Dynamics and Stability AE3-914

Sample problem—Week 3

Centrifuge in Star City

Statement

A centrifuge capable of producing accelerations up to $8g$ is available for cosmonaut training in Star City. The centrifuge can be seen as a massless rotating arm of length R supporting a simple pendulum of length l and mass m. A constant moment M is applied to the arm. The acceleration of gravity is g.

a. Set up the Lagrangian.

b. Are there any ignorable coordinates?

c. Are there any integrals of motion?

When sufficient velocity has been reached the moment M is removed.

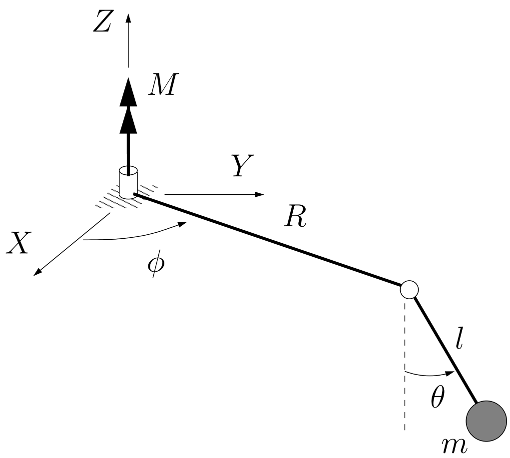
d. Are there any ignorable coordinates now? Which integrals of motion do they introduce?

e. Set up the Routhian and find the equation of motion for the explicit coordinate.

f. What are the conditions for steady motion?

Model

The centrifuge is schematised as shown, with the generalised coordinates θ and ϕ, which provide a proper description of the motion. The arm of length R remains horizontal and the pendulum is always contained within a vertical plane.



a. Set up the Lagrangian

In order to state the Lagrangian function expressions for the kinetic energy T and the potential V must be found. In this example the absolute velocity of the mass will be expressed in terms of the XYZ-coordinate system for a given configuration of the generalised coordinates and the time derivatives will be obtained. The absolute position of the particle is

$$\begin{aligned} X &= (R + l\sin\theta)\cos\phi \\ Y &= (R + l\sin\theta)\sin\phi \\ Z &= -l\cos\theta \end{aligned} \tag{1}$$

and the corresponding time derivatives are

$$\begin{aligned} \dot{X} &= l\cos\theta\cos\phi\,\dot{\theta} - (R + l\sin\theta)\sin\phi\,\dot{\phi} \\ \dot{Y} &= l\cos\theta\sin\phi\,\dot{\theta} + (R + l\sin\theta)\cos\phi\,\dot{\phi} \\ \dot{Z} &= l\sin\theta\,\dot{\theta} \end{aligned} \tag{2}$$

The kinetic energy is now elaborated as

$$\begin{aligned} T &= \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) \\ &= \frac{1}{2}m\big((R + l\sin\theta)^2\dot{\phi}^2 + l^2\dot{\theta}^2\big) \end{aligned} \tag{3}$$

Alternatively, inspection of the motion of the mass reveals that the velocity vector has two orthogonal components that can easily be identified. Other possibility is to consider a moving xyz-coordinate system such that the x-axis coincides with the line joining the hinge and the mass, an angular velocity vector can be stated in terms of $\dot{\theta}$ and $\dot{\phi}$ and the absolute velocity of the mass can be found.

A gravitational potential energy is identified as

$$V_g = -mgl\cos\theta \tag{4}$$

with the zero level at the XY-plane and a moment M is applied to the system. The virtual work is immediately found to be

$$\delta W = M\delta\phi, \tag{5}$$

which signifies that the generalised force associated with ϕ is

$$Q_\phi = M. \tag{6}$$

A generalised potential can be found by simple integration for this generalised force as

$$V_{gen} = -M\phi, \tag{7}$$

which enables us to express the potential function of the system as

$$\begin{aligned} V &= V_g + V_{gen} \\ &= -mgl\cos\theta - M\phi \end{aligned} \tag{8}$$

The Lagrangian is now expressed as

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\big((R + l\sin\theta)^2\dot{\phi}^2 + l^2\dot{\theta}^2\big) + mgl\cos\theta + M\phi. \end{aligned} \tag{9}$$

b. Are there any ignorable coordinates?

Both generalised coordinates θ and ϕ are present in the Lagrangian and consequently none of them is ignorable

c. Are there any integrals of motion?

The applied moment M is carrying out work and therefore the mechanical energy $\mathbb{E}$ is not conserved. In other words, $\mathbb{E}$ is not an integral of motion.

We have a Lagrangian system, because all forces are conservative or can be derived from a generalised potential. Moreover the Lagrangian function exhibits no explicit dependence on time. Thus, the Jacobi energy integral h *is* an integral of motion. It is expressed as

$$\begin{aligned} h &= \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} + \dot{\phi}\frac{\partial L}{\partial\dot{\phi}} - L \\ &= \frac{1}{2}m\big((R + l\sin\theta)^2\dot{\phi}^2 + l^2\dot{\theta}^2\big) - mgl\cos\theta - M\phi \end{aligned} \tag{10}$$

The moment M is removed

When sufficient velocity has been reached the moment M is removed. Consequently, the Lagrangian adopts the form

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\big((R + l\sin\theta)^2\dot{\phi}^2 + l^2\dot{\theta}^2\big) + mgl\cos\theta. \end{aligned} \tag{11}$$

d. Are there any ignorable coordinates now?

The Lagrangian (11) does not depend on the generalised coordinate ϕ explicitly. It can then be stated that ϕ is an ignorable coordinate. The corresponding integral of motion is the generalised momentum conjugate to ϕ, i.e.,

$$\frac{\partial L}{\partial\dot{\phi}} = m(R + l\sin\theta)^2\dot{\phi} = C_\phi. \tag{12}$$

The physical interpretation of (12) is the conservation of angular momentum.

e. Set up the Routhian and find the equation of motion for the explicit coordinate

The ignorable coordinate can be removed from the Lagrangian by means of the Routh formalism. First, the ignorable coordinate needs to be expressed as an explicit function of its corresponding integral of motion,

$$\dot{\phi} = \frac{C_\phi}{m(R + l\sin\theta)^2}. \tag{13}$$

This is substituted in the expression of the Routhian to get

$$\begin{aligned} R &= C_\phi \dot{\phi} - L \\ &= -\frac{1}{2} m l^2 \dot{\theta}^2 + \frac{C_\phi^2}{2m(R + l\sin\theta)^2} - mgl\cos\theta \end{aligned} \tag{14}$$

The equation of motion for the explicit coordinate is now obtained from

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0. \tag{15}$$

The generalised momentum is obtained as

$$\frac{\partial R}{\partial \dot{\theta}} = -ml^2\dot{\theta} \tag{16}$$

and its rate of change is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) = -ml^2\ddot{\theta}. \tag{17}$$

The non-inertial term reads

$$\frac{\partial R}{\partial \theta} = -\frac{C_\phi^2}{m(R + l\sin\theta)^3} l\cos\theta + mgl\sin\theta. \tag{18}$$

Substituting (17–18) into (15) and rearranging terms we obtain the equation

$$\ddot{\theta} - \frac{C_\phi^2}{m^2 l(R + l\sin\theta)^3}\cos\theta + \frac{g}{l}\sin\theta = 0. \tag{19}$$

Notice that the presence of the integral of motion C_ϕ in (19) ensures a coupling between $\dot{\phi}$ and θ as stated by (12).

f. What are the conditions for steady motion?

Steady motion is attained when the generalized velocity and the rate of change of the generalised momentum of the explicit coordinate θ vanish, which in view of (17) is simply written as

$$\dot{\theta} = \ddot{\theta} = 0 \qquad \theta = \text{constant}. \tag{20}$$

A direct consequence of this, together with the integral of motion (12), is that

$$\dot{\phi} = \text{constant}. \tag{21}$$

Since the rate of change of the generalised momentum vanishes, the condition for steady motion is directly obtained from

$$\frac{\partial R}{\partial \theta} = 0 \tag{22}$$

which, considering (18) is written as

$$\frac{C_\phi^2}{m(R + l\sin\theta)^3} l\cos\theta = mgl\sin\theta. \tag{23}$$

Since both θ and $\dot{\phi}$ remain constant in steady motion conditions, the integral of motion C_ϕ can be explicitly substituted in (23) to get

$$\dot{\phi}^2 = \frac{g\tan\theta}{R + l\sin\theta}. \tag{24}$$

Thus, if the initial conditions (i.e., the conditions at the instant that the moment M is removed) for θ and $\dot{\phi}$ meet the above expression, together with $\dot{\theta} = 0$, the motion will be steady. Steady motion can be viewed as an equilibrium state of the non-ignorable (aka explicit) coordinates.