Dynamics and Stability AE3-914

Sample problem—Week 2

Bungee jumping

Statement

Consider a keen bungee jumper of mass m. The bungee can be viewed as a spring with elastic constant k and natural length l when it is loaded in tension and it cannot carry any force in compression. The acceleration of gravity is g and the dimensions of the platform can be neglected. Derive the equations of motion for the jumper in the following cases:

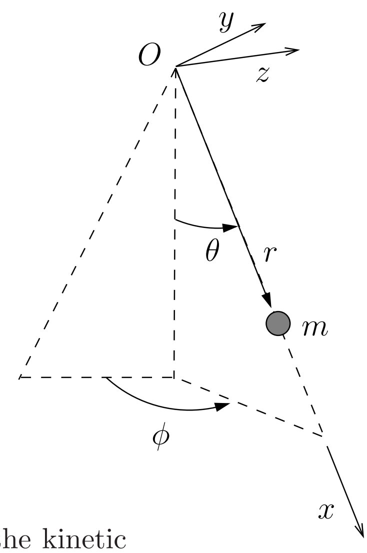
a. The bungee is loose.

b. The bungee is taut.

Indicate the range of coordinates for which each situation takes place.

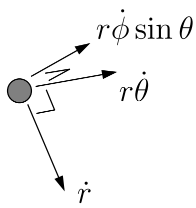
Model

The platform is taken as a fixed reference point O. Since the motion will probably start from there and the bungee is attached to it, it seems reasonable to choose a set of spherical coordinates (r, θ, ϕ) as generalised coordinates, according to the figure. Coordinate r measures the distance to the platform, coordinate θ represents the deviation from the vertical line and coordinate ϕ represents the angle between the planes formed by, on one hand, the crane and the vertical line and, on the other hand, the vertical line and the mass.



Lagrangian function

In order to state the Lagrangian function expressions for the kinetic energy T and the potential energy V must be found. Inspection of the motion of the mass reveals that the velocity vector has the orthogonal components shown in the figure. Alternatively, a moving xyz-coordinate system can be used such that the x-axis coincides with the line joining the platform O and the mass, an angular velocity vector



can be stated in terms of $\dot{\theta}$ and $\dot{\phi}$ and the absolute velocity of the mass can be found. The kinetic energy reads

$$\begin{aligned} T &= \frac{1}{2}m\big(\dot{r}^2 + (r\dot{\theta})^2 + (r\dot{\phi}\sin\theta)^2\big) \\ &= \frac{1}{2}m\big(\dot{r}^2 + r^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)\big). \end{aligned} \tag{1}$$

For the potential energy V two different cases need to be considered, corresponding to either a loose or a taut bungee.

If the bungee is loose it is not providing any force to the mass and only gravity needs to be considered. Taking the platform as zero level of gravitational potential energy one has

$$V = -mgr\cos\theta. \tag{2}$$

Since the natural length of the bungee is l, the expression (2) of the potential energy is valid as long as

$$r < l. \tag{3}$$

When the bungee becomes taut a term involving the elastic potential energy is added to the expression (2) of V,

$$V = -mgr\cos\theta + \frac{1}{2}k(r-l)^2, \tag{4}$$

which will consequently be valid when

$$r \geq l. \tag{5}$$

This leads to the following piece-wise expression for the Lagrangian

$$\begin{aligned} L &= T - V \\ &= \begin{cases} \frac{1}{2}m\big(\dot{r}^2 + r^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)\big) + mgr\cos\theta & \text{if } r < l; \\ \frac{1}{2}m\big(\dot{r}^2 + r^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)\big) + mgr\cos\theta - \frac{1}{2}k(r-l)^2 & \text{if } r \geq l. \end{cases} \end{aligned} \tag{6}$$

Equations of motion

The equations of motion are derived from the Lagrange formalism that, for a conservative system, reads

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \qquad i = 1, \dots n. \tag{7}$$

From equation (6) the generalised momenta are, in all cases, obtained as

$$\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r}; \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta}; \\ \frac{\partial L}{\partial \dot{\phi}} &= mr^2\dot{\phi}\sin^2\theta. \end{aligned} \tag{8}$$

The rate of change of these generalised momenta is, consequently,

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= m\ddot{r}; \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}); \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= m(r^2\ddot{\phi}\sin^2\theta + 2r\dot{r}\dot{\phi}\sin^2\theta + r^2\dot{\phi}\dot{\theta}\sin 2\theta).
\end{aligned} \tag{9}$$

The non-inertial force terms are elaborated as

$$\begin{aligned}
\frac{\partial L}{\partial r} &= \begin{cases} mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mg\cos\theta & \text{if } r < l; \\ mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mg\cos\theta - k(r-l) & \text{if } r \geq l; \end{cases} \\
\frac{\partial L}{\partial \theta} &= \frac{1}{2}mr^2\dot{\phi}^2\sin 2\theta - mgr\sin\theta; \\
\frac{\partial L}{\partial \phi} &= 0.
\end{aligned} \tag{10}$$

Substituting (9–10) into (7) yields

$$\begin{aligned}
0 &= \begin{cases} m\ddot{r} - mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mg\cos\theta & \text{if } r < l; \\ m\ddot{r} - mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mg\cos\theta + k(r-l) & \text{if } r \geq l; \end{cases} \\
0 &= r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - \frac{1}{2}r^2\dot{\phi}^2\sin 2\theta + gr\sin\theta; \\
0 &= r^2\ddot{\phi}\sin^2\theta + 2r\dot{r}\dot{\phi}\sin^2\theta + r^2\dot{\phi}\dot{\theta}\sin 2\theta,
\end{aligned} \tag{11}$$

which can be cleaned up as

$$\begin{aligned}
\ddot{r} &= r(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + g\cos\theta - \begin{cases} 0 & \text{if } r < l; \\ \dfrac{k}{m}(r-l) & \text{if } r \geq l; \end{cases} \\
\ddot{\theta} &= -\frac{2\dot{r}\dot{\theta}}{r} + \frac{1}{2}\dot{\phi}^2\sin 2\theta - \frac{g\sin\theta}{r} \\
\ddot{\phi} &= -\frac{2\dot{r}\dot{\phi}}{r} - \frac{2\dot{\phi}\dot{\theta}}{\tan\theta}.
\end{aligned} \tag{12}$$

Notice that the mass of the jumper only plays a role when the bungee is taut.