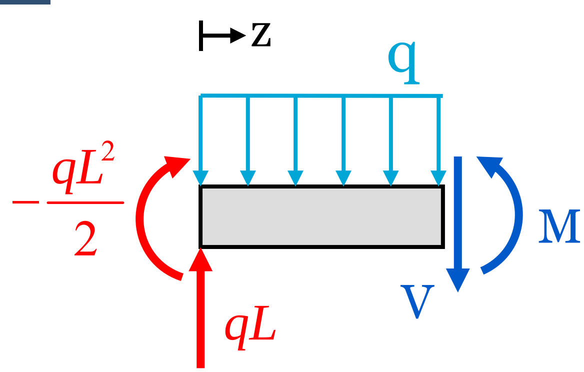
$$\sum M_A^{cw+} = 0 = M_A + qL\frac{L}{2} \Rightarrow M_A = -\frac{qL^2}{2}$$

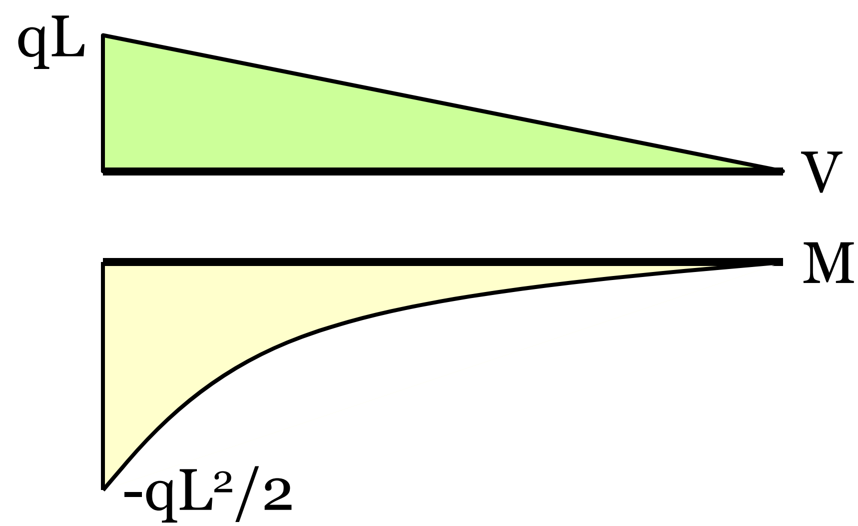
Example 1a

Solution

2) Determine M and V @ z







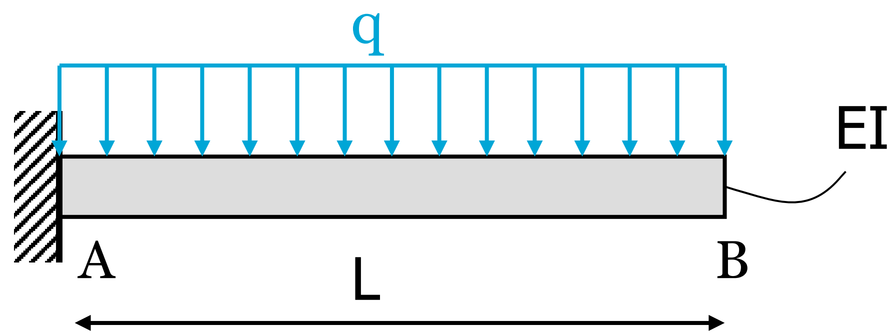
$$\sum \overset{\uparrow +}{F} = 0 = qL - qz - V \Rightarrow V = q\left(L - z\right)$$

$$\sum M_z^{ccw+} = 0 = M + \frac{qL^2}{2} - qLz + qz\frac{z}{2} \Rightarrow M = q\left(Lz - \frac{L^2}{2} - \frac{z^2}{2}\right)$$

Example 1a

Solution

3) Boundary Conditions



At z = 0:

$$\underbrace{v = 0, \quad v' = 0}_{\text{Two boundary conditions}}$$

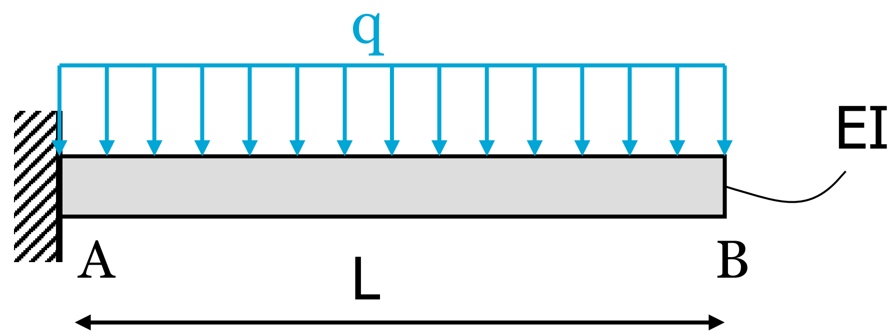
Thus can solve by integrating: $M = -EI\dfrac{d^2v}{dz^2}$

$$\Rightarrow \frac{d^2v}{dz^2} = -\frac{1}{EI}M$$

Example 1a

Solution

4) Solve Differential Equation



$$M = -\frac{qz^2}{2} + qLz - \frac{qL^2}{2}$$

$$\frac{d^2v}{dz^2} = \frac{-1}{EI}\left(-\frac{qz^2}{2} + qLz - \frac{qL^2}{2}\right)$$

$$\frac{dv}{dz} = \frac{-1}{EI}\left(-\frac{qz^3}{6} + \frac{qLz^2}{2} - \frac{qL^2z}{2} + \cancelto{0}{C_1}\right)$$

Boundary Condition

BC: At z = 0, θ = 0

=> $C_1 = 0$

$$v = \frac{-1}{EI}\left(-\frac{qz^4}{24} + \frac{qLz^3}{6} - \frac{qL^2z^2}{4} + \cancelto{0}{C_2}\right)$$

BC: At z = 0, v = 0

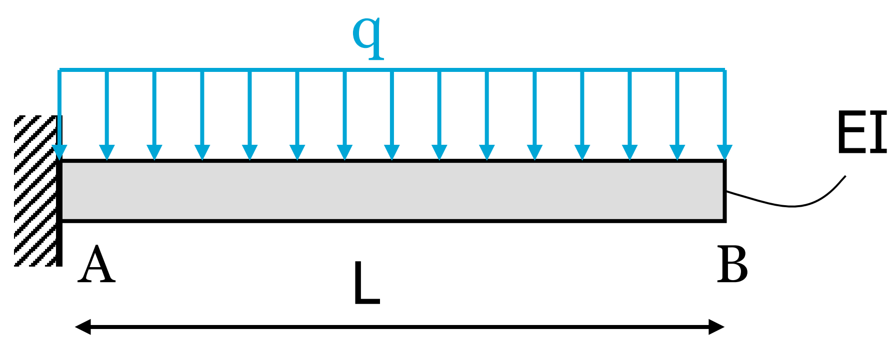
=> $C_2 = 0$

$$\boxed{v = \frac{qz^2}{24EI}\left(z^2 - 4Lz + 6L^2\right)}$$

$$\boxed{\theta = \frac{dv}{dz} = \frac{qz}{6EI}\left(z^2 - 3Lz + 3L^2\right)}$$

Example 1a

Solution



5) Calculate slopes and deflections

Determine deflection and slope at B:

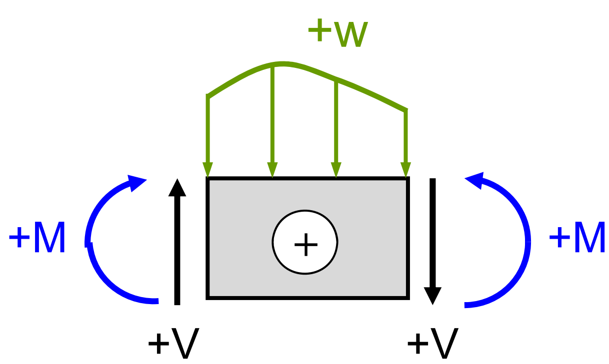
$$v = \frac{qz^2}{24EI}\left(z^2 - 4Lz + 6L^2\right) \qquad \theta = \frac{qz}{6EI}\left(z^2 - 3Lz + 3L^2\right)$$

$$v_B = v_{(z=L)} = \frac{qL^4}{8EI} \qquad \theta_B = \theta_{(z=L)} = \frac{qL^3}{6EI}$$

Relating Deformation to Loading

Shear Force-Moment Diagram Relationships

- Recall from Statics
  (refer to Hibbler Ch. 6.2 for refresher, but be careful of coordinate system)



$$\frac{dV}{dz} = -w$$

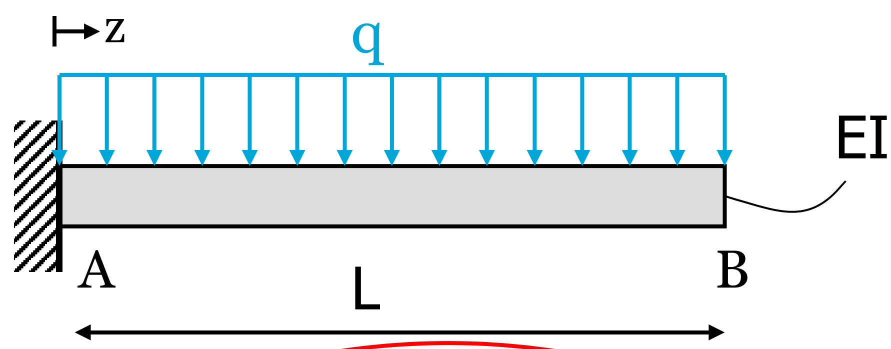
$$\frac{dM}{dz} = V$$

Moment-Curvature Relationship (Eq. 10.1)

$$M = -EI\frac{d^2v}{dz^2}$$

$$\boxed{\begin{aligned} \frac{d^4v}{dz^4} &= \frac{w(z)}{EI} \equiv v'''' \\ \frac{d^3v}{dz^3} &= -\frac{V(z)}{EI} \equiv v''' \\ \frac{d^2v}{dz^2} &= -\frac{M(z)}{EI} \equiv v'' \end{aligned}}$$

Example 1b



We can also solve Example 1 in an alternative way:

$$\frac{d^4v}{dz^4} = \frac{w(z)}{EI} \equiv v''''$$

$$\frac{d^3v}{dz^3} = -\frac{V(z)}{EI} \equiv v'''$$

$$\frac{d^2v}{dz^2} = -\frac{M(z)}{EI} \equiv v''$$

Watch negative sign!

$$-EIv'''' = -q = -w(z)$$

$\int$

$$-EIv''' = -qz + C_1 = V(z)$$

$\int$

$$-EIv'' = -\frac{qz^2}{2} + C_1 z + C_2 = M(z)$$

$\int$

$$-EIv' = -\frac{qz^3}{6} + C_1\frac{z^2}{2} + C_2 z + C_3 = -\theta(z)EI$$

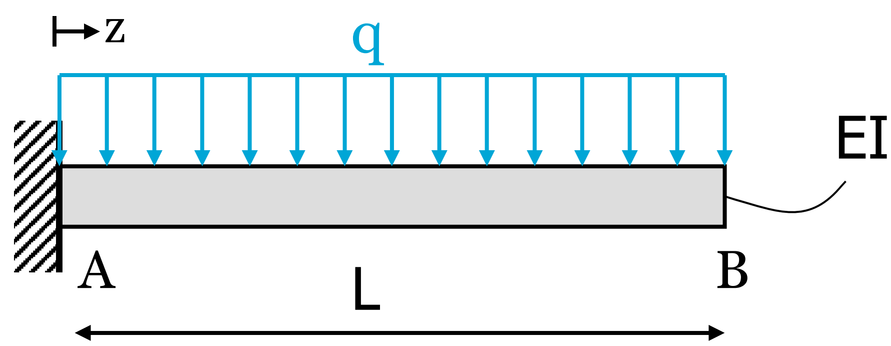
$\int$

$$-EIv = -\frac{qz^4}{24} + C_1\frac{z^3}{6} + C_2\frac{z^2}{2} + C_3 z + C_4$$

We have 4 unknown constants of integration, thus need 4 BCs

Example 1b

We can also solve this problem an alternative way:



$$-EIv''' = -qz + C_1 \quad = V(z)$$

$$-EIv'' = -\frac{qz^2}{2} + \underbrace{qL}_{C_1}z + C_2 \quad = M(z)$$

$$-EIv' = -\frac{qz^3}{6} + \frac{qL}{2}z^2 - \underbrace{\frac{qL^2}{2}}_{C_2}z + C_3 = -\theta(z)EI$$

Boundary Condition:

At $z = L$, $V = o$

$$\Rightarrow C_1 = qL$$

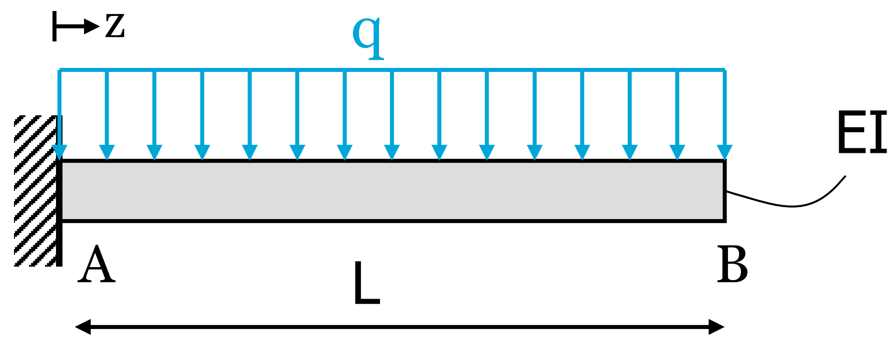
At $z = L$, $M = o$

$$\Rightarrow C_2 = -\left(-\frac{qL^2}{2} + qL^2\right) = -\frac{qL^2}{2}$$

At $z = o$, $\theta = o$

$$\Rightarrow C_3 = 0$$

Example 1b



We can also solve this problem an alternative way:

$$-EIv = -\frac{qz^4}{24} + \frac{qL}{6}z^3 - \frac{qL^2}{4}z^2 + 0 + C_4$$

Boundary Condition:

At $x = o$, $\mathrm{v} = o$

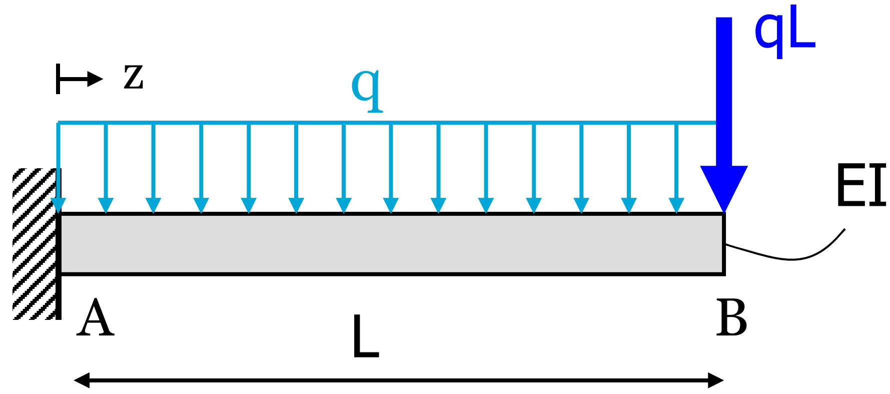
$$\Rightarrow C_4 = 0$$

$$\boxed{v = \frac{qz^2}{24EI}\left(z^2 - 4Lz + 6L^2\right)}$$

$$\boxed{v' = \theta = \frac{qx}{6EI}\left(z^2 - 3Lz + 3L^2\right)}$$

Same result as before!

Example 2



Determine deflection and slope at B:

$$-EIv'''' = -q = -w(z)$$

$$\int$$

$$-EIv''' = -qz + C_1 = V(z)$$

$$\int$$

$$-EIv'' = -\frac{qz^2}{2} + C_1 z + C_2 = M(z)$$

$$\int$$

$$-EIv' = -\frac{qz^3}{6} + C_1\frac{z^2}{2} + C_2 z + C_3 = -\theta(z)EI$$

$$\int$$

$$-EIv = -\frac{qz^4}{24} + C_1\frac{z^3}{6} + C_2\frac{z^2}{2} + C_3 z + C_4$$

We will apply Approach 2

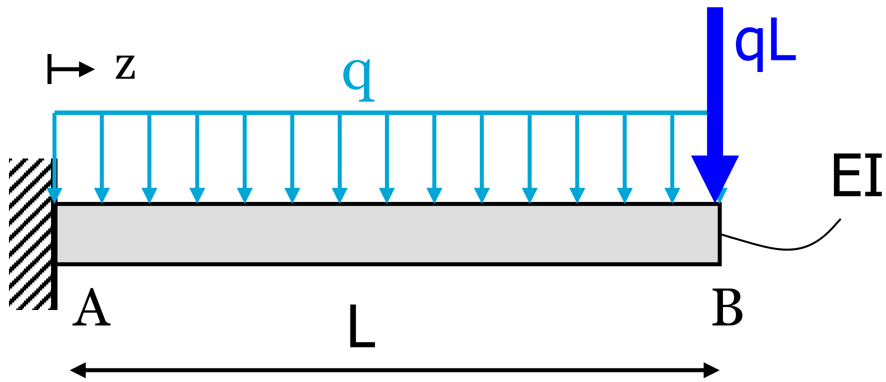
Exact same differential equations as before!!

What makes the problem different?

Boundary Conditions!

Example 2

Determine deflection and slope at B:

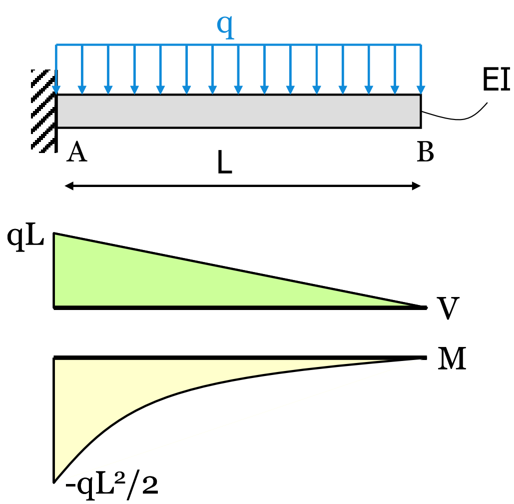


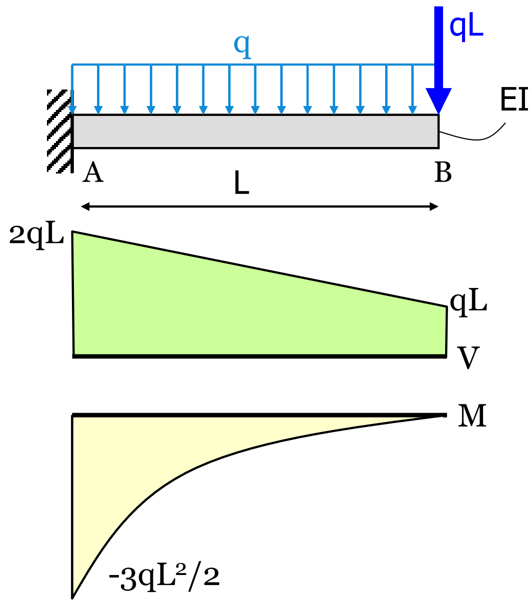
$$-EIv''' = -qz + C_1 = V(z)$$

Boundary Condition:

At $z = L$, $V = qL$

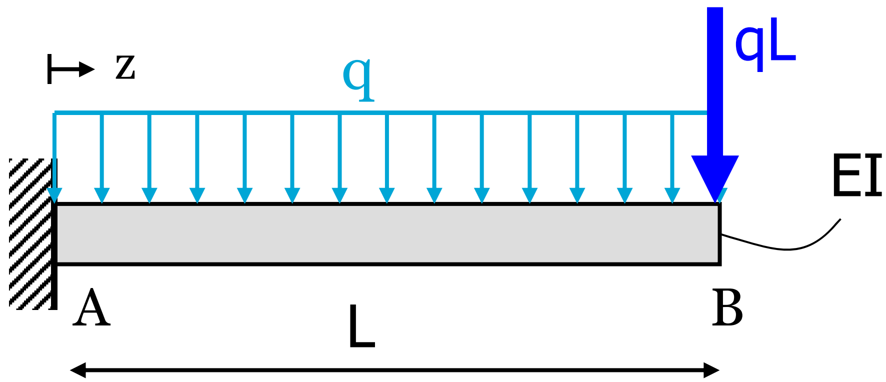
$$\Rightarrow C_1 = 2qL$$





Example 2

Determine deflection and slope at B:



$$-EIv''' = -qz + C_1 = V(z)$$

$$-EIv'' = -\frac{qz^2}{2} + \underbrace{2qL}_{C_1}z + C_2 = M(z)$$

$$-EIv' = -\frac{qz^3}{6} + qLz^2 - \underbrace{\frac{3qL^2}{2}}_{C_2}z + C_3 = -\theta(x)EI$$

Boundary Condition:

At $z = L$, $V = qL$

$$\Rightarrow C_1 = 2qL$$

At $z = L$, $M = o$

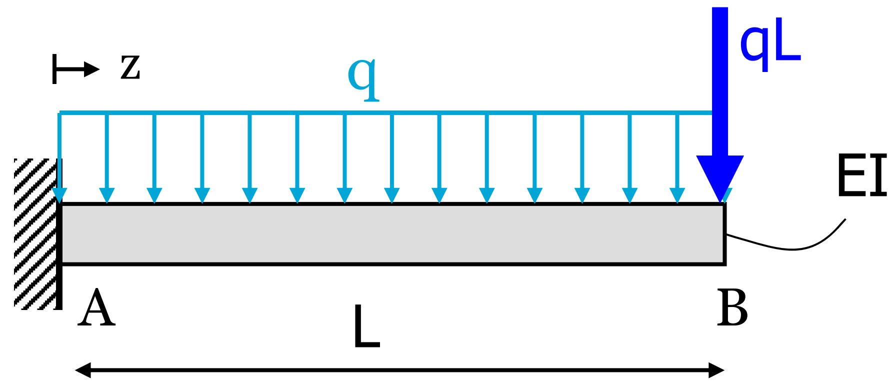
$$\Rightarrow C_2 = -\left(-\frac{qL^2}{2} + 2qL^2\right) = -\frac{3qL^2}{2}$$

At $z = o$, $\theta = o$

$$\Rightarrow C_3 = 0$$

Example 2

Determine deflection and slope at B:



$$-EIv = -\frac{qz^4}{24} + \frac{qL}{3}z^3 - \frac{3qL^2}{4}z^2 + 0 + C_4$$

<u>Boundary Condition:</u>

At $z = o$, v = o

$$\Rightarrow C_4 = 0$$

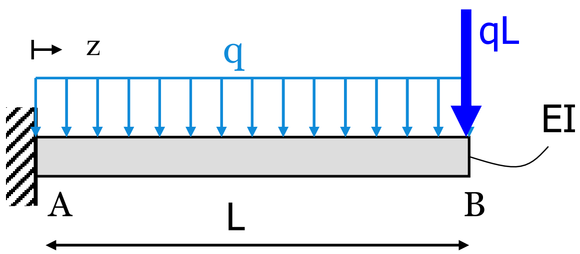
$$\boxed{v = \frac{qz^2}{24EI}\left(z^2 - 8Lz + 18L^2\right)}$$

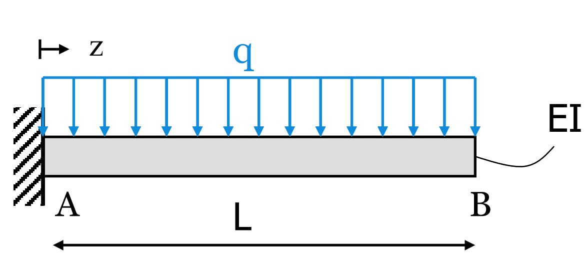
$$\boxed{v' = \frac{qz}{6EI}\left(z^2 - 6Lz + 9L^2\right)}$$

$$v_B = v_{(z=L)} = \frac{11qL^4}{24EI}$$

$$\theta_B = v'_{(z=L)} = \frac{2qL^3}{3EI}$$

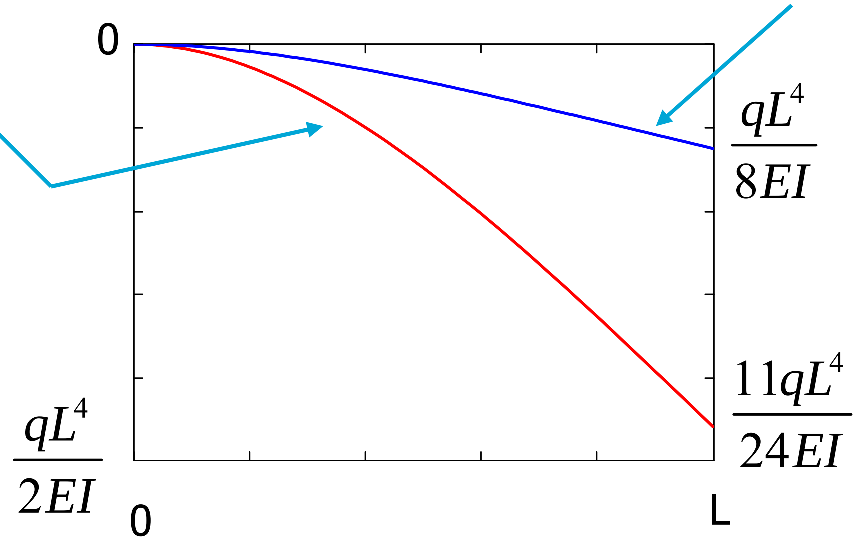
Example 2



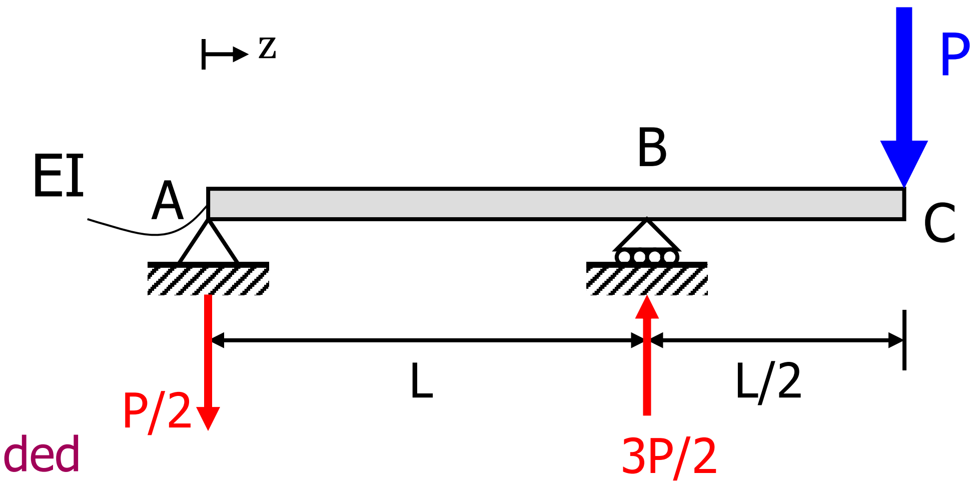


$$v = \frac{qz^2}{24EI}\left(z^2 - 8Lz + 18L^2\right)$$

$$v = \frac{qz^2}{24EI}\left(z^2 - 4Lz + 6L^2\right)$$



Example 3



Determine deflection at C in terms of EI:

To save time, reactions are provided

Since reaction forces act at B (discontinuity), we must split the differential equation into parts for AB and BC

We can easily see by inspection that:

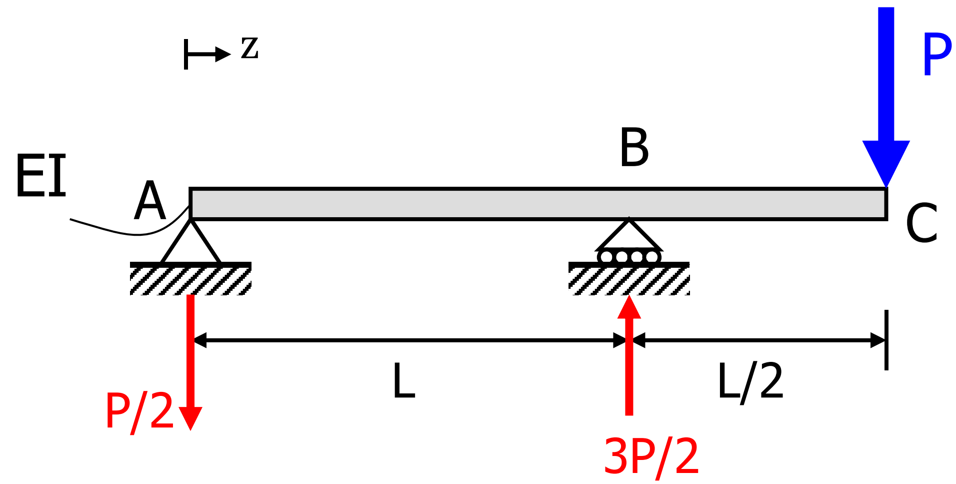
$$-EIv''' = V = -\frac{P}{2} \qquad (0 < z < L)$$

$$-EIv''' = V = P \qquad (L < z < 3L/2)$$

Integrate to find M

Example 3

Determine deflection at C:



Moments:

$$M = -EIv'' = -\frac{P}{2}z + C_1 \qquad (0 \le z \le L)$$

$$M = -EIv'' = Pz + C_2 \qquad (L \le z \le 3L/2)$$

Moment BC's:

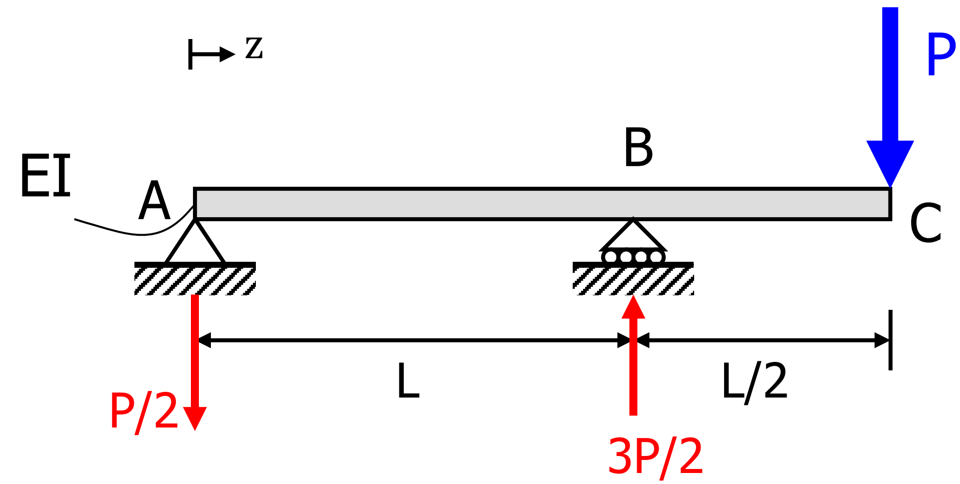
At $z = 0, M = 0$ $\quad \Rightarrow C_1 = 0$

At $z = 3L/2, M = 0$ $\quad \Rightarrow C_2 = -\frac{3PL}{2}$

Integrate to find θ

Example 3

Determine deflection at C:



Slopes:

$$-EIv' = -\frac{P}{4}z^2 + C_3 \qquad (0 \le z \le L)$$

$$-EIv' = \frac{P}{2}z^2 - \frac{3PL}{2}z + C_4 \qquad (L \le z \le 3L/2)$$

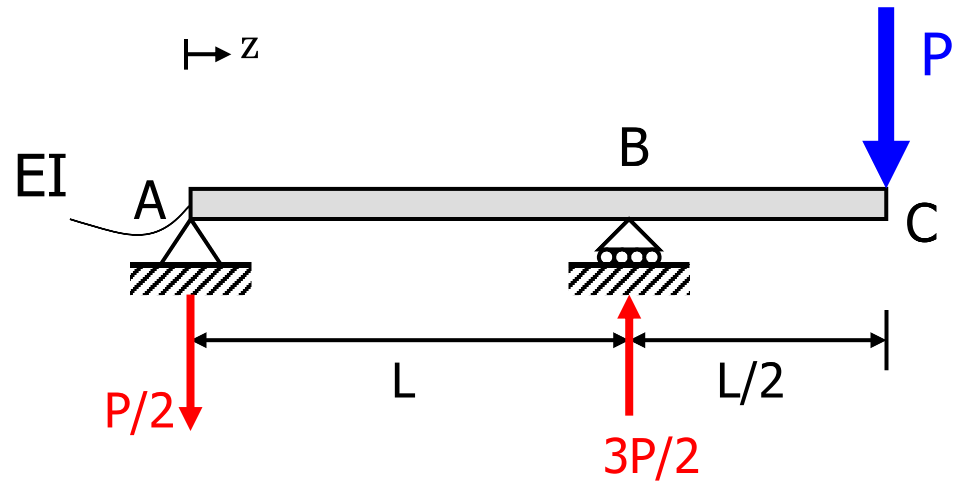
Slope Continuity Condition:

$$\text{At } z = L,\ \theta_{AB} = \theta_{BC} \quad \Rightarrow -\frac{PL^2}{4} + C_3 = -PL^2 + C_4$$

Integrate to find v

Example 3

Determine deflection at C:



Deflections:

$$-EIv = -\frac{P}{12}z^3 + C_3 z + C_5 \qquad (0 \le z \le L)$$

$$-EIv = \frac{P}{6}z^3 - \frac{3PL}{4}z^2 + C_4 z + C_6 \qquad (L \le z \le 3L/2)$$

From last slide

$$\boxed{-\frac{PL^2}{4} + C_3 = -PL^2 + C_4}$$

Deflection BC's:

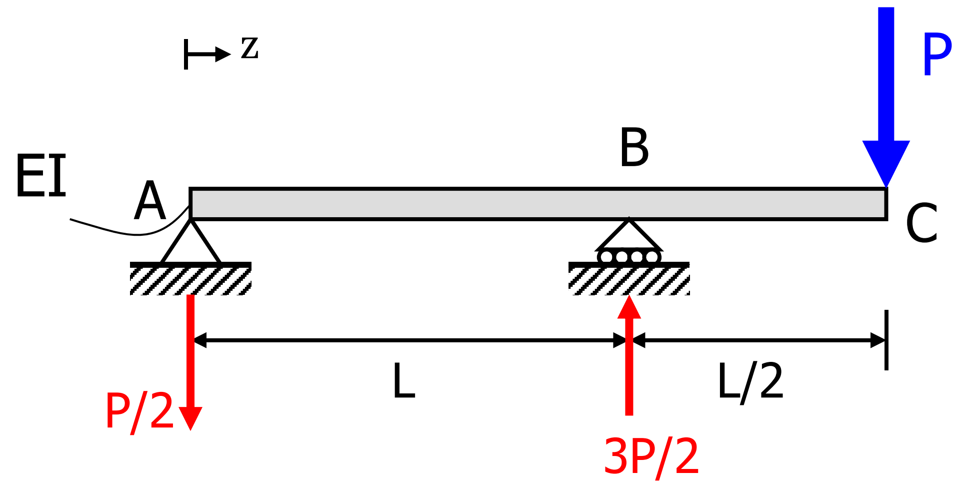
At $z = 0$, v = 0 $\qquad \Rightarrow C_5 = 0$

At $z = L$, v = 0 $\qquad \Rightarrow C_3 = \frac{PL^2}{12}$

$$C_4 = \frac{5PL^2}{6} \qquad C_6 = -\frac{PL^3}{4}$$

Example 3

Determine deflection at C:

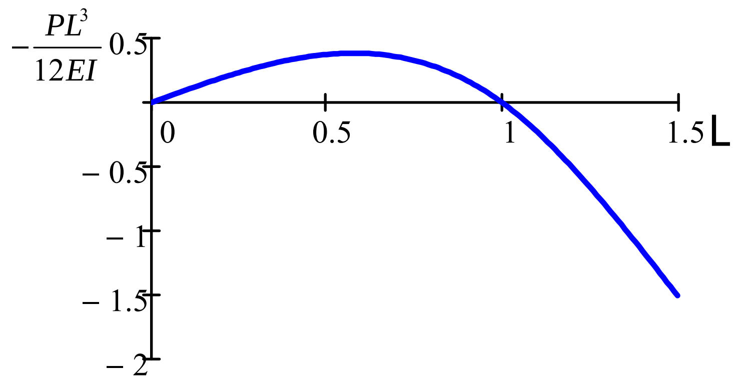


Deflections:

$$v = -\frac{Pz}{12EI}\left(L^2 - z^2\right) \qquad (0 \le z \le L)$$

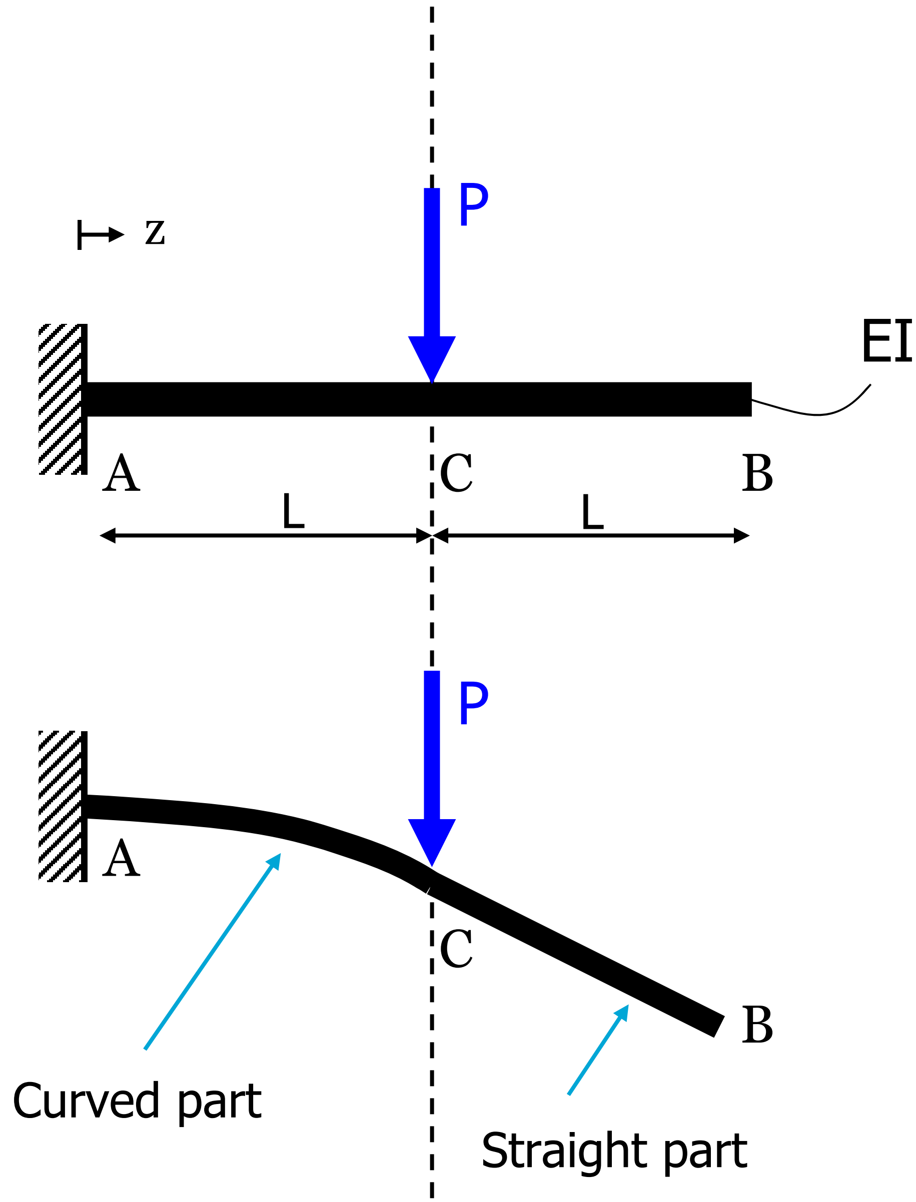
$$v = \frac{P}{12EI}\left(3L^3 - 10L^2 z + 9Lz^2 - 2z^3\right) \qquad (L \le z \le 3L/2)$$

$$v(C) = v\left(\frac{3L}{2}\right) = \frac{PL^3}{8EI}$$



Example 4

What about beams with a non loaded free end?

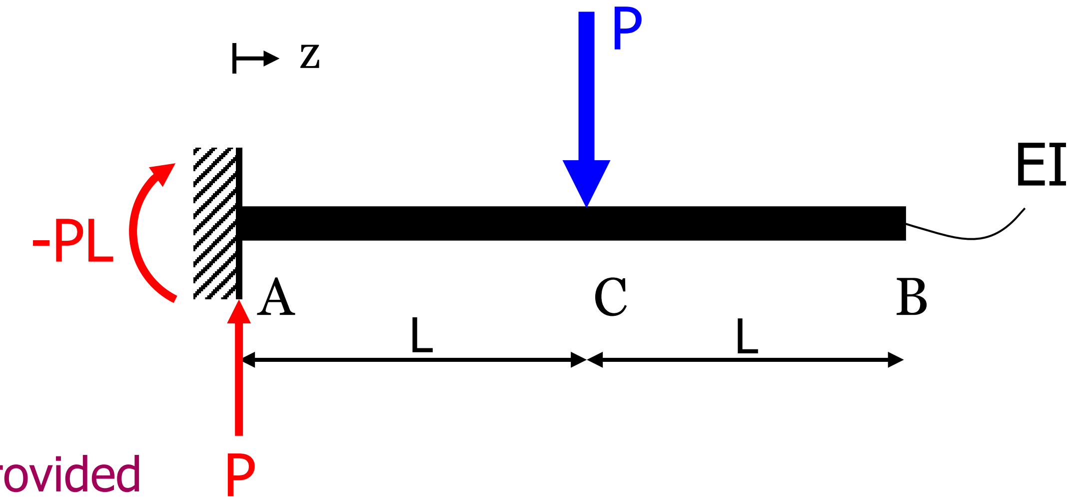


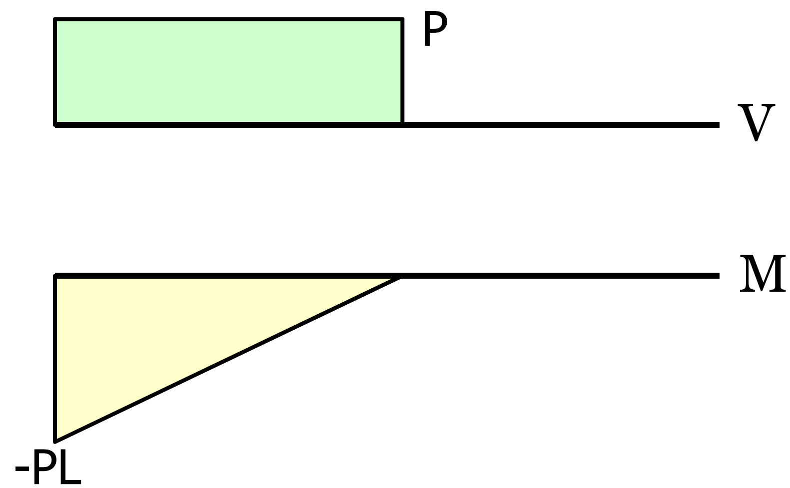
Will it work itself out?

Example 4

What about beams with a non loaded free end?

To save time, reactions are provided





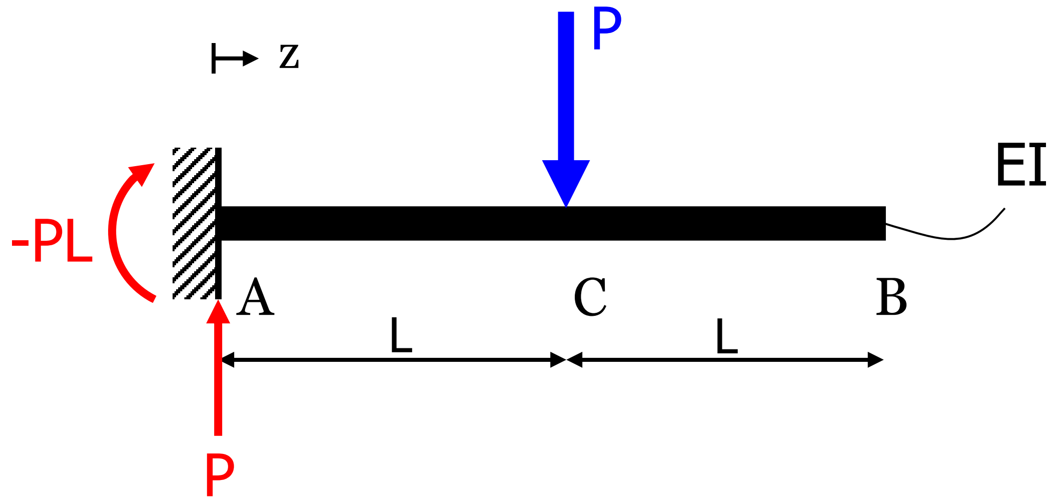
Moments:

$$M = -EIv'' = P\left(z - L\right) \quad (0 \le z \le L)$$

$$M = -EIv'' = 0 \quad (L \le z \le 2L)$$

Example 4

What about beams with a non loaded free end?



Slopes:

$$-EIv' = \frac{P}{2}z^2 - PLz + C_1 \qquad (0 \le z \le L)$$

$$-EIv' = C_2 \qquad (L \le z \le 2L)$$

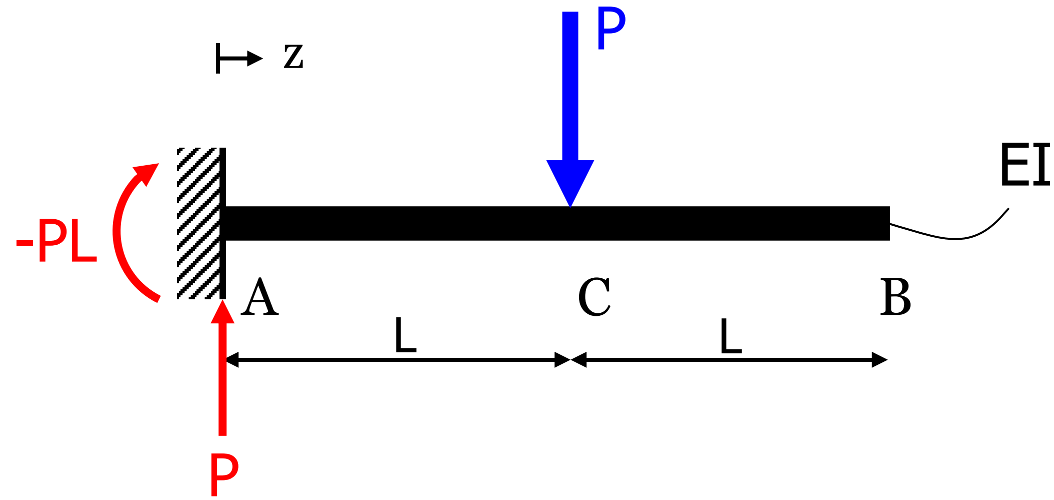
Slope BC's:

At $z = 0$, $\theta = 0$ $\quad \Rightarrow C_1 = 0$

Slope CC's:

At $z = L$, $\theta_{AC} = \theta_{CB}$ $\quad \Rightarrow C_2 = -\frac{PL^2}{2}$

Example 4



What about beams with a non loaded free end?

Displacements:

$$-EIv = \frac{P}{6}z^3 - \frac{PL}{2}z^2 + C_3 \qquad (0 \leq z \leq L)$$

$$-EIv = -\frac{PL^2}{2}z + C_4 \qquad (L \leq z \leq 2L)$$

Displacement BC's:

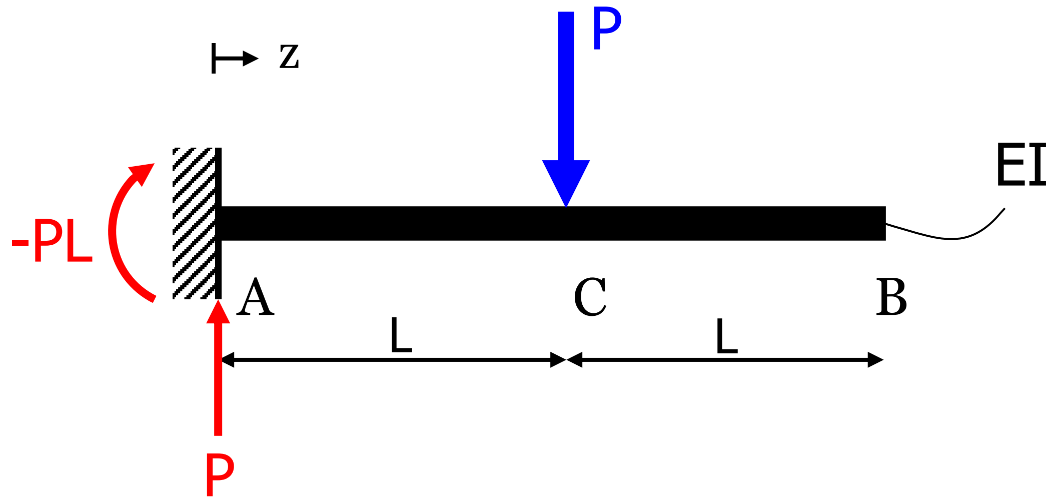
At $z = 0$, $v = 0$ $\qquad \Rightarrow C_3 = 0$

Displacement CC's:

At $z = L$, $v_{AC} = v_{CB}$ $\qquad \Rightarrow C_4 = \frac{PL^3}{6}$

Example 4

What about beams with a non loaded free end?

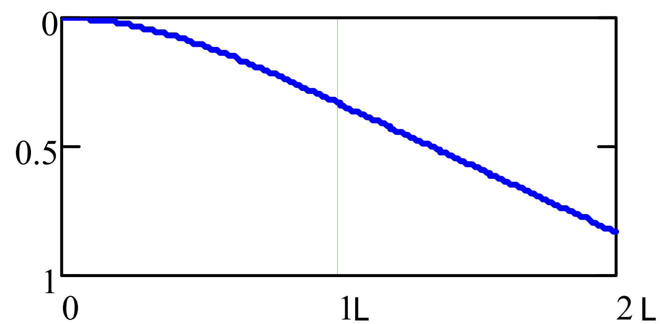


Displacements:

$$v = -\frac{Pz^2}{6EI}\left(z - 3L\right) \qquad (0 \leq z \leq L)$$

$$v = \frac{PL^2}{2EI}\left(z - \frac{L}{3}\right) \qquad (L \leq z \leq 2L)$$

Formula for a straight line!



No curvature, it does work out!

For next time

	First Lecture		Second Lecture		COZ		Instruction
Week	Topics	Preparation	Topics	Preparation	Assignment	Due Date	Activity
3.1	Introduction	None	Stress, Strain, Hooke's Law	L.U. 1 (all)	none	none	no instruction
3.2	Axial loading and static indetermenacy	L.U. 2 (all)	Torsion of circular shafts	L.U. 3.1-3.2	COZ1	18/02/2016	Mock exam 1
3.3	Torsion of thin-walled shafts	L.U. 3.3	Bending stresses in beams	L.U. 4 (all)	COZ2	25/02/2016	Peer grading 1
3.4	Transverse shear stresses in beams	L.U. 5.1	Shear stresses in thin-walled beams	L.U. 5.2	COZ3	03/03/2016	Mock exam 2
3.5	Combined loading	L.U. 6 (all)	Stress transformations & Failure criteria	L.U. 7 (all)	COZ4	10/03/2016	Peer Grading 2
3.6	Beam deflections by integration	L.U. 8.1	Discontinuity functions and	L.U. 8.2-8.3	COZ5	17/03/2016	Mock exam 3
3.7	Statically indeterminate beams	L.U. 8.4	Review	None	COZ6	24/03/2016	Peer grading 3
3.8	Study for Exam						
3.9	Exam - Friday April 8th @ 13:30						

L.U. = Learning unit (refer to blackboard site for learning units)