Chapter 11

Longitudinal driving taks models

Summary of chapter - The description of a traffic stream at microscopic level is about the movements of individual driver-vehicle combinations or driver-vehicle elements (DVE's). Microscopic models describe the interactions between the DVE's and sometimes are called manoeuver models.

An important aspect ot these microscopic model is how they describe the longitudinal driving task, both with respect to the road, and with respect to the other vehicles in the traffic stream. These are discussed in this chapter. Other manoeuver models concern e.g. overtaking on a road with oncoming vehicles; entering the roadway of a motorway from an on-ramp; crossing a road; weaving, etc. These will in part be discussed in the following chapter.

We have seen that the longitudinal driving task pertains both to the interaction with the roadway and the interaction with the other vehicles in the traffic stream. With respect to the former, the longitindal roadway subtask will rougly describe how vehicles will accelerate towards their free speed (or desired speed), when no other vehicles are directly in front of them. The longitudinal interaction task rougly describes how vehicles interact with other, generally slower vehicles. In particular, it describes the car-following behaviour. This section in particular describes the interaction subtask.

List of symbols

s_i	m	distance headway of vehicle i
T_r	s	reaction time
v_i	-	speed of vehicle i
τ	s	reaction time
a_i	m/s^2	acceleration of vehicle i
x_i	m	position of vehicle i
κ	$1/s$	sensitivity
u	-	mean speed

11.1 Model classification

In presenting the different modelling approaches that exist to describe the longitudinal driving task, the following approaches are considered[1]:

1. Safe-distance models

2. CA-models

[1] This list is certainly not exhaustive, but merely identifies the main modelling approaches to describe the longitudinal driving tasks.

3. Stimulus-response models
4. Psycho-spacing models
5. Optimal velocity models / optimal control models
6. Fuzzy-logic / rule-based models

Safe distance models are static and therefore do not descibe the dynamics of vehicle interactions. CA-models are discrete (in time and space) dynamic models, which provide a very coarse (but fast) description of traffic flow operations. Stimulus-response models, Pshycho-spacing models, and optimal control models provide a continuous time description of traffic flow.

Headway distribution models can also be considered microscopic models. These statistical models describe the headway distributions, and are generally based on very simple assumptions regarding the behaviour of drivers. Section 3.2 presented several distribution models as well as several applications. For a discussion on headway distribution models, the reader is referred to section

11.2 Safe-distance models

The first car-following models were developed in [44]. He stated that a good rule for following another vehicle at a safe distance S is to allow at least the length of a car between your vehicle and the vehicle ahead for every ten miles per hour of speed at which the vehicle is travelling.

$$s_i = S(v_i) = S_0 + T_r v_i \tag{11.1}$$

where S_0 is the effective length of a stopped vehicle (including additional distance in front), and T_r denotes a parameter (comparable to the reaction time). A similar approach was proposed by [20]. Both Pipes' and Forbes' theory were compared to field measurements. It was concluded that according to Pipes' theory, the minimum headways are slightly less at low and high velocities than observed in empirical data. However, considering the models' simplicity, agreement with real-life observations was amazing (cf. [43]).

Leutzbach [33] discusses a more refined model describing the spacing of constrained vehi-cles in the traffic flow. He states that the overall reaction time T_r consists of:

- perception time (time needed by the driver to recognise that there is an obstacle);
- decision time (time needed to make decision to decelerate), and;
- braking time (needed to apply the brakes).

In line with the terminology presented in chapter 10, the *perception-response time PRT* consists of te perception time and the decision time; the *control movement time* is in this case equal to the braking time.

The braking distance is defined by the distance needed by a vehicle to come to a full stop. It thus includes the reaction time of the driver, and the maximal deceleration. The latter is a function of the weight and the road surface friction μ, and eventual acceleration due to gravity g (driving up or down a hill). The total safety distance model assumes that drivers consider braking distances large enough to permit them to stop without causing a rear-end collision with the preceding vehicle if the latter come to a stop instantaneously. The safe distance headway equals

$$S(v_i) = S_0 + T_r' v_i + \frac{v_i^2}{2\mu g} \tag{11.2}$$

A similar model was proposed by [30]. Consider two successive vehicles with approximately equal braking distances. We assume that the spacing between the vehicles must suffice to avoid a collision when the first vehicle comes to a full stop (the so-called reaction time distance model). That is, if the first vehicle stops, the second vehicle only needs the distance it covers during the overall reaction time T_r' with unreduced speed, yielding Forbes' model. Jepsen [30] proposes that the gross-distance headway S effectively occupied by vehicle i driving with velocity v_i is a function of the vehicle's length L_i, a constant minimal distance between the vehicles d_{min}, the reaction time T_r' and a speed risk factor F

$$S(v_i) = (L_i + d_{\min}) + v_i \left(T_r' + v_i F\right) \tag{11.3}$$

Experienced drivers have a fairly precise knowledge of their reaction time T_r. For novice drivers, rules of thumb apply ("stay two seconds behind the vehicle ahead", "keep a distance of half your velocity to the vehicle ahead"). From field studies, it is found that the delay of an unexpected event to a remedial action (perception-response time + control movement time; see chapter 10) is in the order of 0.6 to 1.5 seconds. The speed-risk factor F stems from the observation that experienced drivers do not only aim to prevent rear-end collisions. Rather, they also aim to minimise the potential damage or injuries of a collision, and are aware that in this respect their velocity is an important factor. This is modelled by assuming that drivers increase their time headway by some factor – the speed-risk factor – linear to v. Finally, the minimal distance headway d_{min}describes the minimal amount of spacing between motionless vehicles, observed at jam density.

Note that this occupied space equals the gross distance headway only if the following vehicle is constrained. In the remainder of this thesis, this property is used. Otherwise, the car-following distance is larger than the safe distance needed. Dijker et al. [16] discuss some empirical findings on user-class specific car-following behaviour in congested traffic flow conditions.

11.3 Stimulus-response models

Stimulus response models (ofter referred to as *car-following models*) are dynamic models that describe the reaction of drivers as a function of the changes in distance, speeds, etc., relative to the vehicle in front. Generally speaking, these models are applicable to relatively busy traffic flows, where the overtaking possibilities are small and drivers are obliged to follow the vehicle in front of them.

One can considerer as an example the cars on the left (or fast) lane of a motorway. They do not want to maintain a distance headway that is so large, that it invites other drivers to enter it. At the same time, most drivers are inclined to keep a safe distance with respect to their leader. As a consequence, the drivers must find a compromise between safety and the encouragement of lane changes.

On a two-lane road (for two directions) drivers that can not make an overtaking, due to the presence of oncoming vehicles or a lack of sight distance, are obliged to follow the vehicle in front. In that case the intruding of other vehicles in the gap in front is less frequent but nevertheless it appears that drivers maintain a rather short distance headway. Also in dense traffic on urban roads drivers are often obliged to follow the vehicle in front.

As traffic grows faster than the road network is expanding, drivers will more and more be engaged in car-following. Especially as capacity is reached, nearly all driver-vehicle units are in a car-following state.

How does a driver carry out his car-following task? He/she must keep a sufficient large distance headway and respond, or at least be prepared to respond, to speed changes (especially speed reductions) of the vehicle in front. This boils down to the fact that a driver must maintain a certain minimum distance headway that will be dependent on the speed. In the sequel we will assume that this is the case and discuss the consequences for dynamic situations.

Which variable can a driver control? At first sight that is the speed, but at second thought he/she operates the gas pedal and the break pedal, in other words the acceleration is controlled.

Measured value	T_r (s)	κ $(1/s)$
minimum	1.00	0.17
average	1.55	0.37
maximum	2.20	0.74

Table 11.1: Parameter estimates for stimulus-response model

Measured value	T_r (s)	κ_0 $(1/s)$
minimum	1.5	40.3
average	1.4	26.8
maximum	1.2	29.8

Table 11.2: Parameter estimates for stimulus-response model

It is evident that drivers have a certain response time, they can not respond immediately but have to go through the cycle of 'observation, processing, deciding, action'. Taking this into account the following *linear car-following model* has been proposed:

$$a_i\left(t+T_r\right)=\gamma\left(x_{i-1}\left(t\right)-x_i\left(t\right)-s_0\right) \tag{11.4}$$

Eq. (11.4) is an example of a so-called stimulus-response model. These stimulus response models assume that drivers control their acceleration, given some response time T_r. This finite response time stems from the observation time (perception and information collection), processing time and determining the control action (decision making), and applying the action (operating the gas-pedal, braking, shifting). In general terms, the stimulus response model is given by the following equation

$$response\left(t+T_r\right)=sensitivity\times stimulus \tag{11.5}$$

The response typically equals the vehicle acceleration. Various definitions for the stimulus, other than the stimulus in Eq. (11.4), have been put forward in the past. As an example, a well known model is the model of [12], using the relative speed $v_{i-1}\left(t\right)-v_i\left(t\right)$ for the stimulus, i.e.

$$\frac{d}{dt}v_i\left(t+T_r\right)=\kappa\left(v_{i-1}\left(t\right)-v_i\left(t\right)\right) \tag{11.6}$$

In eqn. (11.6), κ denotes the sensitivity. Field experiments were conducted to quantify the parameter values for the reaction time T_r and the sensitivity κ. The experiment consisted of two vehicles with a cable on a pulley attached between them. The leading driver was instructed to follow a pre-specified speed pattern, of which the following driver was unaware. Tab. 11.1 shows the results of these experiments.

It turned out that the sensitivity depended mainly on the distances between the vehicles: when the vehicles were close together, the sensitivity was high. When the vehicles were far apart, the sensitivity was small. Hence, the following specification for the sensitivity κ was proposed

$$\kappa=\frac{\kappa_0}{x_{i-1}\left(t\right)-x_i\left(t\right)} \tag{11.7}$$

Tab. 11.2 depicts the resulting parameter estimates.For this simple model, the steady-state will occur when the speed of the preceding vehicle $i-1$ is constant, and the acceleration of i is zero.

11.3.1 Model stability

One of the main points in any control model is its stability, i.e. will small disturbances damp out or will they be amplified. Whether the control model is stable will depend on the charac-ter and the parameters of the control model (11.6). For car-following models, two types of stability can be distinguished:

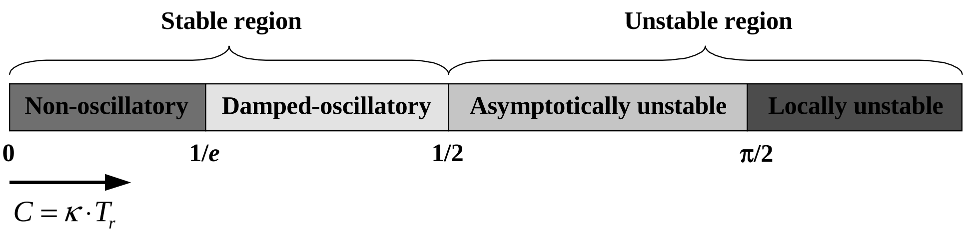

Figure 11.1: Relation between $C = \kappa T_r$ and stability

1. *Local stability*, concerning only the response of a driver on the leading vehicle $i-1$.
2. *Asymptotic stability*, concerning the propagation of a disturbance along a platoon of vehicles that are (car-) following each other.

The second type of stability is of much more practical importance than the local stability. If a platoon of vehicles in asymptotically unstable, a small disturbance of the first vehicle is amplified as it is passed over to the next vehicle, which in turn can lead to dangerous situations. Fig. 11.1 shows the relation between the local and asymptotic stability of the car-following model, and the parameter $C = \kappa T_r$ Let us briefly consider both.

Application of Laplace transforms to local stability analysis

There are various approaches to investigate stability. The most straightforward approach is to apply techniques from conventional system's control theory. The local and asymptotic stability of the model depends on the sensitivity κ and the reaction time T_r. It can be proven that the model is *locally stable* if

$$C = \kappa T_r \leq \frac{\pi}{2} \tag{11.8}$$

Proof. Let us consider the simple car-following model eqn. (11.6) and apply the Laplace transform to it. Recall that the Laplace transform is defined as follows

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{11.9}$$

Let us also recall the following important properties of the Laplace tranform

$$\mathcal{L}(f(t-t_0)) = e^{-t_0 s}\mathcal{L}(f(t)) \tag{11.10}$$

$$\mathcal{L}\left(\frac{d}{dt} f(t)\right) = s\mathcal{L}(f(t)) \tag{11.11}$$

Let $V_i(s) = \mathcal{L}(v_i(t))$. In applying the Laplace transformation to both the left-hand-side and the right-hand-side of Eq. (11.6), we get

$$se^{T_r s} V_i(s) = \kappa\left(V_{i-1}(s) - V_i(s)\right) \tag{11.12}$$

This expression enables us to rewrite the Laplace transform of the speed $V_i(s)$ of vehicle i as a function of the Laplace transform $V_{i-1}(s)$ of the leading vehicle i, i.e.

$$V_i(s) = H(s)\, V_{i-1}(s) = \frac{\kappa e^{-T_r s}}{s + \kappa e^{-T_r s}} V_i(s) \tag{11.13}$$

In control theory, the function $H(s) = \frac{\kappa e^{-T_r s}}{s+\kappa e^{-T_r s}}$ is often referred to as the unit-response (since it describes how the system would respond to a unitary signal ($V_{i-1}(s) = 1$)) or *transfer function.*

The transfer function describes how the output $V_i(s)$of the control system relates to the input function $V_{i-1}(s)$.

The stability of the control system can be studied by studying the properties of the transfer function $H(s)$. To ensure stability of the system the so-called *poles* of the transfer function $H(s)$ must be on the left hand side of the imaginary plane. The poles are defined by the points where the denominator $D(s)$ of $H(s)$ are equal to zero, i.e.

$$D(s) = s + \kappa e^{-T_r s} = 0 \tag{11.14}$$

To study stability, let $s = x + jy$, where $j^2 = -1$ defines the unit imaginary number. The denominator $D(s)$ then equals

$$D(s) = x + jy + \kappa e^{-T_r(x+jy)} = x + \kappa e^{-T_r} \cos(T_r y) + j\left(y - \kappa e^{-T_r x} \sin(T_r y)\right) \tag{11.15}$$

where we have used that $e^{-jT_r y} = \cos(T_r y) - j\sin(T_r y)$. To determine the boundary of local instability, consider the limiting case $x = 0$. Then, the necessary condition for the pole is given by the following expression

$$\kappa \cos(T_r y) + j(y - \kappa \sin(T_r y)) = 0 \tag{11.16}$$

Since both the real part of eqn. (11.16) and the imaginary part of eqn. (11.16) must be equal to zero, we have for the real part

$$\kappa \cos(T_r y) = 0 \to y_k = \frac{1}{T_r}\left(\frac{\pi}{2} + k\pi\right) \tag{11.17}$$

For the imaginary part, we then have

$$\begin{aligned} y_k - \kappa \sin(T_r y_k) &= \frac{1}{T_r}\left(\frac{\pi}{2} + k\pi\right) - \kappa\left(\frac{\pi}{2} + k\pi\right) \quad (11.18) \\ &= \frac{1}{T_r}\left(\frac{\pi}{2} + k\pi\right) - \kappa(-1)^k = 0 \quad (11.19) \end{aligned}$$

implying

$$\left(\frac{\pi}{2} + k\pi\right) = \kappa T_r (-1)^k \tag{11.20}$$

Consider $k = 0$. Eqn. (11.20) will only have a solution when

$$\kappa T_r = \frac{\pi}{2} \tag{11.21}$$

This implies that eqn. (11.21) defines the boundary between stable and unstable parameter values. It can also be shown that for

$$\kappa T_r > \frac{\pi}{2} \tag{11.22}$$

the model is both locally and (thus also) asymptotically unstable. ■

In illustration, Fig. 11.2 shows the behaviour of the car-following model in case of local unstable parameter settings. Clearly, the amplitude of the oscillations grows over time, eventu-ally leading to a collision of the lead car and the following car.

The local stability region is further divided into two regions: non-oscilatory ($C < 1/e$) and damped oscillatory ($C > 1/e$). In the latter case, the response of the following vehicle oscillates, but these oscillations damp out over time.

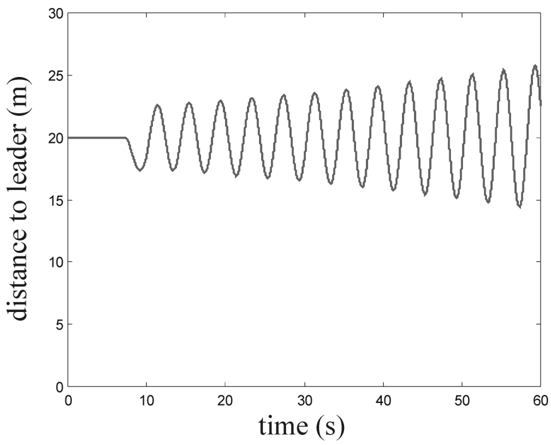

Figure 11.2: Gap between vehicle $i-1$ and vehicle i, assuming that leader $i-1$ brakes suddenly at $t = 10s$ for $T_r = 1s$ and $\kappa = 1.6s^{-1}$

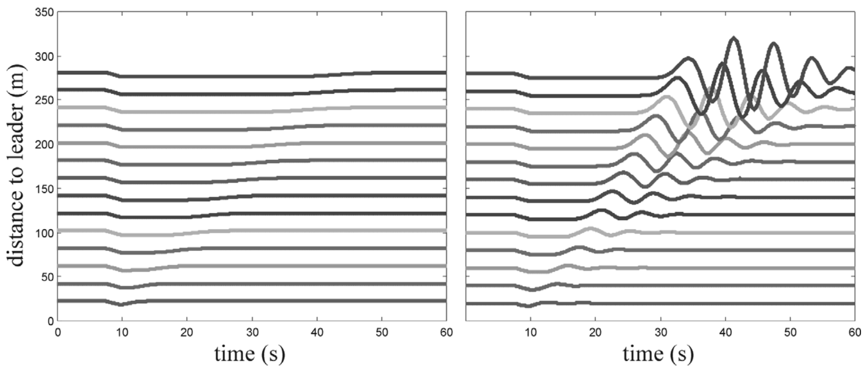

Figure 11.3: Position of vehicles with respect to platoon leader, when leader brakes suddenly at $t = 10s$ in case: $T_r = 1.00s$ and $\kappa = 0.4s^{-1}$ (left); $T_r = 1.00s$ and $\kappa = 0.7s^{-1}$ (right)

Asymptotic stability

A platoon of vehicles is asymptotically unstable if

$$C = \kappa T_r > \frac{1}{2} \tag{11.23}$$

Fig. 11.3 (left) shows a simulation example of a platoon of 15 vehicles, of which the leader brakes abruptly at $t = 10$ s. In this case, the parameters yield a locally and asymptotically stable system. Fig. 11.3 (right) shows the same case, but for parameter settings yielding a locally stable but asymptotically unstable system. See how the amplitude of the disturbance grows when it passes from one vehicle to the next.

Note that the local stability is less critical than the asymptotic stability. If the controller is locally stable, it can still be asymptotically unstable. Furthermore, observe that eqn. (11.23) shows that the car-following model becomes unstable for large response times as well as large sensitivity values. Reconsidering the empirical values of the response time T_r and the sensitivity κ shows that the average values satisfy the local stability condition. However, the conditions for asymptotic stability are not met. Moreover, the minimum values are neither asymptotically, nor locally stable.

Proof. The study of asymptotic stability of the car-following model is based on the analysis of periodic functions of the form $A_i e^{j\omega}$. In this respect, let us first note that we can approximate any periodic function $f(t)$ of time by a series of such signals, i.e. we can write $f(t) = \sum_k a_k e^{j\omega_k t}$.

This thus also holds for the behaviour of the leading vehicle $i-1$, i.e.

$$v_{i-1}(t) = \sum_k a_k^{(i-1)} e^{j\omega_k t} \tag{11.24}$$

(Fourier series). Here, $\omega_k = \frac{2\pi k}{T}$ denotes the frequency of the signal, T denotes the period of the function $f(t)$ (defined by $f(t) = f(t+T)$), and a_k denotes the amplitude of the contribution $e^{j\omega_k t}$. To study asymptotic stability, we consider one of these terms $a_k^{(i-1)} e^{j\omega_k t}$. The speed of the following $v_i(t)$ is the solution of a delayed differential equation (11.6). This solution can be written in the following form

$$v_i(t) = \sum_k a_k^{(i)} e^{j\omega_k t} \tag{11.25}$$

Let us now substitute If we now substitute Eq. (11.25) into Eq. (11.6):

$$\frac{d}{dt} v_i(t+T_r) = \kappa \left(v_{i-1}(t) - v_i(t) \right) \tag{11.26}$$

$$\Leftrightarrow$$

$$\sum_k j\omega_k a_k^{(i)} e^{j\omega_k (t+T_r)} = \kappa \left(\sum_k \left(a_k^{(i-1)} - a_k^{(i)} \right) e^{j\omega_k t} \right) \tag{11.27}$$

If this equation holds for all t, then Eq. (11.26) implicates

$$j\omega_k a_k^{(i)} e^{j\omega_k (t+T_r)} = \kappa \left(\left(a_k^{(i-1)} - a_k^{(i)} \right) e^{j\omega_k t} \right) \text{ for all } k \geq 0 \tag{11.28}$$

Let us now determine what happens to the amplitude $a_k^{(i)}$ as it passes from vehicle $i-1$ to vehicle i. By rewriting Eq. (11.28), we can easily show that

$$\frac{a_k^{(i)}}{a_k^{(i-1)}} = \frac{\kappa}{j\omega_k e^{j\omega_k T_r} + \kappa} \tag{11.29}$$

To ensure asymptotic stability, it is required that the amplitude of the speed profile *decreases* as it passes from vehicle $i-1$ to vehicle i (and from vehicle i to vehicle $i+1$, etc.), implying that

$$\left| \frac{a_k^{(i)}}{a_k^{(i-1)}} \right| = \left| \frac{\kappa}{j\omega_k e^{j\omega_k T_r} + \kappa} \right| < 1 \Leftrightarrow \kappa < \left| j\omega_k e^{j\omega_k T_r} + \kappa \right| \tag{11.30}$$

Rewriting

$$j\omega_k e^{j\omega_k T_r} + \kappa = j\omega_k \left(\cos(\omega_k T_r) + j \sin(\omega_k T_r) \right) \tag{11.31}$$

$$= j\omega_k \cos(\omega_k T_r) - \omega_k \sin(\omega_k T_r) \tag{11.32}$$

and substituting the result in eqn. (11.30), we get

$$\left| j\omega_k e^{j\omega_k T_r} + \kappa \right| = \left| j\omega_k \cos(\omega_k T_r) - \omega_k \sin(\omega_k T_r) \right| \tag{11.33}$$

$$= \sqrt{\left(\kappa - \omega_k \sin(\omega_k T_r) \right)^2 + \omega^2 \cos^2(\omega_k T_r)} \tag{11.34}$$

$$= \sqrt{\kappa^2 + \omega_k^2 - 2\kappa\omega_k \sin(\omega_k T_r)} \tag{11.35}$$

We can show that this the condition becomes critical for very small ω_k. Hence, we may use the approximation

$$\sin(\omega_k T_r) \approx \omega_k T_r \tag{11.36}$$

yielding

$$\omega_k^2 > 2\kappa\omega_k^2 T_r \to \kappa T_r < \frac{1}{2} \tag{11.37}$$

Situation	Reponse time (s/100)
Cue free, i.e. left to driver	50
Driver has to concentrate on brake lights vehicle in front	63
Cue free; first car is only one with functioning brake lights	101

Table 11.3: Effect of 'cue' (stimulus) on response timel

Asymptotic stability thus requires $C = \kappa T_r < \frac{1}{2}$. ■

On the test track of General Motors around 1960 much research has been carried out concerning the car-following model has been investigated. It was found that the response time T_r varied between 1.0 to 2.2 s and the sensitivity κ_0 from 0.2 to 0.8 s^{-1}. It appeared that often platoons were nearly asymptotic unstable. Two examples of experiments:

- Effect on response time

The situation concerns a platoon of 11 cars that follow each other closely. The first car brakes strongly at an arbitrary moment.

The conclusion is that a driver uses not only the brake lights to anticipate a needed speed reduction ($0.63 > 0.50$) but that brake lights as such are useful ($1.01 > 0.50$).

- Effect of extra information at rear side of car

The brake lights were modified as follows:

- acceleration $\geq 0 \rightarrow$ blue;
- no break and no gas (coasting) $\rightarrow$ yellow;
- braking $\rightarrow$ red.

The effect of this modification, that offers more information, was: response time smaller; sensitivity larger; product of both hardly changed, which means the same stability; a smaller distance headway at the same speed ($\rightarrow$ smaller time headway too). An interpretation of this is a gain in capacity but not in safety.

This simple model has several undesirable and unrealistic properties. For one, vehicles tend to get dragged along when the vehicle in front is moving at a higher speed. Furthermore, when the distance $s_i(t)$ is very large, the speeds can become unrealistically high. To remedy this deficiency, the sensitivity κ can be defined as a decreasing function of the distance. In more general terms, the sensitivity can be defined by the following relation

$$\kappa = \kappa_0 \frac{v_i\left(t + T_r\right)^m}{\left[x_{i-1}\left(t\right) - x_i\left(t\right)\right]^l} \tag{11.38}$$

where κ_0 and T_r denote positive constants. Equation (11.38) implies that, the following vehicle adjusts its velocity $v_i(t)$ proportionally to both distances and speed differences with delay T_r. The extent to which this occurs depends on the values of κ_0, l and m. In combining eqns. (11.6) and (11.38), and integrating the result, relations between the velocity $v_i\left(t + T_r\right)$ and the distance headway $x_{i-1}(t) - x_i(t)$ can be determined. Assuming stationary traffic conditions, the following relation between the equilibrium velocity $u\left(k\right)$ and the density k results

$$u\left(k\right) = u_f \left(1 - \left(\frac{k}{k_j}\right)^{l-1}\right)^{\frac{1}{1-m}} \tag{11.39}$$

for $m \neq 1$ and $l \neq 1$.

Proof. In combining eqns. (11.6) and (11.38), we get

$$\frac{d}{dt} v_i(t+T_r) = \kappa_0 \frac{v_i(t+T_r)^m}{[x_{i-1}(t) - x_i(t)]^l} (v_{i-1}(t) - v_i(t)) \tag{11.40}$$

which can be, by re-arranging the different terms, and noticing that $v_i(t) = \frac{d}{dt} x_i(t)$, written as follows

$$\frac{\frac{d}{dt} v_i(t+T_r)}{v_i(t+T_r)^m} = \kappa_0 \frac{\frac{d}{dt}(x_{i-1}(t) - x_i(t))}{[x_{i-1}(t) - x_i(t)]^l} \tag{11.41}$$

Assuming $m \neq 1$ and $l \neq 1$, we have

$$\frac{1}{m-1} \frac{d}{dt} \left(\frac{1}{v_i(t+T_r)^{m-1}} \right) = \frac{\frac{d}{dt} v_i(t+T_r)}{v_i(t+T_r)^m} \tag{11.42}$$

and

$$\frac{\kappa_0}{l-1} \frac{d}{dt} \left(\frac{1}{[x_{i-1}(t) - x_i(t)]^{l-1}} \right) = \kappa_0 \frac{\frac{d}{dt}(x_{i-1}(t) - x_i(t))}{[x_{i-1}(t) - x_i(t)]^l} \tag{11.43}$$

yielding

$$v_i(t+T_r)^{1-m} = C + \kappa_0 \frac{1-m}{1-l} [x_{i-1}(t) - x_i(t)]^{1-l} \tag{11.44}$$

i.e.

$$v_i(t+T_r) = \left(C + \kappa_0 \frac{1-m}{1-l} [x_{i-1}(t) - x_i(t)]^{1-l} \right)^{\frac{1}{1-m}} \tag{11.45}$$

where C is an integration constant. Under stationary conditions, both $s_i(t) = x_{i-1}(t) - x_i(t)$ and $v_i(t)$ will be time-independent. Furthermore, under stationary conditions, the speeds and the distances headways of all vehicles i must be equal to u and $s = 1/k$. We then have

$$u = u(k) = \left(C + \kappa_0 \frac{1-m}{1-l} k^{l-1} \right)^{\frac{1}{1-m}} \tag{11.46}$$

For $k = 0$, we have $u(0) = u_f$ (mean free speed), and thus

$$C^{m-1} = u_f \tag{11.47}$$

For $k = k_j$, we have $u(k_j) = 0$ (speed equals zero under jam-density conditions), and thus

$$\frac{\kappa_0}{C} \frac{m-1}{l-1} k_j^{l-1} = -1 \rightarrow k_j = \left(\frac{C}{\kappa_0} \frac{l-1}{1-m} \right)^{\frac{1}{l-1}} \tag{11.48}$$

In sum, we have determined the following relation between the mean speed u and the density

$$u(k) = u_0 \left(1 - \left(\frac{k}{k_j} \right)^{l-1} \right)^{\frac{1}{1-m}} \tag{11.49}$$

■

Car-following models have been mainly applied to single lane traffic (e.g. tunnels, cf. [40]) and traffic stability analysis ([25], [36]; chapter 6). That is, using car-following models the limits of local and asymptotic stability of the stream can be analysed.

The discovery of the possibility of asymptotic instability in traffic streams was considered to explain the occurring of congestion without a clear cause, 'Stau aus dem Nichts' (*phantom jams*); Fig. 11.4 (from [53]) shows a famous measurement from those years in which the first generation of car-following models was developed. The vehicle trajectories have been determined from aerial photos, taken from a helicopter, and exhibit a shock wave for which there seems to be no reason. It can also be noted that the shock wave more or less fades out, al least becomes less severe at the right hand side of the plot. From the plot can be deduced a shock wave speed of approximately -20 km/h.

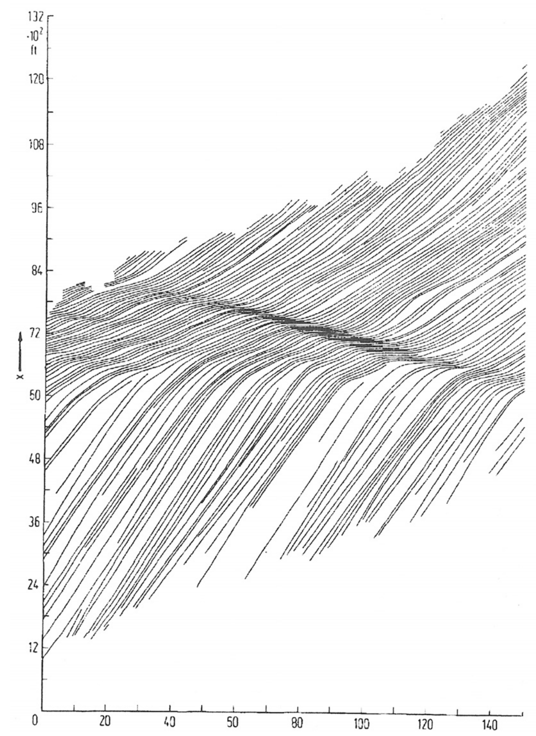

Figure 11.4: Traffic stream with shock wave at lane of motorway

11.4 Psycho-spacing models / action point models

The car-following models discussed so far have a rather mechanistic character. The only human element in the models so far is the presence of a finite response time, but for the rest a more or less perfect driver is assumed. In reality a driver is not able to:

1. observe a stimulus lower than a given values (perception threshold);
2. evaluate a situation and determine the required response precisely, for instance due to observation errors resulting from radial motion observation;
3. manipulate the gas and brake pedal precisely;

and does not want to be permanently occupied with the car-following task.

This type of considerations has inspired a different class of car-following models, see e.g. [33]. In these models the car-following behaviour is described in a plane with relative speed and headway distance as axis. The model is illustrated with Fig. 11.5. It is assumed that the vehicle in front has a constant speed and that the potential car-following driver catches up with a constant relative speed v_r'. As long as the headway distance is larger than s_g, there is no response.

Moreover, if the absolute value of the relative speed is smaller than a boundary value v_{rg}, then there is also no response because the driver can not perceive the relative speed. The boundary value is not a constant but depends on the relative speed. If the vehicle crosses the boundary, it responds with a constant positive or negative acceleration. This happens in Fig. 11.5 first at point A, then at point C, then point B, etc.

Leutzbach [33] has introduced the term 'pendeling' (the pendulum of a clock) for the fact that the distance headway varies around a constant value, even if the vehicle in front has a constant speed. In this action-point model the size of the acceleration is arbitrary in the first instance, whereas it was the main point of the earlier discussed car-following models.

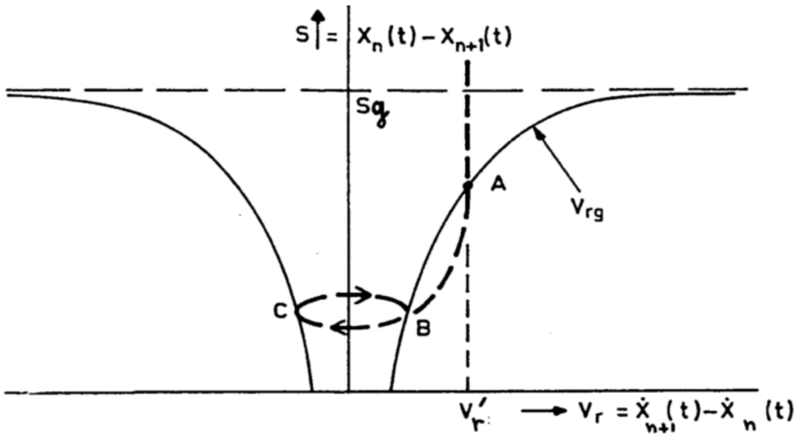

Figure 11.5: Basic action-point car-folllowing model

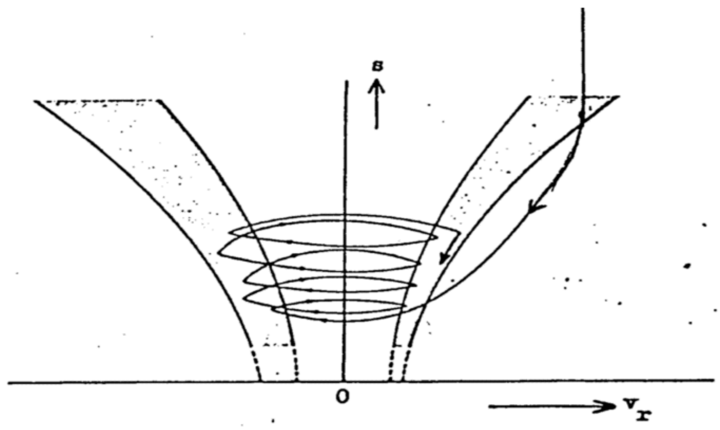

Figure 11.6: Action-point model with variable perception thresholds and a response time

The first version of the basic action point model has been extended with the following

1. a separate threshold for catching up (= approaching from a large distance);
2. different perception thresholds for negative and positive v_r;
3. a zone in stead of a function for the perception threshold;
4. an extra response if the vehicle in front shows its brake lights;

These extensions are illustrated in Fig. 11.6, and Fig. 11.7 where the following points are indicated:

- d_1: distance at speed 0
- d_2: minimum desired distance at small v_r
- d_3: maximum s at pendeling
- d_4: threshold if catching up
- d_5: threshold at pendeling

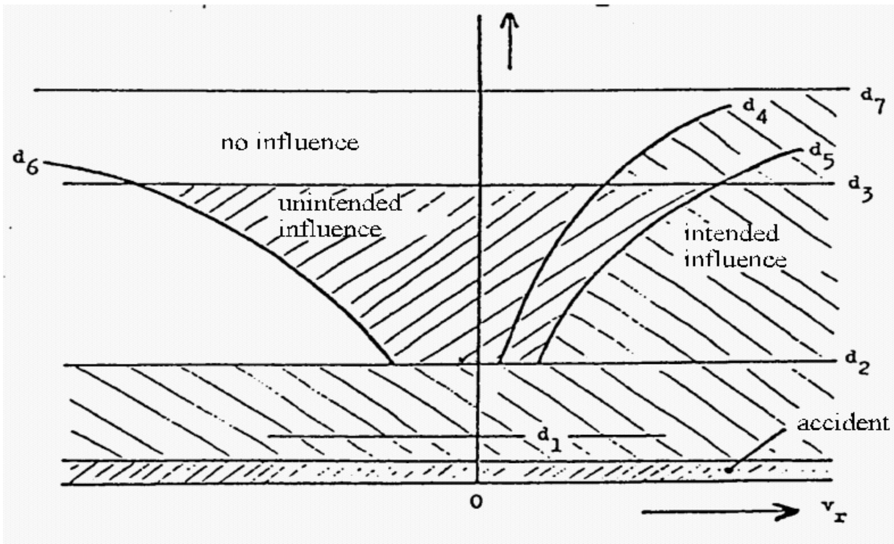

Figure 11.7: Thresholds in the $v_r - s$ plane used in a simulation model

- d_6: threshold at pendeling if $v_r < 0$
- d_7: no response at al, if $v_r > 0$.

The action point models form the basis for a large number of contemporary microscopic traffic flow models, such as FOSIM, AIMSUN2 and VISSIM.

11.5 Optimal control models

The conceptual model discussed in chapter 10 can also be applied to establish a mathematical model. Since we are only considering the longitudinal driving task, the state $\mathbf{x}(t)$ of the system can be described by the locations $r_i(t)^2$ and the speeds $v_i(t)$ of the drivers i,i.e.

$$\mathbf{x} = (r_1, ..., r_n, v_1, ..., v_n) \tag{11.50}$$

where n is the number of vehicles in the traffic system. Let us consider driver i. The prediction model used by i to predict the conditions of the traffic system are very simple

$$\frac{d}{dt}r_j = v_j \quad \text{and} \quad \frac{d}{dt}v_j = a_j \quad \text{for} \quad j = 1, ..., n \tag{11.51}$$

where a_j denotes the acceleration of vehicle j. In fact, the accelerations can be considered the controls u of the traffic system (i.e. accelerating and braking). Note that driver i can only directly influence the acceleration a_i, i.e. $u = a_i$; in principle, he/she can *indirectly* influence the control behaviour of the other drivers $j \neq i$. For the sake of simplicity, we assume that driver i predicts that the other drivers will not accelerate/decelerate during the considered time period, i.e. $a_j = 0$ for $j \neq i$.

Having specified the prediction model Eq. (10.5), we need to specifying the driving cost function $J^{(i)}$ incurred by driver i. We assume that the driving cost are of the following form

$$J^{(i)}\left(u_{[t,\infty)}|t, \hat{x}(t)\right) = \int_t^\infty e^{-\eta s} L(\mathbf{x}(s), u(s))\, ds \tag{11.52}$$

subject to the prediction model

$$\dot{\mathbf{x}}(s) = \mathbf{f}(\mathbf{x}(s), u(s)) \quad \text{for } s > t \quad \text{with} \quad \mathbf{x}(t) = \hat{\mathbf{x}}(t) \tag{11.53}$$

[2] We have used the notation $r_i(t)$ to describe the location of driver i to avoid confusion with the state $\mathbf{x}(t)$.

where $\eta > 0$ is the so-called *temporal discount factor*, describing that drivers will discount the running cost $L(x, u)$ over time (i.e. the cost incurred in the near future are more important than the cost incurred in the far future).

The running cost reflect the cost that driver i will incurr during a very short interval $[t, t+dt)$. The total cost $J^{(i)}$ are thus determined by integrating the running cost L with respect to time. In this section, we will assume that the running cost will consist of three different factors, namely

1. running cost incurred due to not driver at the free speed
2. running cost incurred when driving too close to the vehicle $i-1$ directly in front
3. running cost incurred due to accelerating/braking

Let $v_f^{(i)}$ denote the free speed of driver i. The running cost due to not driving at the free speed is modelled as follows

$$\frac{c_1}{2}\left(v_f^{(i)} - v\right)^2 \tag{11.54}$$

where $c_1 > 0$ is a parameter expressing the relative importance of this cost component.

Similarly, the running cost incurred due to driving too close to the vehicle $i-1$ directly in front (so-called proximity cost) equals

$$c_2\Phi\left(r_{i-1} - r_i, v_{i-1} - v_i\right) = c_2 e^{-(r_{i-1}-r_i)/S_0} \tag{11.55}$$

where $c_2 > 0$ again denotes the relative importance of this cost component, and where S_0 is some scaling parameter. Eq. (11.55) shows that as the distance between vehicle $i-1$ and vehicle i increases, the running cost decreases and vice versa.

Finally, we propose using the following expression to describe the running cost incurred due to acceleration/deceleration

$$\frac{c_3}{2}u^2 \tag{11.56}$$

where $c_3 > 0$ is the relative importance of this cost component.

Now, all the components are in place to derive the model. To mathematicall derive the model, we define the so-called *Hamilton function* H as follows

$$H = e^{-\eta t}L + \boldsymbol{\lambda}'\mathbf{f} \tag{11.57}$$

The vector $\boldsymbol{\lambda}$ denotes the so-called shadow-costs. The shadow cost describe the marginal changes in the total cost $J^{(i)}$ due to small changes in the state $\mathbf{x}$. We can show that for the control u to be optimal, it has to satisfy the so-called optimality condition

$$\frac{\partial H}{\partial u} = 0 \rightarrow u^* = -\frac{1}{c_3}e^{\eta t}\lambda_{v_1}(t, \mathbf{x}) \tag{11.58}$$

where $\lambda_{v_i}(t, \mathbf{x})$ denotes the marginal cost of the speed v_i of driver i. Eq. (11.58) shows that when the marginal cost $\lambda_{v_i}(t, \mathbf{x})$ of the speed v_i is positive, the control is negative, meaning that the driver will decelerate ($u^* < 0$). On the contrary, when the marginal cost of the speed is negative, it makes sense to further increase the speed and thus to acceletate ($u^* > 0$). When the marginal cost $\lambda_{v_i}(t, \mathbf{x}) = 0$, driver i will keep driving at constant speed.

The remaining problem is to determine the shadow cost. From optimal control theory, it is well known that the marginal cost satisfy

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{11.59}$$

The fact that we are considering a time-independent discounted cost problem implies furhtermore that $\dot{\boldsymbol{\lambda}} = -\eta\boldsymbol{\lambda}$. Let us first consider the marginal cost of the location r_i. We can easily determine that

$$\eta\lambda_{r_i} = \frac{\partial H}{\partial r_i} = e^{-\eta t}\frac{\partial L}{\partial r_i} = e^{-\eta t}\frac{\partial \Phi}{\partial r_i} \tag{11.60}$$

This equation shows that the marginal cost of the location of driver i only depends on the proximity cost function. Note that generally, $\frac{\partial \Phi}{\partial r_i} > 0$, that is, the proximity cost will increase with increasing r_i. For $\Phi = c_2 e^{-(r_{i-1}-r_i)/S_0}$, we would have

$$\frac{\partial \Phi}{\partial r_i} = \frac{c_2}{S_0} e^{-(r_{i-1}-r_i)/S_0} \tag{11.61}$$

We can then easily determine the marginal cost of the speed v_i of driver i. We find

$$\begin{aligned}
\eta\lambda_{v_i} &= \frac{\partial H}{\partial v_i} = e^{-\eta t}\frac{\partial L}{\partial v_i} + \lambda_{r_i} && (11.62)\\
&= e^{-\eta t}\left(c_1\left(v_f^{(i)} - v_i\right) + \frac{\partial \Phi}{\partial v_i}\right) + \lambda_{r_i} && (11.63)\\
&= e^{-\eta t}\left(c_1\left(v_f^{(i)} - v_i\right) + \frac{\partial \Phi}{\partial v_i} + \frac{1}{\eta}\frac{\partial \Phi}{\partial r_i}\right) && (11.64)
\end{aligned}$$

This equation shows how the marginal cost of the speed v_i of driver i depends on the difference in the free speed of driver i, the function Φ (proximity cost) and the marginal cost of the position of driver i.

We can now very easily determine that for the optimal acceleraton, the following equation holds

$$\begin{aligned}
\frac{d}{dt}v_i &= u^* = \frac{v_f^{(i)} - v_i}{\tau} - A_0\left(\frac{\partial \Phi}{\partial r_i} + \eta\frac{\partial \Phi}{\partial v_i}\right) && (11.65)\\
&= \frac{v_f^{(i)} - v_i}{\tau} - A_0 e^{-(r_{i-1}-r_i)/S_0} && (11.66)
\end{aligned}$$

where the acceleration time τ and the interaction factor A_0 are respectively defined by

$$\frac{1}{\tau} = \frac{c_1}{\eta c_3} \text{ and } A_0 = \frac{c_2}{\eta^2 c_3} \tag{11.67}$$

Eq. (11.65) shows how the acceleration of driver i is determined by the driver's aim to driver at the free speed $v_f^{(i)}$, and a factor that describes effect of driving behind another vehicle at a certain distance $s_i = r_{i-1} - r_i$. In appendix E we present the NOMAD pedestrian flow model, which has been derived using the optimal control paradigm discussed in this chapter.

Let us consider the stationary case, where the speed $v_i(t)$ is time-independent. Stationarity implies that the acceleration $\frac{d}{dt}v_i = 0$. This implies

$$v_i = v_f^{(i)} - \tau A_0 e^{-s_i/S_0} \tag{11.68}$$

This equation shows the relation between the stationary speed v_i, the free speed $v_f^{(i)}$ and the distance s_i between vehicle $i-1$ and vehicle i.

The following model was developed by [2]

$$\frac{d}{dt}v_i(t) = \kappa_0\left[V'(s_i(t)) - v_i(t)\right] \tag{11.69}$$

where κ_0 describes the sensitivity of the driver reaction to the stimuli, in this case the difference between the speed $V(s_i(t))$ - a function of the distance between vehicle i and its leader $i-1$ -

and the current speed $v_i(t)$. This model is equal to the optimal control model describes by Eq. (11.65), assuming that

$$V'(s_i(t)) = v_f^{(i)} - A_0\tau\left(\frac{\partial}{\partial r_i} + \eta\frac{\partial}{\partial v_i}\right)\Phi \tag{11.70}$$

and $\kappa_0 = 1/\tau$. In [2], the following relation is used

$$V'(s) = \tanh(s-2) + \tanh 2 \tag{11.71}$$

For this particular choice, it turns out that traffic flow becomes unstable at critical densities and that small distrurbances will cause stop-and-waves. For small *and* large vehicle densities, the flow is however stable. More precisely, the traffic flow is unstable when

$$\frac{d}{ds}V' > \frac{\kappa_0}{2} \tag{11.72}$$

Proof. To determine the instability criterion (11.72), we first linearise Eq. (11.69) around the spatially homogeneous solution

$$x_i^e(t) = x_0 - is + V'(s)\,t \tag{11.73}$$

For the variable

$$\delta x_i(t) = x_i(t) - x_i^e(t) \tag{11.74}$$

with $|\delta x_i(t)| \ll s$, we can then derive the following dynamic equation

$$\frac{d^2(\delta x_i(t))}{dt^2} = \kappa_0\left\{\frac{dV'}{ds}\left[\delta x_{i-1}(t) - \delta x_i(t)\right] - \frac{d}{dt}(\delta x_i(t))\right\} \tag{11.75}$$

$\delta x_i(t)$ can then be written as a Fourier-series

$$\delta x_i(t) = \frac{1}{N}\sum_{k=1}^{N} c_{k-1}\exp\left(2\pi j\frac{i-1}{N}(k-1) + (\lambda - j\omega)\,t\right) \tag{11.76}$$

again with $j^2 = -1$, which can in turn be substituted into the dynamic equation, yielding

$$(\lambda - j\omega)^2 + \kappa_0(\lambda - j\omega) - \kappa_0\frac{dV'}{ds}\left[\exp\left(-2\pi j\frac{k-1}{N}\right)\right] = 0 \tag{11.77}$$

It can subsequently be shown that when we determine solutions for $k = 1, 2, ..., N$, these solutions $\tilde{\lambda}(k) = \lambda(k) - j\omega(k)$ will have a positive damping rate $\lambda(k) > 0$ when Eq. (11.72) holds. Hence, the solutions are not stable. ■

11.6 Fuzzy Logic Models

Some researchers recognized that the reactions of the following vehicle to the lead vehicle might be based on a set of approximate driving rules developed through experience. Their approach to modeling these rules consisted of a fuzzy inference system with membership sets that could be used to describe and quantify the behavior of following vehicles. However, the logic to define the membership sets is subjective and depends totally on the judgment and approximation of the researchers. Furthermore, no field experiments were conducted to calibrate and validate these fuzzy membership sets under real driving conditions. While we agree with the premise of the paper, and seek to resolve similar issues, the methodology employed does not warrant any further explanation. Some researchers may argue over semantics, but there are no quantitative problems in fuzzy logic that cannot be solved in an equivalent manner using classical methods.

11.7 Cellular automata models

The CA models (*Cellular Automata*) divide the roadways into small cells. For instance, in the model of [38], these cells have a constant length of $\Delta x = 7.5$m. The cells can either be occupied by one vehicle or not. Besides the location of the vehicles, also their speeds v is discretised, and can only attain discrete values

$$v = \hat{v}\frac{\Delta x}{\Delta t} \quad \text{with} \quad \hat{v} = 0, 1, ..., \hat{v}_{\max} \tag{11.78}$$

With respect to the discretisation in time, the time-step is chosen such that a vehicle with speed $\hat{v} = 1$ precisely moves 1 cell ahead during one time step. Thus, in case of a time-step $\Delta t = 1$s, a maximum speed of 135 km/h holds. Despite this rather coarse represation, the CA-model describes the dynamics of traffic flow fairly well.

The updating of the vehicle dynamics is achieved using the following rules

1. *Acceleration.* If a vehicle has not yet reached its maximum speed $\hat{v}_{\max}$, and if the leading vehicle is more that $\hat{v} + 1$ cells away, then the speed of the vehicle is raised by one, i.e. $\hat{v} := \hat{v} + 1$.

2. *Braking.* If the vehicle driving with speed $\hat{v}$ has a distance headway Δj with $\Delta j \leq \hat{v}$, then the speed of the vehicle is reduced to $(\Delta j - 1)$. As a result, the minimum safe distance of $(\hat{v} + 1)\,\Delta x$ is maintained.

3. *Randomisation.* With a probability $\hat{p}$ will the speed be reduced with 1, i.e. $\hat{v} := \hat{v} - 1$. This describes the fact that the vehicle is not able to perfectly follow its predecessor.

4. *Convection.* The vehicle will move ahead with $\hat{v}$ cells during a single time step.

The updating of the vehicles can be achieved in a number of ways, e.g. in a random order, in the direction of travel, or just in the opposite direction. For the particular model here, the updating order does not affect the model behaviour.

Due to the simplicity of the model, complex networks with a large number of vehicles can be simulated in real-time. By varying the parameters $\hat{v}_{\max}$ and $\hat{p}$, different fundamental diagrams can be established, approximating real-life traffic flow at different levels of accuracy. The CA model describes the spontaneous formation of traffic congestion (unstable traffic conditions) and stop-and-go waves.

A large number of modifications to the basic CA model of [38] have been established, e.g. including lane changing. Also analytical results were obtained for simplified model dynamics.

11.7.1 Deterministic CA model

Let us briefly consider some of the properties of the CA model by excluding the randomisation rule. In that situation, all initial conditions eventually lead to one of the following two regimes (depending on the overall density of the system):

1. *Free-flow traffic.* All vehicles move with speed $v_{\max}$ and the gap between the vehicles is either $v_{\max}$ or larger. As a result, the flow rate in this regime equals

$$q = kv_{\max} \tag{11.79}$$

2. *Congested traffic.* If the density k is larger than the critical density k_c, not all vehicles will be able to move at maximum speed. In that case, the average speed of the drivers equals

$$u = \frac{1}{k} - 1 \tag{11.80}$$

and the average flow thus equals

$$q = 1 - k \tag{11.81}$$

The two regimes meet at the critical density k_c, which equals $k_c = \frac{1}{k_j+1}$. The capacity equals $q_c = \frac{v_{\max}}{v_{\max}+1}$.