Part II

Traffic Flow Characteristics

Chapter 2

Microscopic and macroscopic traffic flow variables

Contents of this chapter. Introduction of vehicle trajectories, time headways (h), distance headways (s), intensity (q), density (k), mean speed (u), local and instantaneous mean speed, harmonic mean speed, relation $q = ku$. Formal definition of stationary and homogeneous states of a traffic flow; definition of q, k and u as a continuous function of position and time; definition of q, k, and u for a surface in the space-time plane; measuring methods, occupancy rate, moving observer method.

List of symbols

x, x_0	m	cross-section, location
t, t_0	s	(initial) time instants
v_i	m/s	speed of vehicle i
a_i	m/s^2	acceleration of vehicle i
γ_i	m/s^3	jerk of vehicle i
h_i	s	headway of vehicle i
s_i	m	distance headway of vehicle i
n	veh	number of vehicles passing cross-section x
T	s	length of time period
q	veh/s	flow rate, intensity, volume
m	veh	number of vehicles on roadway section at instant t
X	m	length of roadway section
k	veh/m	traffic density
u_L	m/s	local speed
u_M	m/s	instantaneous speed
$f_L(v)$, $f_M(v)$	$-$	local / instantaneous speed probability density function
σ_L^2	m^2/s^2	variance local speeds
σ_M^2	m^2/s^2	variance instantaneous speeds
$N(x,t)$	veh	cumulative vehicle count at cross-section x and instant t
β	$-$	occupancy rate
n_{active}	veh	number of active vehicle passings
$n_{passive}$	veh	number of passive vehicle passings

2.1 Introduction

In general, a traffic network consists of intersections and arterials. On arterials of sufficient length the traffic will no longer be influenced by the intersections, and drivers are mainly con-

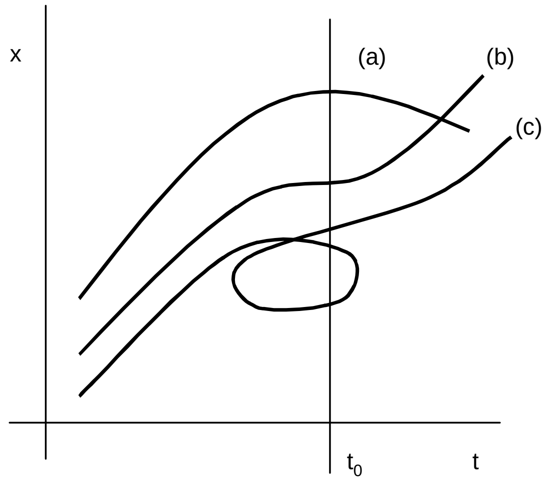

Figure 2.1: Time-space curves: (a) and (b) are vehicle trajectories; (c) is not.

cerned about traffic on the same roadway, either driving in the same or the opposing direction. In these lecture notes it is mainly this situation that is being discussed. The main microscopic variables are *trajectories*, *time headways*, and *distance headways.* The main macroscopic characteristics of a traffic flow are *intensity*, *density*, and *speed.* These variables and some related ones will be discussed in this chapter.

Vehicle trajectories

Very often in the analysis of a particular transportation operation one has to track the position of a vehicle over time along a 1-dimensional guideway as a function of time, and summarize the relevant information in an understandable way. This can be done by means of mathematics if one uses a variable x to denote the distance along the guideway from some arbitrary reference point, and another variable t to denote the time elapsed from an arbitrary instant. Then, the desired information can be provided by a function $x(t)$ that returns an x for every t.

2.1.1 Trajectory of a single vehicle

Definition 2 *A graphical representation of $x(t)$ in the (t, x) plane is a curve which we call a* trajectory*.*

As illustrated by two of the curves Fig. 2.1 (adapted from [14]), trajectories provide an intuitive, clear and complete summary of vehicular motion in one dimension. Curve (a), for example, represents a vehicle that is proceeding in the positive direction, slows down, and finally reverses direction. Curve (b) represents a vehicle that resumes travelling in the positive direction after stopping for some period of time. Curve (c) however is not a representation of a trajectory because there is more than one position given for some t's (e.g. t_0). Valid vehicle trajectories must exhibit one and only one x for every t.

Vehicle trajectories or rather, a *set of trajectories*, provide nearly all information concerning the conditions on the facility. As we will see in the ensuing of this chapter, showing multiple trajectories in the (t, x) plane can help solve many problems.

The definition of a trajectory is not complete in the sense that it does not specify which part of the vehicle the position of the vehicle refers to. In fact, none of the traffic flow theory handbooks explicitly specifies whether we consider the front bumper, the rear bumper or the centre of the vehicle. In the remainder of this reader, we will use the *rear bumper of a vehicle* as the reference point, unless explicitly indicated otherwise.

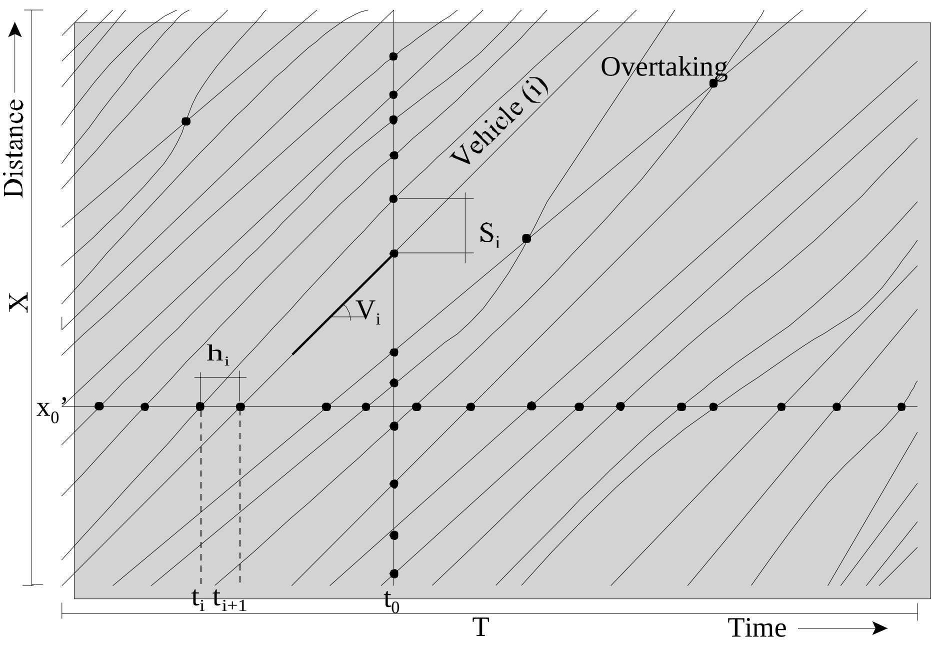

Figure 2.2: Vehicle trajectories.

A second issue is the fact that only a one-dimensional case is considered here. In fact, the position of a vehicle (or a pedestrian, a cyclist) consists of three dimensions x, y, and z.

For vehicular traffic (e.g. cars on a motorway or a bidirectional roadway), the coordinates generally do not refer to real-life coordinates, but are taken *relative to the roadway*, i.e. including the curvature of the latter. That is, x describes the longitudinal position with respect to the roadway, generally in the direction of the traffic. The y dimension, which is only seldom known/shown, describes the lateral position of the vehicle with respect to the roadway. This information thus includes the lane the vehicle is driving on. In fact, for any traffic system where the infrastructure largely determines the main direction of travel, the x and y direction respectively describe the longitudinal and lateral position of the vehicles along the roadway.

For traffic systems where this is not the case – consider for instance pedestrian walking infrastructure – the x and y (and sometimes the z coordinate as well) are given in Cartesian coordinates relative to some reference point $x = y = 0$. In those cases, the definition of x and y directions is more or less arbitrary.

Remark 3 *In traffic flow theory it is customary to show the position on the vertical axis and the time on the horizontal axis. However, in public transport, time is usually displayed vertically (increasing downwards) and position horizontally.*

See Fig. 2.2 for a set of vehicle trajectories for one-way traffic, i.e. the longitudinal position of vehicles along the roadway[1]. All information that the traffic analyst requires can be determined from the trajectories: individual speeds and acceleration, overtakings – where trajectories cross – but also macroscopic flow characteristics, such as densities, intensities, etc.

The speed of a vehicle is the tangent in a point of the trajectory; $v_i = dx_i/dt$; the acceleration of a vehicle is defined by $a_i = d^2x_i/dt^2$. Although these relations are well known, it is important to emphasize that steeply increasing (decreasing) sections of $x_i\left(t\right)$ denote a rapidly advancing (receding) vehicle; horizontal positions of $x_i\left(t\right)$ denote a stopped vehicle and shallow segments a slow moving vehicle. Straight line segments depict constant speed motion (with no acceleration)

[1]In the remainder of this reader, we will generally only describe the one-dimensional case (unless explicilty indicated). However, the discussed notions are in most cases easily extended to two or three dimensions.

and curving sections denote accelerated motion; here, the higher the curvature, the higher the absolute value of the acceleration. Concave downwards curves denote deceleration and concave upward (convex) curves denote accelerated motion.

2.1.2 Trajectories of multiple vehicles

On the spot $x = x_0$, i.e. a cross-section, one can observe the time instants that vehicles pass. The differences between successive moments are 'time headways' (h_i), and the speeds at a cross-section are 'local' speeds (v_i) (or spot speeds). Time headways can pertain to the leading vehicle directly in front (i.e. on the same lane), or vehicles which have a different lateral position (i.e. on another lane). For two or three dimensional flows, the definition of time headways is more involved.

On the moment $t = t_0$, one can observe positions of vehicles. The differences between successive positions are 'distance headways' (s_i), and the speeds at a moment are 'instantaneous' speeds v_i. For two or three dimensional flows, the notion of distance headway is less useful, for one since there is no direct relation with the density.

Remark 4 *In English, an important distinction is made between the speed and the velocity of a vehicle. In general, the speed is a scalar describing the absolute speed of a vehicle. The velocity is a one-, two-, or three-dimensional vector that also includes the direction of the vehicle. The latter direction is usually taken relative to the main direction of travel. For one-dimensional flows, the speed is generally equal to the velocity. For two- or three-dimensional flows, the speed is generally not equal the velocity. If $\mathbf{v}(t)$ denotes the velocity of a pedestrian i, then his/hers speed is defined by $v(t) = \|\mathbf{v}(t)\| = \sqrt{v_1^2(t) + v_2^2(t)}$.*

Let us close off by illustrating the applications of the use of trajectories, first by recalling an example of [14] showing how the use of the (t, x) plane can help in finding errors in the solution approach.

2.1.3 Applications of trajectories in traffic problem solving

Three friends take a long trip using a tandem bicycle for 2 persons. Because the bike riders travel at 20 km/h, independent of the number of riders, and all three persons walk at 4 km/h, they proceed as follows: to start the journey, friends A and B ride the bicycle and friend C walks; after a while, friend A drops off friend B who starts walking, and A rides the bicycle alone in the reverse direction. When A and C meet, they turn the bicycle around and ride forward until they catch up with B. At that moment, the three friends have complete a basic cycle of their strategy, which they then repeat a number of times until they reach their destination. What is their average travel speed?

The answer to this question is not straightforward, unless one plots the trajectories of the four moving objects on the (t, x) diagram. One finds by inspection that the average speed is 10 km/h. The proof of this is left to the reader as an exercise.

2.1.4 Application of trajectories to scheduling problems

From [14]. This problem illustrates the use of the time-space diagram to analyze the interaction of ships in a narrow canal. The canal is wide enough for only one ship, except for a part in the middle ('the siding'), which is wide enough for two ships so passing is possible. Ships travel at a speed of 6 km/h and should be at least 1.5 km apart when they are moving – expect when traveling in a convoy. When stopped in the siding, the distance between the ships is only 0.25 km. Westbound ships travel full of cargo and are thus given high priority by the canal authority over the eastbound ships, which travel empty. Westbound ships travel in four convoys which are regularly scheduled every 3.5 hours and do not stop at the siding. The problem is now the following

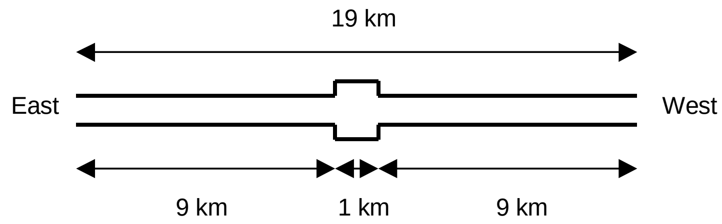

Figure 2.3: Sketch of a canal with an intermediate siding for crossing ships.

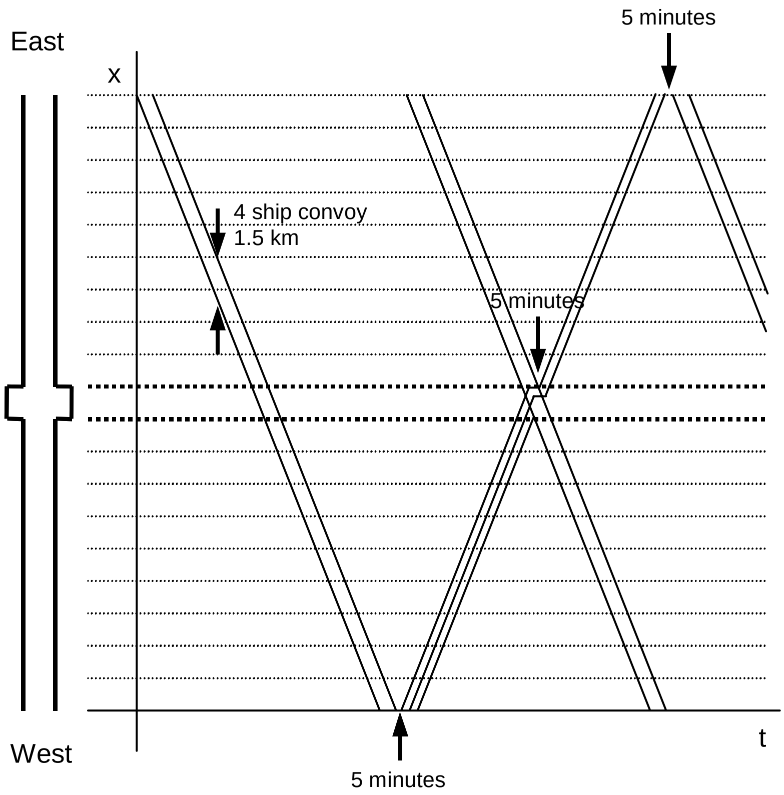

Figure 2.4: Time-space diagram in case of 1 km siding.

1. What is the maximum daily traffic of eastbound ships, and

2. What is the maximum daily traffic of eastbound ships if the siding is expanded to one km in length on both sides to a total of three km.

Note: we assume that eastbound ships wait exactly five minutes to enter either one of the one way sections after a westbound convoy has cleared it. We do not take into account that the ships do not accelerate instantaneously.

To solve the problem, we start by drawing the time-space diagram with the trajectories of the high-priority westbound convoys. See Fig. 2.4. The convoy leaves at the western end at 03:25. Note that since we neglect the size of the ships, the 4 ship convoy takes up 1.5 km distance. The second step is to draw the trajectory of a ship entering the western end of the canal at 3:30, which is the first time a ship can enter given the 5 minute time headway.

Note that the first ship must stop at the eastern part of the siding to yield the right of way to the last ship in the westbound convoy; note also how it makes it within the 5 minute allowance to the eastern end of the canal. The same process is follows successfully with the second trajectory. The third ship will however not be able to arrive at the western bound of the siding within the 5 min. allowance and it cannot be dispatched. Thus, we find:

$$\text{capacity} = 2(\text{ships per 3.5 hours}) = 13.71 \text{ ships/day} \tag{2.1}$$

It is left as an exercise to determine the capacity in case of the wider siding.

2.1.5 Mathematical description of trajectories and vehicle kinematics

In this section, we recall the mathematical equations describing the kinematics of a vehicle i as a function of time by means of ordinary differential equations. The starting point of our description is the trajectory $x_i(t)$ of vehicle i; the speed v, acceleration a and the jerk γ are respectively defined by the following expressions

$$v_i(t) = \frac{d}{dt} x_i(t) \tag{2.2}$$

$$a_i(t) = \frac{d}{dt} v_i(t) = \frac{d^2}{dt^2} x_i(t) \tag{2.3}$$

$$\gamma_i(t) = \frac{d}{dt} a_i(t) = \frac{d^2}{dt^2} v_i(t) = \frac{d^3}{dt^3} x_i(t) \tag{2.4}$$

Given the initial conditions of vehicle i (in terms of its position, speed, and acceleration), at time $t = t_0$, we can easily determine the following equations of motion:

$$\begin{aligned}
x_i(t) &= x_i(t_0) + \int_{t_0}^{t} v_i(s)\, ds && (2.5)\\
&= x_i(t_0) + (t - t_0) v_i(t_0) + \int_{t_0}^{t} \int_{t_0}^{s} a_i(s')\, ds' ds && (2.6)\\
&= x_i(t_0) + (t - t_0) v_i(t_0) + \frac{1}{2}(t - t_0)^2 a_i(t_0) && (2.7)\\
&\quad + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{s'} \gamma_i(s'')\, ds'' ds' ds && (2.8)
\end{aligned}$$

The motion of a vehicle can also be described as functions of the position x or the speed v. For instance

$$v(x) = \frac{1}{dt/dx} \Leftrightarrow dt = \frac{dx}{v(x)} \tag{2.9}$$

yielding

$$t_i(x) = t_0 + \int_{x_0}^{x} \frac{1}{v_i(y)} dy \tag{2.10}$$

Alternatively, we can use different definitions to describe the process at hand. For instance, rather that the speed (which by definition describes changes in the position as a function of changes in time), we can define the *slowness* (describing the changes in time per unit distance)

$$w(x) = \frac{dt(x)}{dx} \tag{2.11}$$

yielding the following equations of motion (analogous to equation (2.5))

$$t_i(x) = t_i(x_0) + \int_{x_0}^{x} w_i(y)\, dy \tag{2.12}$$

The kinematics of a vehicle can be modelled by considering the *different forces that act on the vehicle*; once the resultant force F_i acting on the vehicle is known, the acceleration a_i of the vehicle can be easily determined by application of Newton's second law $F_i = m_i a_i$, where m_i denotes the mass of vehicle i. Amongst the most important force terms are the following [14]:

1. Propulsion force F_p: the force that the guideway exerts on the vehicle. It usually varies with time as per the 'driver' input, but is always limited by engine power and the coefficient of friction in the following way

$$\frac{F_p}{m} = a_p \leq g \min\left\{f, \frac{\kappa}{v}\right\} \tag{2.13}$$

where g is the acceleration of gravity, f is a dimensionless coefficient of friction, and κ is the power to weight ratio of the vehicle.

2. Fluidic (air) resistance F_f: the force that air / water exerts on the vehicle. A good approximation is the following

$$\frac{F_f}{m} = -\alpha v_r^2 \tag{2.14}$$

where v_r is the vehicle speed relative to the air of the fluid, and α is the coefficient of drag.

3. Rolling resistance F_r: a force term that is usually modelled as a linear relation with the speed, but is not as important for higher speeds.

4. Braking resistance F_b: this force depends on the force with which the brakes are applied, up to a maximum that depends on the friction coefficient between the wheels and the guideway. Thus, we can write

$$\frac{F_b}{m} \geq -gf \tag{2.15}$$

Note that generally, $F_b = 0$ when $F_p > 0$ (brake and throttle rarely applied simultaneously).

5. Guideway resistance F_g describing the effects of the acceleration due to the earths gravity. When the vehicle is at an upgrade, this force is negative; when the vehicle is on a downgrade, this force is positive.

Let us finally note that for simulation traffic on a digital computer, the continuous time scale generally used to describe the dynamics of traffic flow needs to be discretised and solved numerically. To this end, the time axis is partitioned into equally sized periods k, defined by $[t_k, t_{k+1})$, with $t_k = k\Delta t$. If we then assume that during the interval $[t_k, t_{k+1})$, the acceleration of vehicle i is constant, the time-discretised dynamics of the speed and the location become

$$v_i(t_{k+1}) = v_i(t_k) + a_i(t_k)\Delta t \tag{2.16}$$

$$x_i(t_{k+1}) = x_i(t_i) + v_i(t_k)\Delta t + \frac{1}{2}a_i(t_k)\Delta t^2 \tag{2.17}$$

Using these approximations, we will make an error of $O(\Delta t)$.

2.2 Time headways

Vehicle trajectories are the single most important microscopic characteristic of a traffic flow. However, only in very special cases, trajectory information is available. In most situations, one has to make due using local observations (i.e. observations at a cross-section x_0).

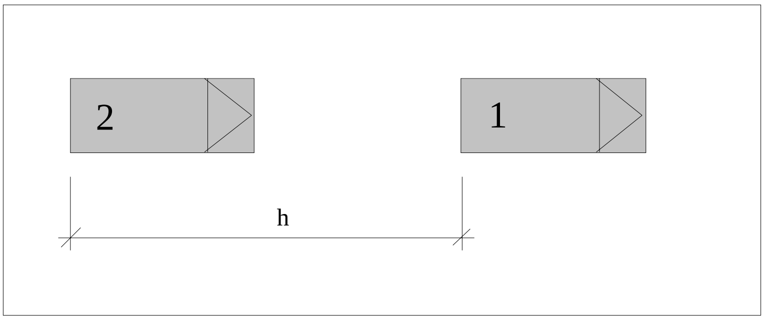

Figure 2.5: Definition of gross time headway h of vehicle 2.

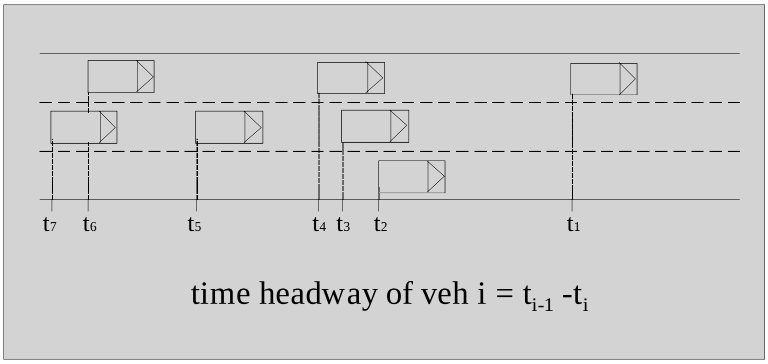

Figure 2.6: Definitions of time headways for a roadway

2.2.1 Time headways

Definition 5 *A time headway of a vehicle is defined as the period between the passing moment of the preceding vehicle and the vehicle considered; see also Fig. 2.2.*

Let h_i denote the time headway of the i th vehicle. The mean time headway equals for a period of length T

$$\overline{h} = \frac{1}{n}\sum_{i=1}^{n} h_i = \frac{T}{n} \tag{2.18}$$

where n denotes the number of vehicles that passed the cross-section during a period of length T. One can distinguish a nett and a gross time headway, as is shown in the definitions below.

Definition 6 *A nett time headway is defined as the period between the passing moments of the rear side of the preceding vehicle and the front of the vehicle considered.*

Definition 7 *A gross time headway (or simply headway) refers to the same reference point of both vehicles, e.g. front or back. Using the rear side of both vehicles has the advantage that the headway of a vehicle is dependent on its own length and not on the length of its predecessor; see Fig. 2.5.*

In traffic flow theory a time headway is usually a gross headway, because then the mean value is known if intensity is known (see Sec. 2.3). Other terms used for time headway are *gap* and *interval*.

Usually headways refer to one lane of traffic. However, they can also be defined for a roadway consisting of two or more lanes. An important consequence of such a definition is that gross and nett headways can be as small as 0.

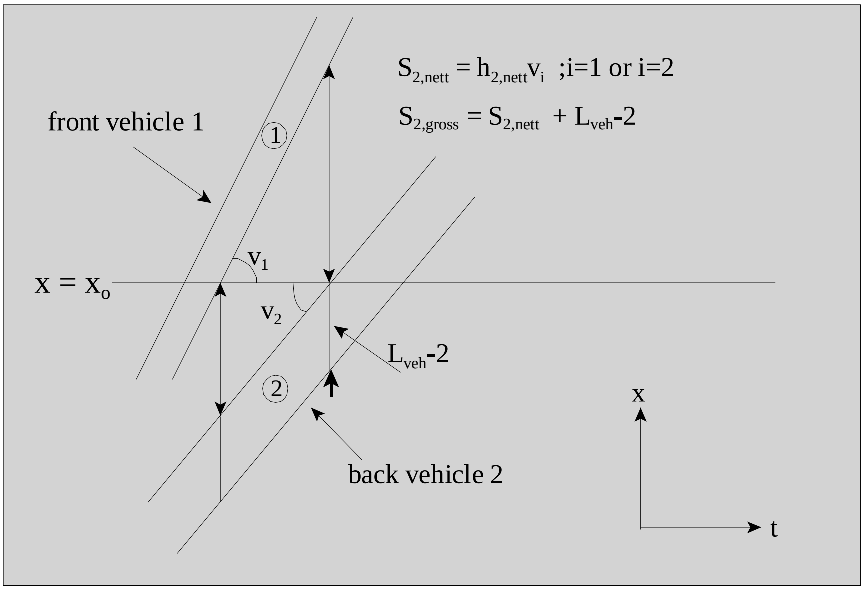

Figure 2.7: Calculation of distance headways from time headways and speeds

2.2.2 Distance headways

Using a similar definition, we can determine the distance headway between two vehicles. On the contrary to the time headway, the distance headway is a *instantaneous variable* defined at a certain time instant.

Definition 8 *A time headway of a vehicle is defined by the distance between the rear bumper of the preceding vehicle and the rear bumper of the considered vehicle at a certain time instant; see also Fig. 2.2.*

If s_j denotes the distance headway of the j th vehicle, then the mean distance headway equals

$$\overline{s} = \frac{1}{m}\sum_{j=1}^{m} s_j = \frac{X}{m} \tag{2.19}$$

where m denotes the number of vehicles that are present on a road of length X at a certain time t.

One can distinguish a nett and a gross distance headway, either including or excluding the length of the vehicle. Fig. 2.7 shows how one can calculate distance headways from local observations, using several possibilities.

2.3 Intensity, density and mean speed

The previous sections described the most important micrscoscopic traffic flow variables. In this section, the main macroscopic – i.e. describing the average behavior of the flow rahter than of each individual vehicle – intensity, density and mean speed, and the relations between them.

2.3.1 Intensity

Definition 9 *The intensity of a traffic flow is the number of vehicles passing a cross-section of a road in a unit of time.*

The intensity can refer to a total cross-section of a road, or a part of it, e.g. a roadway in one direction or just a single lane. Any unit of time may be used in connection with intensity, such as 24 h, one hour, 15 min, 5 min, etc. Hour is mostly used.

Apart from the unit of time, the time interval over which the intensity is determined is also of importance, but the two variables should not be confused. One can express the number of vehicles counted over 24 h in the unit veh/second. The intensity (or flow) is a local characteristic that is defined at a cross-section x for a period T, by

$$q = \frac{n}{T} \quad \text{[number of vehicles / unit of time]} \tag{2.20}$$

From Fig. 2.2 can be deduced that the period T is the sum of the time headways h_i of the n vehicles.

$$q = \frac{n}{T} = \frac{n}{\sum_i h_i} = \frac{1}{\frac{1}{n}\sum_i h_i} = \frac{1}{\bar{h}} \quad \text{where } \bar{h} = \text{mean time headway} \tag{2.21}$$

It goes without saying that in using Eq. 2.21 one should use units that correspond with each other; e.g. one should not use the unit veh/h for q in combination with the unit s for h_i.

The definition of intensity is easily generalized to two or three dimensional flows. It is however important that one realizes that the flow q for a two or three dimensional system is in fact a vector. For instance, for a two dimensional flow, $\mathbf{q} = (q_1, q_2)$ describes the flow q_1 in the longitudinal direction and the flow q_2 in the lateral direction. The respective elements of the flow vector can be determined by considering a lines (for a two-dimensional flow) or a plane (for a three dimensional flow) perpendicular to the direction of the considered element of the flow vector. For instance, when considering the longitudinal component q_1, we need to consider a line perpendicular to this direction (i.e. in the lateral direction) and count the number of vehicles passing this line during time a time period of length T.

2.3.2 Density

Definition 10 *The density of a traffic flow is the number of vehicles present on a unit of road length at a given moment. Just like the intensity the density can refer to a total road, a roadway, or a lane. Customary units for density are veh/km and veh/m.*

Compared to intensity, determining the density is far more difficult. One method is photography or video from either a plane or a high vantage point. From a photo the density is simply obtained by counting m = the number of vehicles present on a given road section of length X.

The density is thus an instantaneous quantity that is valid for a certain time t for a region X. The density is defined by:

$$k = \frac{m}{X} \quad \text{[number of vehicles / unit of length]} \tag{2.22}$$

From figure 2.2 follows that the road length X equals the sum of the 'distance headways' s_i

$$k = \frac{m}{X} = \frac{m}{\sum_i s_i} = \frac{1}{\frac{1}{m}\sum_i s_i} = \frac{1}{\bar{s}} \quad \text{where } \bar{s} = \text{mean distance headway} \tag{2.23}$$

For two or three dimensional flows, the density can be defined by considering either an area or a volume and counting the number of vehicles m that occupy this area or volume at a certain time t. The mean distance headway can in these cases be interpreted as the average area / volume that is occupied by a vehicle in the two or three dimensional flow.

Remark 11 *Strictly speaking, Eqns. (2.21) and (2.23) are only valid if the period T and section length X are precisely equal to an integer number of headways. Practically, this is only relevant when relatively short periods T or section lengths X are used.*

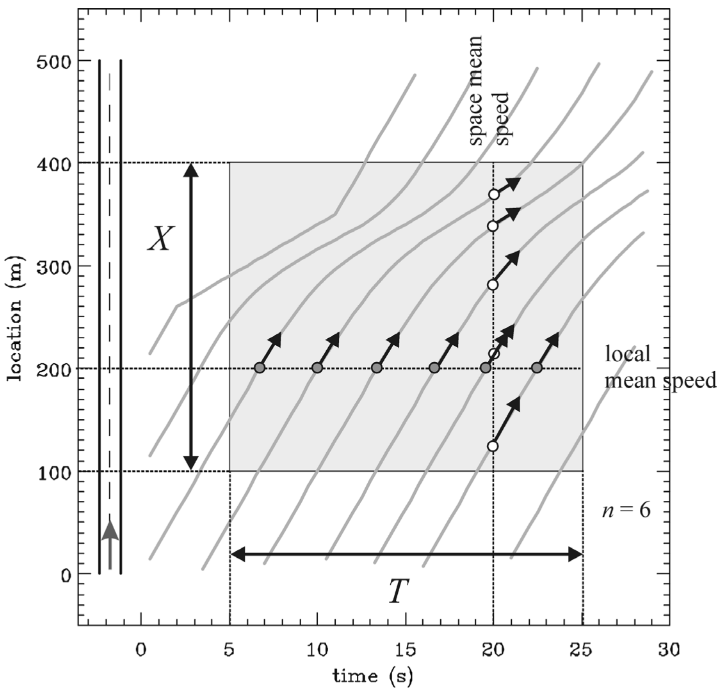

Figure 2.8: Defintion of local mean speed (or time mean speed) and space mean speed

Remark 12 *Intensity and density are traditionally defined as local and instantaneous variables. In the sequel of this chapter, generalized definition of both variables will be discussed. For these generalizations, the relation with the time and distance headways is not retained, at least not for the classical definition of the latter microscopic variables.*

2.3.3 Mean speed

The mean speed can be determined in several ways:

- Suppose we measure the speeds of vehicles *passing a cross-section during a certain period.* The arithmetic mean of those speeds is the so called 'local mean speed' (or mean spot speed; denoted with index L, referring to local).

$$u_L = \frac{1}{n}\sum_{i=1}^{n} v_i \tag{2.24}$$

- Suppose we know the speeds of the vehicles, v_j, that are *present on a road section at a given moment.* The arithmetic mean of those speeds is the-so called 'instantaneous mean speed' (denoted with index M, referring to moment), or 'space mean speed'.

$$u_M = \frac{1}{m}\sum_{j=1}^{m} v_j \tag{2.25}$$

Fig. 2.8 shows the difference between the definitions of the local mean speed (or local mean speed) and the space mean speed. Instantaneous speeds can be determined by reading the positions of vehicles from two photos taken a short time interval apart (e.g. 1 s). This is an expensive method, but it is possible to estimate the instantaneous mean speed from local speeds, as discussed in section 2.5.1.

Note that the notions of local and instantaneous speeds are only meaningful when a number of vehicles is considered; for a single vehicle, both speeds are equal.

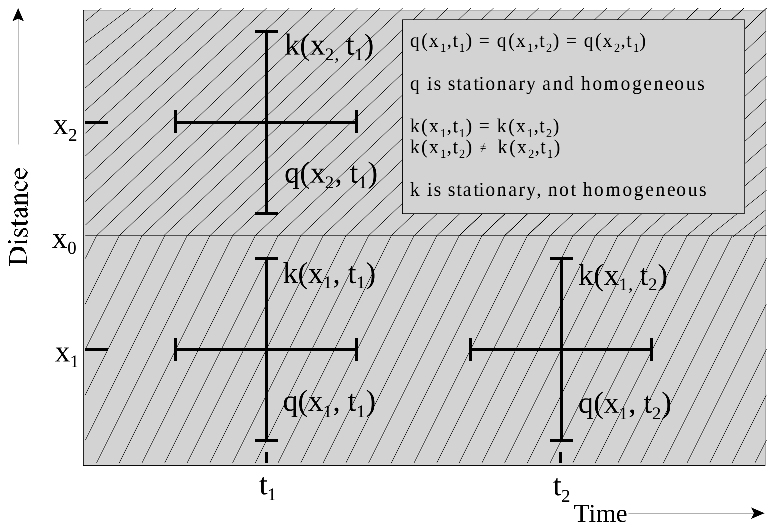

Figure 2.9: Effects of drastic change of road profile at position x_0

2.4 Homogeneous and stationary flow conditions

A traffic flow is composed of vehicles. Movements of different vehicles are a function of position and time (each vehicle has its own trajectory). The characteristics of a traffic flow, such as intensity, density, and mean speed, are an aggregation of characteristics of the individual vehicles and can consequently also be dependent on position and time.

Consider a variable $z(x,t)$. We define this variable z to be:

- Homogeneous, if $z(x,t) = z(t)$; i.e. the variable z does not depend on position.
- Stationary, if $z(x,t) = z(x)$; i.e. the variable z is independent of time.

Example 13 *Figure 2.9 presents a schematic image of vehicle trajectories. At the spot $x = x_0$ the road profile changes drastically, and as a result all vehicles reduce their speed when passing x_0. In this case the distance headways change but the time headways remain the same. This means that intensity q is stationary and homogeneous and density k is stationary but not homogeneous. Figure 2.10 presents vehicle trajectories in another schematized situation. At the moment to the weather changes drastically. All vehicles reduce their speed at that moment. Then the time headways change but the distance headways remain the same.*

2.4.1 Determination of periods with stationary intensity

Intensity is a characteristic that influences many other properties of the traffic flow. When studying such an influence, e.g. on the parameters of a headway distribution, it is advantageous to have periods with a constant or stationary intensity. To determine stationary periods one can apply formal statistical methods, but a practical engineering method is also available.

The number of vehicles that pass a cross-section after a given moment is drawn as a function of time. This can be done using passing moments of every vehicle, but it can also be done with more aggregate data, e.g. 5-minute intensities. A straight part of the cumulative curve corresponds to a stationary period. The next question is ‘what is straight enough’ but it turns out in practice that this is not problematical. One should choose the scale of the graph with some care; it should not be too large because on a detailed scale no flow looks stationary. *Fig. 1.4 presents an example of application of the method.* Three straight sections and two transition periods between them can be distinguished.

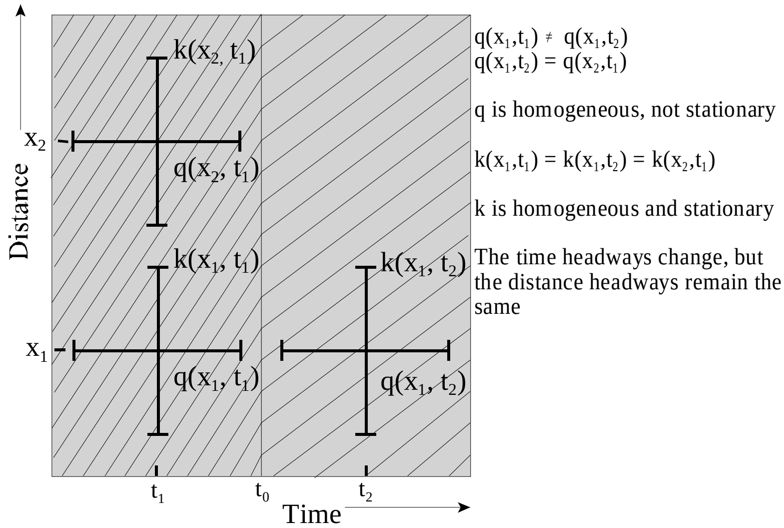

Figure 2.10: Effects of substational weather change at moment t_0

2.5 Relation between local and instantaneous characteristics

Considering a traffic flow in a stationary and homogeneous 'state', the following relation (referred to as the *fundamental relation*) is valid:

$$q = ku \tag{2.26}$$

In words: The number of particles, passing a cross-section per unit of time (q), equals the product of:

- The number of particles present per unit of distance (k); and
- The distance covered by those particles per unit of time (u)

From this general formulation it follows that the relation will be valid for all types of flows, e.g. liquids, gasses, pedestrians, etc. Clearly, when two or three dimensional flows are considered, both the flow and the speed (or rather, velocity) are vectors describing the mean intensity and speed in a particular direction.

2.5.1 Relation between instantaneous and local speed distribution

Let $f_L(v)$ and $f_M(v)$ respectively denote the local and instantaneous speed distribution. Consider a region from x_1 to x_2. Let us assume that the traffic state is homogeneous and stationary, i.e.

$$q(x,t) = q \quad \text{and} \quad k(x,t) = k \tag{2.27}$$

The probability that in the region $[x_1, x_2]$ we observe a vehicle with a speed in the interval $[v, v + dv)$ (where dv is very small) at time instant t, equals by definition

$$(x_2 - x_1)\, k f_M(v) dv \tag{2.28}$$

Consider the period from t_1 to t_2. Then, the probability that during the period $[t_1, t_2]$ a vehicle passes the cross-section x having a speed in the interval $[v, v + dv)$ equals

$$(t_2 - t_1)\, q f_L(v) dv \tag{2.29}$$

Now, consider a vehicle driving with speed v passing cross-section x_1 at time t_1. This vehicle will require $(x_2 - x_1)/v$ time units to travel from x_1 to x_2. Hence:

$$t_2 - t_1 = \frac{x_2 - x_1}{v} \tag{2.30}$$

As a result, the probability that during the interval $[t_1, t_1 + (x_2 - x_1)/v]$ a vehicle driving with speed v passes x_1 is equal to the probability that a vehicle with speed v is present somewhere in $[x_1, x_2]$ at instant t_1. From Eq. 2.29, we can calculate that the probability that a vehicle passes x_1 during the period $[t_1, t_1 + (x_2 - x_1)/v]$:

$$\frac{x_2 - x_1}{v} q f_L(v) dv \tag{2.31}$$

which is in turn equal to eq. 2.28, implying

$$v k f_M(v) = q f_L(v) \tag{2.32}$$

Integrating 2.32 with respect to the speed v yields the following relation between concentration and intensity

$$\int v k f_M(v) dv = k \int v f_M(v) dv = k \langle v \rangle_M = \int q f_L(v) dv = q \tag{2.33}$$

where we have used the following notation to describe the mean-operator with respect to the probability density function of the instantaneous speeds

$$\langle A(v) \rangle_M = \int A(v) f_M(v) dv \tag{2.34}$$

Note that $\langle v \rangle_M$ denotes the mean instantaneous speed u_M.

At the same time, we can rewrite eq 2.32 as $k f_M(v) = q f_L(v)/v$. Again integrating with respect to the speed v, we find the following relation

$$k = q \left\langle \frac{1}{v} \right\rangle_L \tag{2.35}$$

where

$$\langle A(v) \rangle_L = \int A(v) f_L(v) dv \tag{2.36}$$

In combining Eqns. 2.32 and 2.35, we get the following relation between the instantaneous speed distribution and the local speed distribution

$$f_M(v) = \frac{1}{v \left\langle \frac{1}{v} \right\rangle_L} f_L(v) \tag{2.37}$$

Fig. 2.11 shows the probability density function of the local speeds collected at a cross-section of a two-lane motorway in the Netherlands during stop-and-go traffic flow conditions. Note that the local speed and the instantaneous speed probability density functions are quite different. Local speeds are collected at two-lane A9 motorway in the Netherlands during peak hours. Note that the differences between the speed distributions are particularly high for low speeds.

2.5.2 Local and instantaneous mean speeds part 1

Consider the case where data is collected using a presence type detection, e.g. an inductive loop. Assume that besides the passage times of the vehicles, also their speeds are determined. Furthermore, assume that during the data collection periods, traffic conditions are stationary and homogeneous.

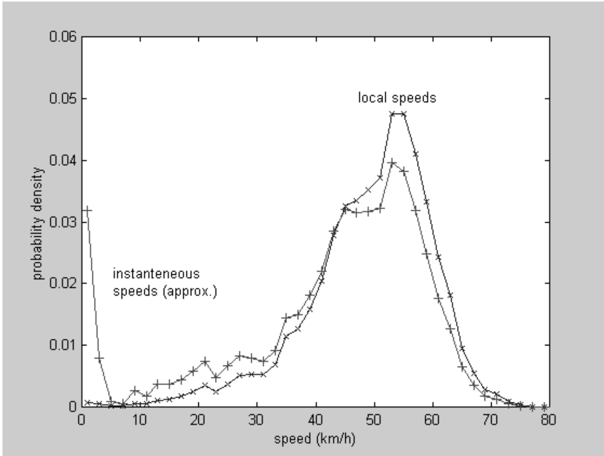

Figure 2.11: Local speed density function and instantaneous speed density function. Local speeds are collected at two-lane A9 motorway in the Netherlands during peak hours.

From the speeds v_i $(i = 1, ..., n)$ collected at the cross-section x, we can determine the so-called *local (or instantaneous) empirical probability density function* $\hat{f}_L$ as follows

$$\hat{f}_L(v) = \frac{1}{n}\sum_{i=1}^{n} \delta(v - v_i) \tag{2.38}$$

where δ is the so-called δ-dirac function implicitly defined by

$$\int a(y)\,\delta(y - x)\,ds = a(x) \tag{2.39}$$

and where n equals the number of vehicles that have passed the cross-section x during the considered period. The local mean speed thus becomes

$$u_L = \langle v \rangle_L = \frac{1}{n}\sum_{i=1}^{n} v_i \tag{2.40}$$

where the mean operator is defined using Eq. (2.36) with $f_L(v) = \hat{f}_L(v)$. According to Eq. (2.37), we now find the following relation between the instantaneous empirical speed distribution $\hat{f}_M(v)$ and the local speed distribution $\hat{f}_L(v)$

$$\hat{f}_M(v) = \frac{1}{v\left\langle \frac{1}{v} \right\rangle_L}\hat{f}_L(v) = \frac{1}{v\frac{1}{n}\sum_{i=1}^{n}\frac{1}{v_i}}\frac{1}{n}\sum_{i=1}^{n}\delta(v - v_i) \tag{2.41}$$

Using this expression, we find the following equation for the instantaneous or space mean speed

$$u_M = \langle v \rangle_M = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{v_i}} \tag{2.42}$$

That is, the space mean speed is equal to the *harmonic average of the speeds collected at a cross-section x during a stationary period.*

The fact that local mean and space mean differ is not only true for speeds *but also for other vehicle characteristics!* We will derive a formula for determining the space mean (of some variable) based on local observations, and also a formula for the reverse procedure. The trick used in these derivations is: define the mean over individual vehicles; divide the traffic flow in uniform sub-flows; use the relation $q = ku$ per sub-flow i; go back to individual values by shrinking the sub-flows to one vehicle.

- From local observations to space mean

Vehicle i passing cross section x has quantitative property z_i. Examples for z are: speed; length; number of passengers; weight; emission rate (= emission per time); etc.

As an introduction, we note that the local mean of z is by definition:

$$z_L = \frac{1}{n}\sum_i z_i \tag{2.43}$$

We can divide the flow in uniform sub-flows q_j , uniform with respect to speed v_j and characteristic z_j. Then the local mean is a mean with intensities q_j as weight:

$$z_L = \frac{\sum_i q_i z_i}{\sum_i q_i} \tag{2.44}$$

Now the derivation: The space-mean value of z is by definition a mean with the densities as weights:

$$z_M = \frac{\sum_i k_i z_i}{\sum_i k_i} \tag{2.45}$$

Replacing k_i by q_i/v_i yields

$$z_M = \frac{\sum_i (q_i/v_i)\, z_i}{\sum_i (q_i/v_i)} \tag{2.46}$$

Take $q_i = 1$, i.e. a sub-flow consists of one vehicle, i.e.

$$z_M = \frac{\sum_i z_i/v_i}{\sum_i 1/v_i} \tag{2.47}$$

Hence the space mean of z can be estimated by taking a weighted sum of local observed characteristics z_i, and the weights are $1/v_i$.

- Reverse, i.e. from space observations to local mean

Assume the following: observed in space are vehicles i with characteristics z_i and v_i. The space mean of z is by definition:

$$z_M = \frac{1}{m}\sum_j z_j \tag{2.48}$$

We divide the total density in uniform subclasses with density k_j, uniform with respect to speed v_j and characteristic z_j.

$$z_L = \frac{\sum_j q_j z_j}{\sum_j q_j} = \frac{\sum_j k_j v_j z_j}{\sum_j k_j v_j} \tag{2.49}$$

Take $k_j = 1$. Then

$$z_L = \frac{\sum_j v_j z_j}{\sum_j v_j} \tag{2.50}$$

Hence the local mean can be estimated from space observations by taking a weighted sum of instantaneous observed characteristics z_j and the weights are v_j.

Example 14 *Many car drivers complain about the high number of trucks on the road during peak hours. Drivers do not observe the traffic flow at a spot but over a road section, whereas road authorities observe and publish truck percentages (TP) based on local observations. Say: local TP = 10%; mean speed of trucks = 80 km/h and mean speed of cars = 120 km/h. Then the TP observed in space is: TPM = (10/80) / {10/80 + 90/120} = 14.3 %. So instead of 1 in 10 vehicles being a truck at a spot, the space proportion is about 1 in 7. The difference is more extreme at a grade where the speed difference between cars and trucks is larger.*

From the example can be concluded that sometimes a characteristic can have a substantially different value at a spot and in space. Which value is more appropriate depends on the application.

2.5.3 Local and instantaneous mean speeds part 2

For general distributions, we have the mean speeds $u_L = \langle v \rangle_L$ and $u_M = \langle v \rangle_M$ are related by:

$$u_L = u_M + \frac{\sigma_M^2}{u_M} \tag{2.51}$$

Proof. This relation can be proven as follows. From eqns. 2.32 and 2.33 we find

$$f_L(v) = \frac{kvf_M(v)}{q} = \frac{kvf_M(v)}{k\langle v \rangle_M} = \frac{vf_M(v)}{u_M} \tag{2.52}$$

By definition, the expected local speed equals

$$u_L = \langle v \rangle_L = \int vf_L(v)dv = \int \frac{v^2 f_M(v)}{u_M} dv \tag{2.53}$$

It can thus be easily shown that

$$u_L = \int \frac{(v-u_M)^2 + 2vu_M - u_M^2}{u_M} f_M(v)dv = \frac{\sigma_M^2}{u_M} + u_M \tag{2.54}$$

where the variance of the instantaneous speeds by definition equals

$$\sigma_M^2 = \int (v-u_M)^2 f_M(v)dv \tag{2.55}$$

■

Relation 2.21 cannot be used to derive u_M from u_L because it contains the parameter σ_M, the standard deviation of the instantaneous speeds, which is not known either. From eq. 2.21 it follows that the local mean speed is larger than the instantaneous mean speed. This fact has got the 'memory aid': 'faster vehicles have a higher probability to pass a cross-section than slower ones'. This memory aid stems from the following situation: Consider a road section on which are present subpopulations of vehicles with the same density k but different speeds u_i. According to the relation $q_i = ku_i$ it is true that vehicles with a higher speed pass more often than those with a lower speed. This situation can be maintained and easily understood if one considers a loop that vehicles drive around and around.

Example 15 *At free flow at a motorway a representative value for the instantaneous mean speed is 110 km/h and for the instantaneous standard deviation (STD) of speeds 15 km/h. Then the local mean speed equals: $110+15^2/110 = 112$ km/h. At congestion the instantaneous mean speed might be 10 km/h and the instantaneous STD 12 km/h. Then the local mean speed equals: $10+12^2/10 = 24$ km/h. Hence at free flow the difference between local and instantaneous mean speed can be neglected but at congestion the difference can be substantial.*

2.5.4 Relation between flow, density and speed revisited

In the end, the question remains which of the average speeds should be used in $q = ku$ (Eq. 2.26): the time-mean speed or the space-mean speed? The answer to this question can be found as follows. Considered a group of vehicles that is driving with speed u_i. If we consider a certain section of the roadway, we will denote the density of this group of vehicles by k_i. Since all vehicles are driving at the same speed, equation 2.26 holds. As a consequence, group i will contribute to the total volume q according to $q_i = k_i u_i$. The total traffic volume thus becomes

$$q = \sum_i q_i = \sum_i k_i u_i \tag{2.56}$$

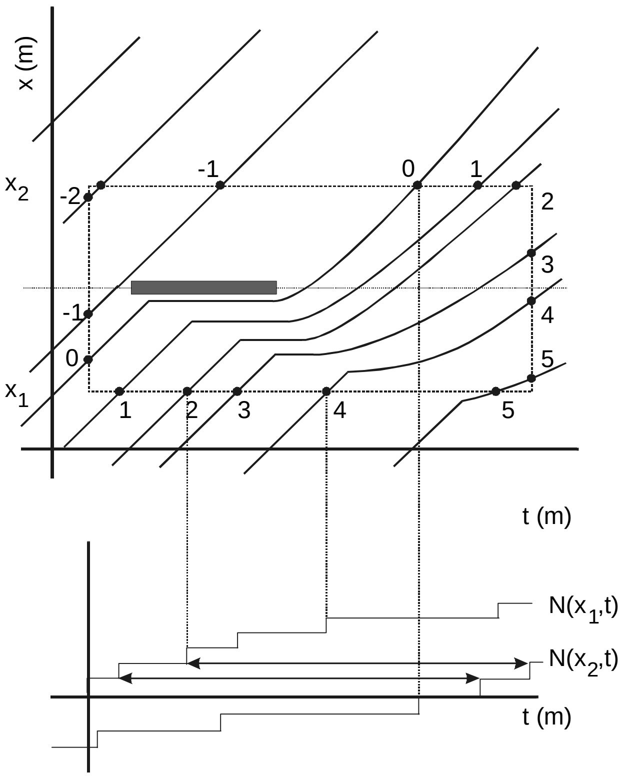

Figure 2.12: Trajectories and cumulative vehicle plot

The total density in the considered roadway section is determined by

$$k = \sum_i k_i \tag{2.57}$$

The mean speed thus equals

$$u = \frac{\sum_i k_i u_i}{\sum_i k_i} = \frac{\sum_i q_i}{\sum_i q_i / u_i} \tag{2.58}$$

Since q_i is the number of vehicles that passes a cross-section per unit time, u is the harmonic average of the speeds collected at the cross-section. Hence, we need to use the space-mean speeds. As the space-mean speed is not easy to obtain an approximation for it has been derived, the harmonic mean of local speeds.

2.6 Cumulative vehicle plots and their applications

A cumulative plot of vehicles is a function $N(x,t)$ that represents the number of vehicles that has passed a cross section x from an arbitrary starting moment. Fig. 2.12 shows a couple of vehicle trajectories which are numbered in increasing order, as well as the cumulative vehicle plots $N(x_1,t)$ and $N(x_2,t)$ determined for two cross-sections x_1 and x_2 as a function of time. Note that each time a vehicle passes either of the cross-section, the respective cumulative vehicle count increases with one. The example shows vehicles being stopped at a controlled intersection. The arrows in the lower figure indicate the travel times of vehicles 1 and 2, including their delay due to the controlled intersection.

Fig. 2.13 shows another plot determined from real-life data.

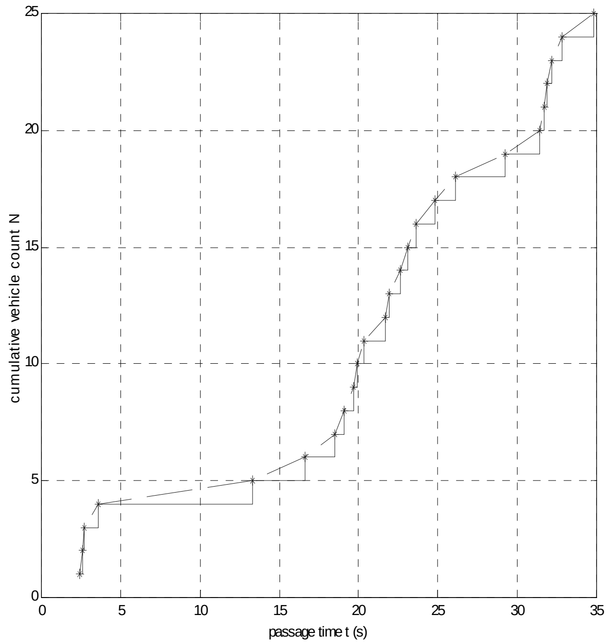

Figure 2.13: Cumulative flow function $N(x,t)$ and smooth approximation $\tilde{N}(x,t)$. Note that for passage times t_j, we have $N(x,t_j) = \tilde{N}(x,t_j)$

2.6.1 Relation between cumulative flow, intensity and density

It is obvious that the intensity measured at a certain cross-section x during period t_1 to t_2 equals:

$$q(x, t_1 \text{ to } t_2) = \frac{N(x,t_2) - N(x,t_1)}{t_2 - t_1} \tag{2.59}$$

Since the vehicle are indivisible objects, $N(x,t)$ is a step function. However, in most practical problems it is not needed to have solutions with an accuracy of one vehicle. This allows us to approximate the step function $N(x,t)$ by a smooth function $\tilde{N}(x,t)$ that is *continuous and can be differentiated*. The smooth approximation $\tilde{N}(x,t)$ is defined such that for the passage times t_j at which the vehicles pass the cross-section x, we have $N(x,t_j) = \tilde{N}(x,t_j)$; see Fig. 2.13 for an example.

Taking the limit of eq. (2.59) for $(t_2 - t_1) \to 0$ results in:

$$q(x,t) = \frac{\partial \tilde{N}(x,t)}{\partial t} \tag{2.60}$$

As the position x is a continuous variable, we have now introduced a concept of a local and instantaneous intensity.

Now consider two cumulative plots at position x_1 and x_2. Then at time instant t, the average density is:

$$k(x_1 \text{ to } x_2, t) = \frac{\tilde{N}(x_1,t) - \tilde{N}(x_2,t)}{x_2 - x_1} \tag{2.61}$$

Taking the limit of eq. (2.61) for $(x_2 - x_1) \to 0$ leads to:

$$k(x,t) = -\frac{\partial \tilde{N}(x,t)}{\partial x} \tag{2.62}$$

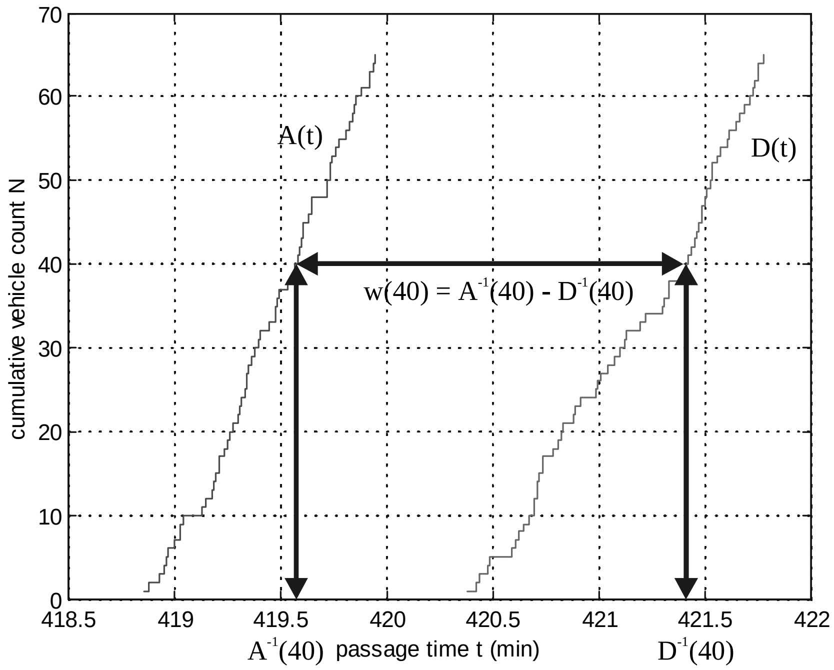

Figure 2.14: Determination of travel time using cumulative vehicle counts

Finally we can define the mean speed at the spot x and instant t as:

$$u(x,t) = q(x,t)/k(x,t) \tag{2.63}$$

Consequently all three main macroscopic characteristics of a traffic flow can be handled as continuous functions of the position x and time t. This property will be used when considering macroscopic traffic flow models.

2.6.2 Trip times and the cumulative flow function

The cumulative flow function has a vast number of applications. It can also be used to determine the *time vehicles need to traverse a certain roadway section* (recall Fig. 2.12). To us further consider this application of the cumulative flow function.

Let $A(t) = N(x_1, t)$ (the *arrival curve*) denote the cumulative vehicle count at the entry of a roadway section x_1; let $D(t) = N(x_2, t)$ (the *departure curve*) denote the cumulative vehicle count at the exit x_2 of a roadway section. The time t at which the N-th vehicle passes the cross-section can be obtained by finding the time t where a horizontal line across the ordinate N meets the crest of a step. Let $N^{-1}(x, N)$ denote the function that returns t for a given N. If vehicles pass the roadway in a *First-In-First-Out* (FIFO) order, then these N-th observations correspond to the same individual, and hence the trip time of the N-th vehicle throught the section $[x_1, x_2]$ equals

$$w(N) = A^{-1}(N) - D^{-1}(N) \tag{2.64}$$

See Fig. 2.14. Note that this relation is not true in case passing is possible.

2.6.3 Delay times and cumulative flow functions

Let us assume that the time τ needed to traverse the empty roadway section is equal for all vehicles. Furthermore, let us assume that vehicle 1 enters an empty roadway. Then, the free trip time τ can be determined by $\tau = D^{-1}(1) - A^{-1}(1)$.

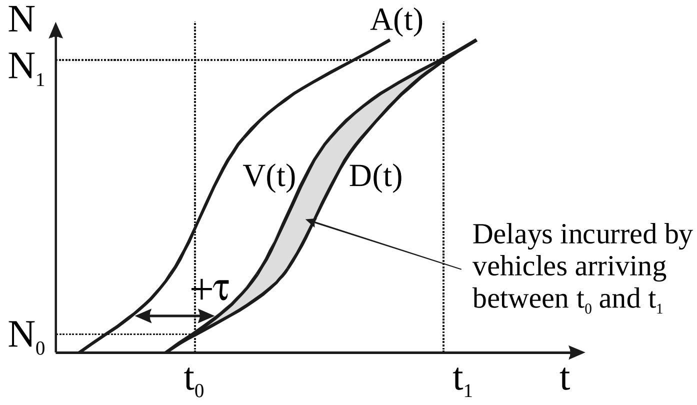

Figure 2.15: Arrival curve $A(t)$ (arrivals at entry), arrival curve $D(t)$ (arrivals at exit), and the virtual departure curve $V(t)$.

Now, let us assume that τ has been determined. We can then define the *virtual arrival curve* (cf. [14]) by translating the arrival curve τ units along the time-axis, i.e.

$$V(t) = A(t - \tau) \tag{2.65}$$

Since horizontal separations in the (t, N) diagram represent time and verticle separations represent accumulation, it is clear that the area enclosed by $V(t)$ and $D(t)$ and any two vertical lines $t = t_0$ and $t = t_1$, is the total delay time incurred by vehicles in the road-section during the period $[t_0, t_1]$. Fig. 2.15 shows the curves $A(t)$, $D(t)$, and $V(t)$, and the total delay W incurred by vehicles arriving at x_2 during the period $[t_0, t_1]$, defined by

$$W = \int_{t_0}^{t_1} [V(t) - D(t)]\, dt \tag{2.66}$$

The average delay for all vehicles that have passed during the period hence becomes

$$\bar{w} = \frac{W}{V(t_1) - V(t_0)} = \left(\frac{W}{t_1 - t_0}\right)\left(\frac{t_1 - t_0}{V(t_1) - V(t_0)}\right) \tag{2.67}$$

See Fig. 2.15. From eqn. (2.67), we can rewrite

$$\bar{Q} = \bar{\lambda}\bar{w} \tag{2.68}$$

where the average number $\bar{Q}$ of vehicles in the system is equal to

$$\bar{Q} = \frac{W}{t_1 - t_0} \tag{2.69}$$

and where the arrival rate $\bar{\lambda}$ is equal to

$$\bar{\lambda} = \frac{V(t_1) - V(t_0)}{t_1 - t_0} \tag{2.70}$$

The chapter describing queuing theory will provide further discussion on the applications of cumulative curves in traffic theory. At his point, let us provide an example involving delays at an controlled intersection.

Example 16 *Let us show an example of the use of cumulative curves to determine the average waiting time at a controlled intersection. Let R and G respectively denote the red and the green time of the cycle; the total cycle time $C = R + G$. Let λ denote the arrival rate at the controlled*

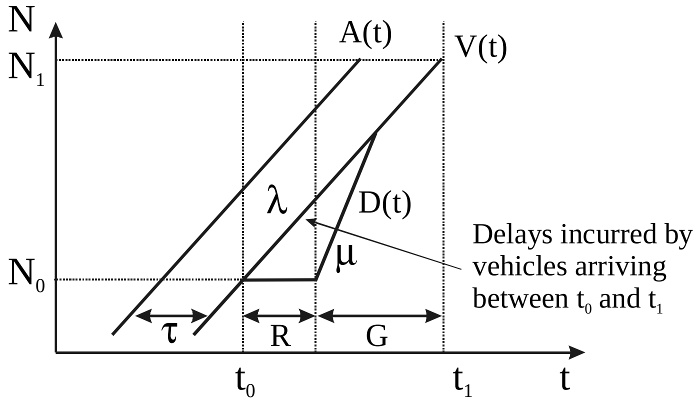

Figure 2.16: Delay at a pre-timed traffic signal

intersection. When the controller is in its red phase, vehicle cannot pass and the queue will grow. During the green phase, μ veh/s can be served. The area between $V(t)$ and $D(t)$ again reflects the total delay W of the vehicles in the system. The total number of vehicles served during one cycle equals $n' = N_1 - N_2$. The number of vehicles that have been delayed is equal to n. From the geometry of the picture, it should be clear that $R = \frac{n}{\lambda} - \frac{n}{\mu}$ and thus that $n = R/\left(\lambda^{-1} - \mu^{-1}\right)$. For a non-saturated intersection, we have $n < n' = \lambda C$, yielding

$$\mu G < \lambda C \tag{2.71}$$

When the signal is over-saturated, this condition is not met and the queue would grow steadily over time. Since $W = nR/2$ and $n = R/\left(\lambda^{-1} - \mu^{-1}\right)$, we obtain $W = \frac{1}{2}\lambda\mu R^2 (\mu - \lambda)$. The long-run average delay per car is thus

$$\bar{w} = \frac{W}{n'} = \frac{1}{2}\frac{\mu R^2}{(\mu - \lambda) C} \tag{2.72}$$

Example 17 *Fig. 2.17 shows a two-lane rural highway in California, with a controlled intersection at its end (Wildcat Canyon Road)[2]. The site has limited overtaking opportunities. More important, there are no entry and exit points. At 8 observation points, passage times of vehicles have been collected and stored. Using these data, the cumulative curves shown in Fig. 2.18 have been determined. In the same figure, we have also indicated travel times. Fig. 2.18 shows how vehicle 1000 experiences a higher travel time than say vehicle 500. Note that at the observer 8 site, a traffic responsive traffic signal is present. Notice how the gradient of the cumulative curve of observer 8 decreases after some time (approximately at 7:15). The reason for this is the growth in the conflicting traffic streams at Wildcat Canyon Road. At approx. 8:40 we see another reduction of the flow. An interesting observation can be made from studying the cumulative curves in more detail. From Fig. 2.19 we can observe clearly the flows during the green phase and the zero flow during the red phase. Notice that before 8:40, the flow during the green phase is approximately constant for all green phases. Reduction in the average flow is caused by a reduction in the length of the green phase as a result of the conflicting streams. From approx. 8:40 onward, the flow during the green phase is reduced. The reason for this is that congestion from downstream spills back over the intersection.Fig. 2.20 shows the shifted cumulative curves. The curves are shifted along the time-axis by the free travel time. The resulting curves can be used to determine the delays per vehicle, as well as the total and average delays.Fig. 2.21 shows*

[2]The data used for this example can be downloaded from the website of Prof. Carlos Daganzo at Berkeley (www.ce.berkeley.edu/~daganzo/spdr.html).

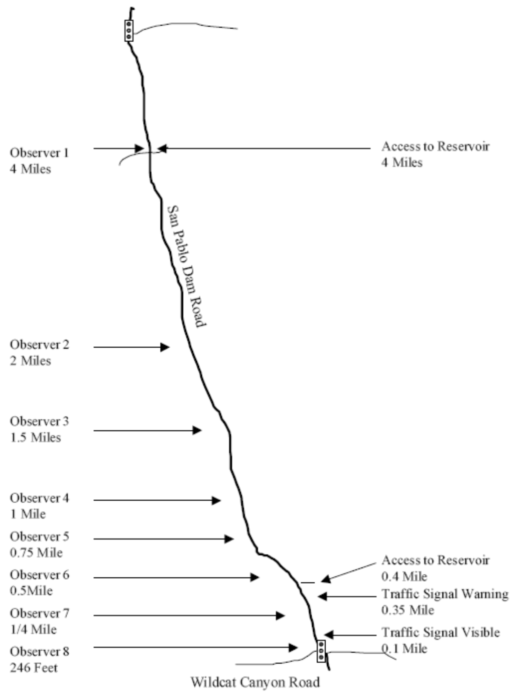

Figure 2.17: Overview of data collection site.

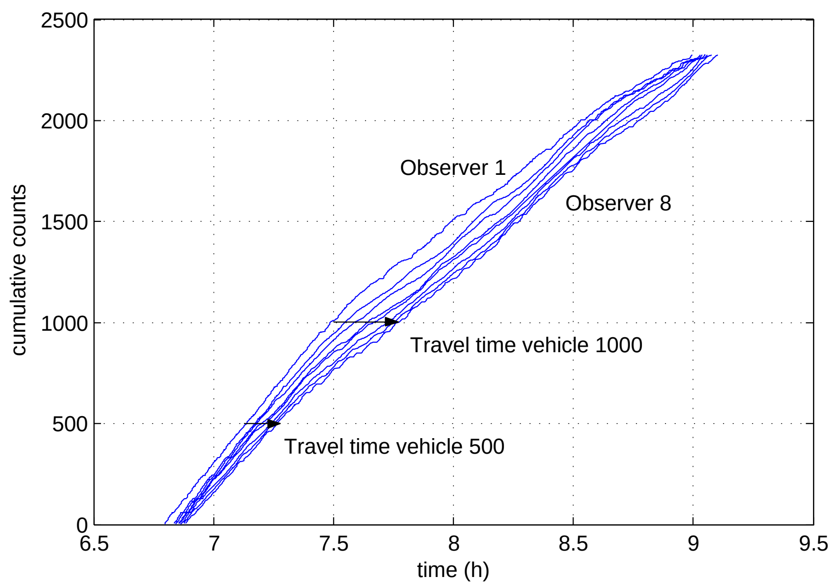

Figure 2.18: Cumulative counts collected at San Pablo site during morning peak period.

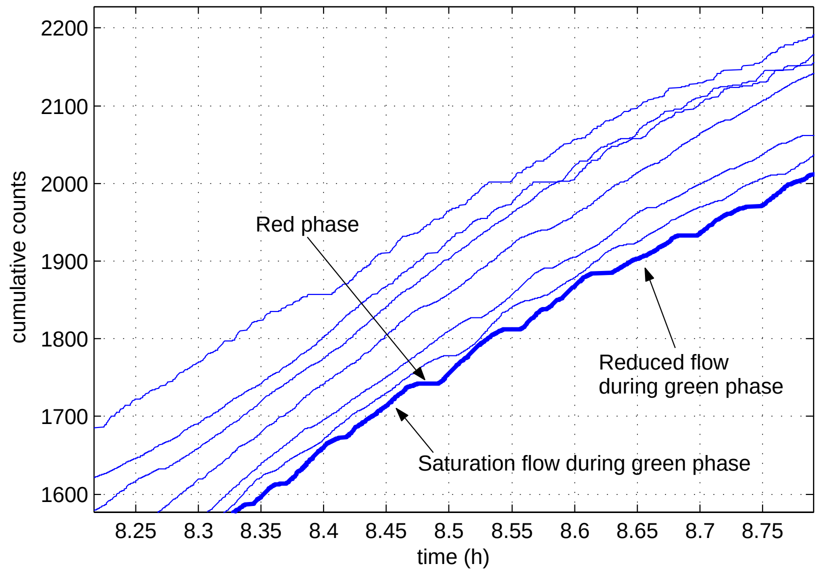

Figure 2.19: Magnification of cumulative curves showing reducting in saturation flow from 8:30 onward.

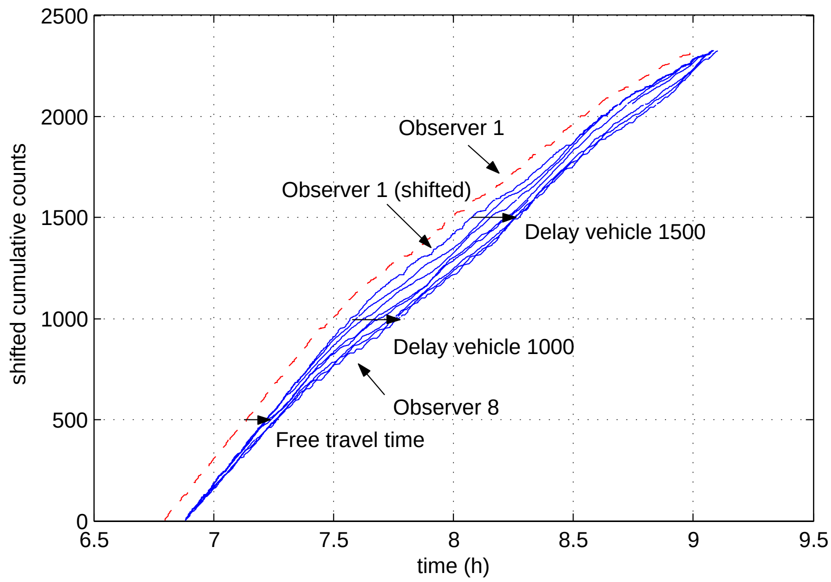

Figure 2.20: Shifted cumulative curves indicating the delays per vehicle.

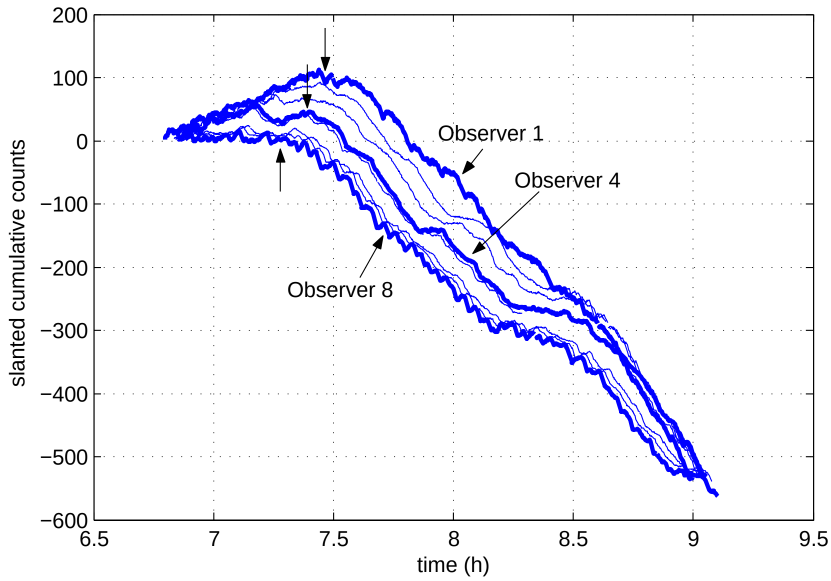

Figure 2.21: Slanted cumulative curves $N'(x_i, t) = N(x_i, t) - q'(t - t_0)$ for $q' = 1300$ veh/h.

the so-called slanted cumulative curves $N'(x, t)$ *defined by* $N'(x, t) = N(x, t) - q'(t - t_0)$ *for some value of* q'. *Using these slanted curves rather than the reguler cumulative curves can be useful to identify changes in the flowrate more easily. The three arrows in Fig. 2.21 show for instance, when the reducting in the flowrate at 7:15 due to increased conflicting flows on the intersection reach observer 4 and observer 1 respectivly. That is, the slanted cumulative curves can be used to identify the speed at which congestion moves upstream. This topic will be discussed in more detail in the remainder of the syllabus.*

2.6.4 Derivation of conservation of vehicle equation using cumulative flow functions

The *conservation of vehicle equations* (also known as the *continuity equation*) is the most important equation in macroscopic traffic flow modelling and analysis. It describes the fact that to no vehicles are lost or generated. The conservation of vehicle equation can be derived using elementary differential calculus. However, the definitions of q and k introduced above, allow a more direct derivation: differentiate eqn. (2.60) to x and (2.62) to t

$$\frac{\partial q(x,t)}{\partial x} = \frac{\partial^2 \tilde{N}(x,t)}{\partial x \partial t} \quad \text{and} \quad \frac{\partial k(x,t)}{\partial t} = -\frac{\partial^2 \tilde{N}(x,t)}{\partial x \partial t} \tag{2.73}$$

and thus yielding the conservation of vehicle equation

$$\frac{\partial q(x,t)}{\partial x} + \frac{\partial k(x,t)}{\partial t} = 0 \tag{2.74}$$

Eq. (2.74) represents one of the most important equations in traffic flow theory. It describes the fact that vehicles cannot be created or lost without the presence of sinks (off-ramps) or sources (on-ramps).

2.7 Definition of q, k and u for a time-space region

The macroscopic traffic flow variables intensity, density and mean speed in the preceding sections referred to either a *cross-section* x *and a period* T (local variable), or, *a road section* X *and*

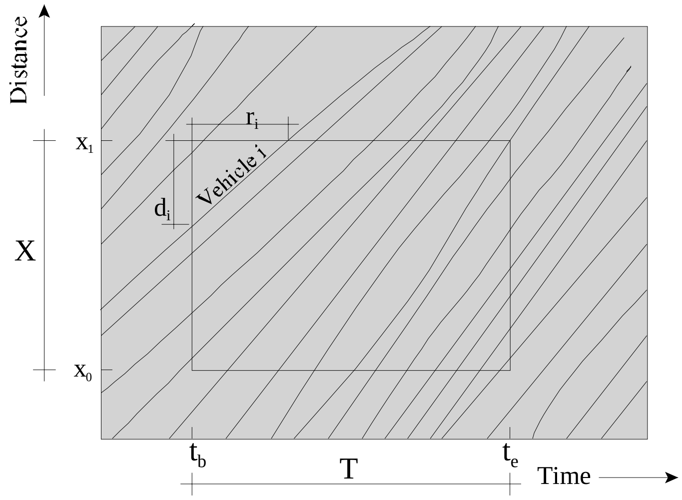

Figure 2.22: Definition of q, k, and u for a time-space region

a moment t (instantaneous variable), or, a *cross-section* x *and a moment* t. Edie [19] has shown that these variables can be defined for a time-space region and that those definitions are consistent with the earlier ones and make sense. Their main property is that small (microscopic) fluctuations have less influence on the values of the macroscopic variables, which is often an advantage in practice.

Fig. 2.22 shows a time-space rectangle of length X and duration T. In the window marked every vehicle covers a distance d_i and is present during a period r_i. Now intensity can be defined as the sum of all distances d_i divided by the area of the window; in formula:

$$q = \frac{\sum_i d_i}{XT} \tag{2.75}$$

Discussion of this definition:

- The dimension is correct, it is: 1/time.
- Define $x_1 = x_0 + \delta x$ and let δx, that is the height of the window, approach 0. Then the distances d_i become approximately the same for all vehicles; Edie approaches the number entering the window through the lower boundary (n) multiplied by δx and in the denominator $X = \delta x$. Consequently (2.75) approaches n/T, the 'ordinary' (local) intensity.
- Above the rectangle has been shrunken into a cross-section. One can also let it shrink into a moment. Define $t_1 = t_b + \delta t$ and let δt, that is the width of the window, approach 0. Then the distances the vehicles cover in this window are $v_i \delta t$ and the denominator becomes $X\delta t$. Hence eq. (2.75) approaches $\sum_i v_i/X$. This seems a strange definition of intensity but it is a consistent one. It is in fact not more artificial that the local density derived from q/u, which is equal to $\sum_i (1/v_i)/T$.

In analogy with the definition for the intensity, the definition for the density becomes:

$$k = \frac{\sum_i r_i}{XT} \tag{2.76}$$

Deduce yourself that if one lets t_e approach t_b, i.e. letting the window approach a vertical line, the definition of k transforms into the ordinary (instantaneous) definition. Also investigate what happens if one lets the window approach a horizontal line.

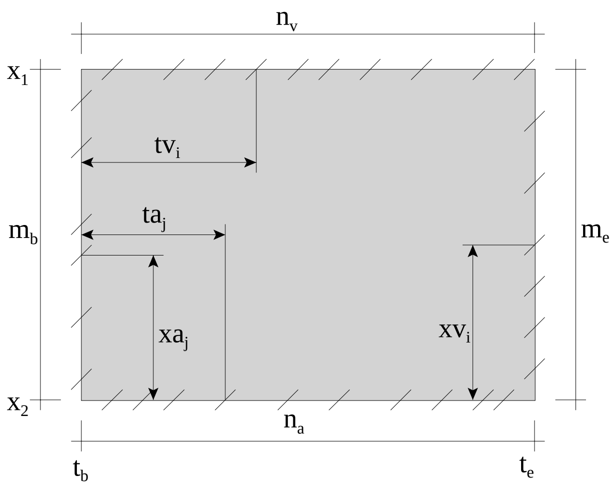

Figure 2.23: Determination of q and k for a time-space region

Finally, the mean speed for a time-space region is defined as:

$$u = \frac{\sum_i d_i}{\sum_i r_i} \tag{2.77}$$

Consequently with these definitions $q = ku$ is valid by definition.

The variable $\sum_i d_i$ is called the 'production' of the vehicles in the time-space region, and $\sum_i r_i$ the total travel time. Customary units are: vehicle-kilometre [veh km] and vehicle-hour [veh h].

Remark 18 *The definitions do not require the window to be a rectangle; they are applicable for an arbitrary closed surface in the time-space plane.*

2.7.1 Calulating the generalised q and k

To determine the generalised intensity and density for a time-space region, only the vehicle coordinates at the border of the area and not the trajectories are needed.

Fig. 2.23 is the time-space area of Fig. 2.22 with only:

- n_a: number of arrivals at cross-section x_0
- n_v: number of departures at cross-section x_1
- m_b: number of positions at moment t_b
- m_e: number of positions at moment t_e

This lead to

$$\text{Production}: \; K = \sum_i x_i = n_v X - \sum_{j=1}^{m_b} xa_j + \sum_{i=1}^{m_e} xv_i \tag{2.78}$$

$$\text{Travel time}: \; R = \sum_i t_i = m_e T - \sum_{j=1}^{n_a} ta_j + \sum_{i=1}^{n_v} tv_i \tag{2.79}$$

The required data (the moments ta_j and tv_j , and their numbers n_a and n_v) are simple to determine at the cross-sections x_0 and x_1.

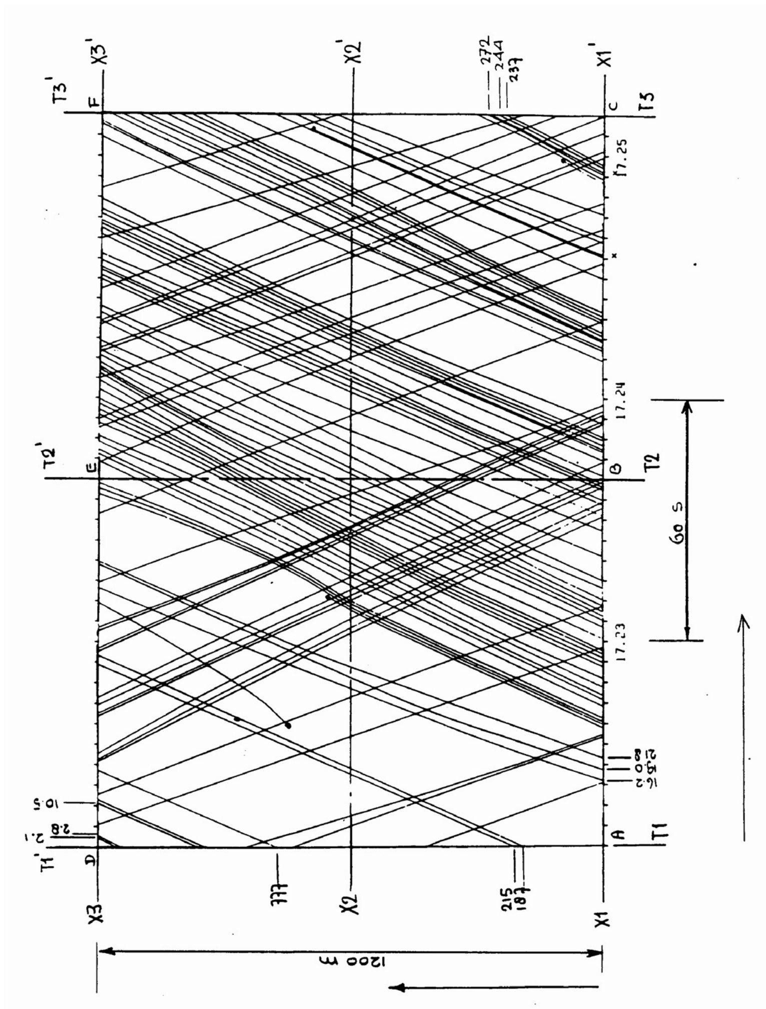

Figure 2.24: Trajectories plot

Determining xa_j and xv_i is clearly more difficult. A possible solution is to identify the vehicles when they pass cross-sections x_0 and x_1, and determine their position at the moments t_b and t_e by means of interpolation. This interpolation becomes more accurate by installing more detector stations between x_0 and x_1 and registering vehicles with an identification.

In principle, vehicles do not appear or disappear unnoticed, and consequently the relation

$$n_a + m_b = n_v + m_e \tag{2.80}$$

holds. This can be used as a check on observations.

Example 19 *Macroscopic characteristics have been derived from a plot of trajectories, depicted in figure 2.24. This is a representation of traffic operation on a two-lane road with vehicles going in both directions. The length of the road section is 1200 m and the considered period lasts 3 minutes. We will consider the traffic in the direction from X1 to X3. Intensities and densities can be determined straightforwardly by counting trajectories crossing lines 'position = a constant' and crossing lines 'time = a constant' respectively. See Table 1 and Table 2 for results.*

Inspecting the numbers shows that especially the density varies substantially between the different moments. The approach of determining intensity and density for a time-space region has as its goal to reduce these strong fluctuations. The moments and positions at the borders of the time-space region have been measured from the plot. The region considered is the rectangle

Cross-section	Period	Period (min)	#veh	q (veh/h)
$X - X1'$	T1 $- T2$	3	65	1300
$X2 - X2'$	$T1 - T3$	3	61	1220
$X3 - X3'$	$T1 - T3$	3	52	1040

Table 2.1: Intensity for 3 cross-sections

Moment	Road section	Length (m)	#veh	k (veh/km)
$T1 - T1'$	X1 $- X3$	1200	7	5.8
$T2 - T2'$	$X1 - X3$	1200	27	22.5
$T3 - T3'$	$X1 - X3$	1200	8	15.0

Table 2.2: Density for 3 cross-sections

ACFD. Result:

$$K = \sum_i x_i = 70.6 \ \textit{veh-km} \quad \textit{and} \quad R = \sum_i t_i \tag{2.81}$$

$$q = \frac{K}{XT} = \frac{70.6}{1.2 \cdot (3/60)} = 1177 \ veh/h \tag{2.82}$$

$$k = \frac{R}{XT} = \frac{0.888}{1.2 \cdot (3/60)} = 14.8 \ veh/km \tag{2.83}$$

$$u = \frac{q}{k} = \frac{1177}{14.8} = 79.5 \ km/h \tag{2.84}$$

Check:

$$n_a + m_b = n_v + m_e \qquad 65 + 7 = 52 + 18 \tag{2.85}$$

2.8 Measuring methods

Real traffic data is one of the most important elements in analysing and improving traffic systems. Different systems have been proposed to collect the different microscopic and macroscopic quantities. This section describes some systems to measure the different traffic flow variables described in this chapter. It is beyond the scope of this course to go into futher detail.

2.8.1 Passage times, time headways and intensity

The *local* traffic variables, passage times, time headways and intensity, can be measured with relative little effort. They are generally determined at a cross-section for each passing vehicle or averaged during a period of time. Detection can be achieved by so-called *infrastructure-based detectors*, which operate at a fixed location. However, moving observer methods are also possible. Let us briefly discuss some well know detection methods.

- *Manual, using a form and a stopwatch.* Advantages of this simple method is that one can use a detailed registration of vehicle types; a disadvantage is the (labour) costs, and the inaccuracies – especially with respect to the microscopic variables.

XT space	n_a	n_v	m_b	m_e	q (veh/h)	k (veh/km)	u (km/h)
ABED	34	14	7	27	938	12.4	75.4
BCEF	31	38	27	18	1415	17.2	82.2

Table 2.3: Time-space q, k and u for smaller areas

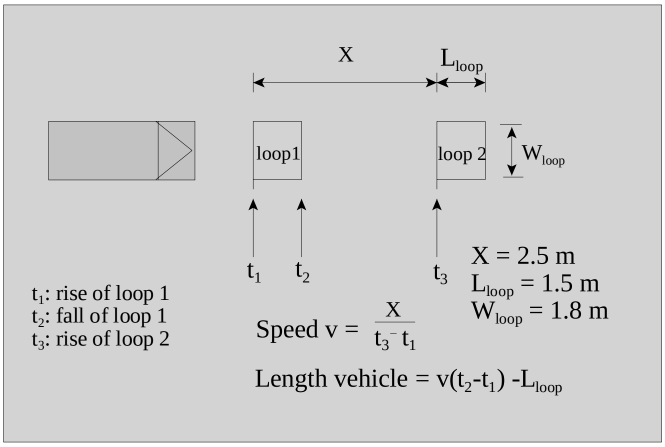

Figure 2.25: Loop configuration as applied on Dutch motorways

- *Pneumatic tube.* The pneumatic tube is one of the oldest traffic detectors available, and consist of a hollow rubber tube that sends a pressure wave when a vehicle runs over it. This device can accurately determine passage times and time headways. A disadvantage is the relatively short time the tubes survive. Note that a tube counts not vehicles but axles. The conversion from axles to vehicles introduces an extra error, especially when many articulate and non-articulate trucks are present in the flow.

- *Induction loops.* The passing of vehicles (iron) changes the magnetic field of the loop that is buried in the road surface. These changes can be detected. The induced voltage shows alternately a sharp rise and fall, which correspond approximately to the passing of the front of the vehicle over the front of the loop and the rear of the vehicle over the rear of the loop; see Fig.2.25. If one installs two loops behind each other on a lane (a 'trap'), then one can determine for each vehicle: passing moment; speed; (electrical) vehicle length. On Dutch motorways it is customary to use two loop detectors; see Fig. 2.25, implying that in principle, individual vehicle variables are available. The figure shows the relation between the individual speed v, the (electrThdsdsdsic) vehicle length, and the different time instant t_1, t_2 and t_3. Fig. 2.26 show how these relations are determined. Also note that there is redundancy in the data, i.e. not all avaiable information is used. The Dutch Ministry only stores the 1-minute or 5-minute *arithmetic* averages (counts and average speeds). This in fact poses another problem, since the arithmetic mean speed may not be an accurate approximation of the instantaneous speed, especially during congested conditions. The Dutch Ministry does however provide the possibility to temporarily use the induction loop for research purposes, and storing individual passing times for each vehicle.
  In the USA often one loop is used because that is cheaper. In the latter case intensity and the so called occupancy rate (see section 2.9) can be measured, but speed information is lacking unless one makes assumptions on the average lengths of the vehicles. In other countries (e.g. France), a combination of single-loop and double-loop detectors is used.
  Non-ideal behaviour of the equipment and of the vehicles (think about the effect of a lane change near the loops) lead to errors in the measured variables. Let us mention that a possible error in speed of 5 % and in vehicle length of 15 %. Especially at low speeds, i.e. under congestion, large errors in the intensities can occur too.

- *Infrared detector.* This family of traffic detectors detects passing vehicles when a beam of light is interrupted. This system thus provides individual passing times, headways and intensities.

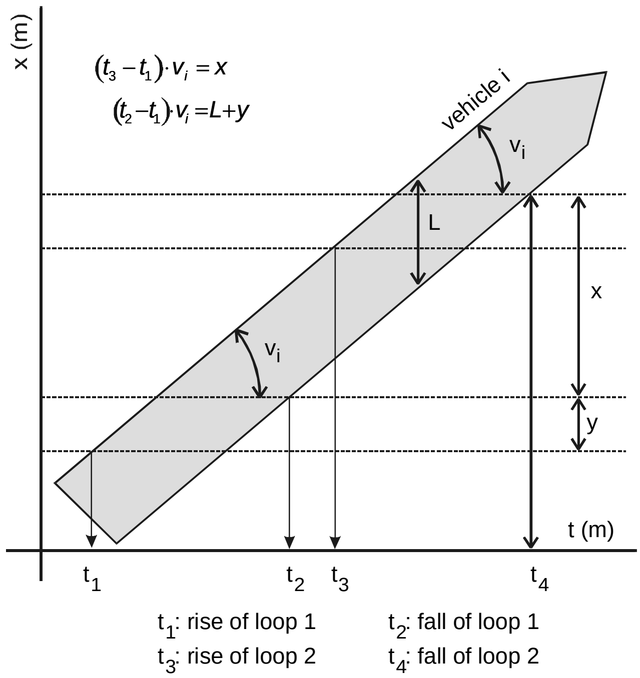

Figure 2.26: Calculating the individual vehicle speed v_i and the vehicle length L from double induction loops.

- *Instrumented vehicles.* Moving observer method, see Sec. 2.10.

2.8.2 Distance headways and density

In principle, distance headways and density can only be measured by means of photos or video recordings from a high vantage point (remote-sensing, see Sec. 2.8.3), that reveal the positions of the vehicles for a particular region X at certain time instants t. That is, the instantaneous variables in general require *non-infrastructure based* detection techniques. However, these method are mostly expensive and therefore seldom used (only for research purposes). Usually distance headways and densities are calculated from time headways, speeds and intensities, or from the the occupancy rate (see section 2.9) is used instead.

2.8.3 Individual speeds, instantaneous and local mean speed

Invidual vehicle speeds can be determined by infrastructure-based detectors and by non-infrastructure based detection methods (probes, remote-sensing). In practice individual vehicle speeds are measured and averaged. Keep in mind that the correct way to average the individual speed collected at a cross-section is using harmonic averages!

- *Radar speedometers at a cross-section.* This method is mostly used for enforcement and incidentally for research.

- *Induction loops or double pneumatic tubes* (discussed earlier under Intensity)

- *Registration of licence number (or other particular characteristics) of a vehicle / vehicle recognition.* Registration / identification at two cross-sections can be used to determine the mean travel speed over the road section in between. Manual registration and matching of the data from both cross-sections is expensive. The use of video and automatic processing of the data is currently deployed for enforcement (e.g. motorway A13 near Kleinpolderplein, Rotterdam).

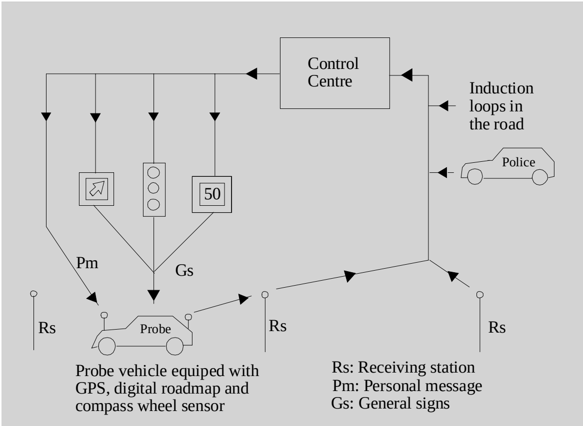

Figure 2.27: Lay-out of probe vehicle system concept

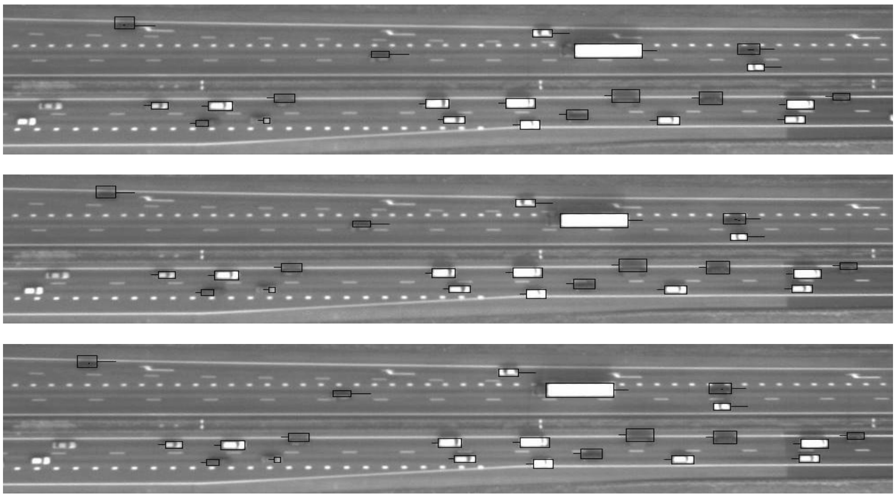

Figure 2.28: Results of vehicle detection and tracking (following objects for consecutive frames) obtained from a helicopter.

- *Probe vehicles.* Probe vehicles are ordinary vehicles with equipment that measures and also emits to receiving stations, connected to a control centre, regularly variables such as position, speed, and travel time over the last (say) 1 km. This results in a very special sample of traffic flow data and its use is under study. Combined with data collected from road stations (with e.g. induction loops), probe data seems very promising for monitoring the state of the traffic operation over a network (where is the congestion?; how severe is it?; how is the situation on the secondary network?; etc); see Fig. 2.27 and [57].

- *Remote-sensing techniques.* Remote sensing roughly pertains to all data collection methods that are done from a distance, e.g. from a plane, a helicopter, or a satelite. In illustration, the Transportation and Traffic Engineering Section of the TU Delft has developed a data collection system where the traffic flow was observed from a helicopter using a digital camera. Using special software, the vehicles where detected and tracked from the footage. By doing so, the trajectories of the vehicles could be determined (and thus also the speeds). It is obvious that this technique is only applicable for special studies, since it is both costly and time consuming. Alternative approaches consist of mounting the system on a high building. Fig. 2.28 shows an example of vehicle detection and tracking.

2.9 Occupancy rate

Earlier it was stated that in the Netherlands induction loops at motorways are installed in pairs but that in the US often one loop stations are used. The use of one loop has brought about the introduction of the characteristic occupancy rate. A vehicle passing over a loop temporarily 'occupies' it, approximately from the moment the front of the car is at the beginning of the loop until its rear is at the end of the loop. Note this individual occupancy period as b_i . For a period T, in which n vehicles pass, the occupancy rate β is defined as:

$$\beta = \frac{1}{T}\sum_{i=1}^{n} b_i \tag{2.86}$$

Per vehicle with length L_i and speed v_i : $b_i = (L_i + L_{loop})/v_i$. Suppose vehicles are approximately of the same length, then b_i is proportional to $1/v_i$, and β can be rewritten to:

$$\beta = \frac{1}{T}\sum_{i=1}^{n}\frac{L_i + L_{loop}}{v_i} = \frac{L_{tot}}{T}\sum_{i=1}^{n}\frac{1}{v_i} = L_{tot}\frac{n}{T}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{v_i} = L_{tot}\frac{q}{u_M} = L_{tot}k \tag{2.87}$$

It appears that β is proportional to the (calculated local) density k if vehicle lengths are equal. However, if a mix of passenger cars and trucks is present, then the meaning of β is less obvious.

Conclusion 20 *If one uses a single induction loop, β is a meaningful characteristic, but if one uses two loops it is better to calculate density k from intensity q and the harmonic mean of the local speeds, u_M.*

2.10 Moving observer method

It is possible to measure macroscopic characteristics of traffic flow by means of so called moving observers, i.e. by measuring certain variables form vehicles driving with the flow. The method is suitable for determining characteristics over a larger area, e.g. over a string of links of a network.

The method produces only meaningful results if the flow does not change drastically. When investigating a road section with an important intersection, where the intensity and/or the vehicle composition change substantially, it is logical to end the measuring section at the intersection.

Moving observer (MO). The MO drives in direction 1 and observes:

- t_1 = travel time of MO over the section
- n_1 = number of oncoming vehicles
- m_1 = number of passive overtakings (MO is being overtaken) minus number of active overtakings (MO overtakes other vehicles)

Remark 21 *If the MO has a lower speed than the stream, then m_1 is positive. m_1 is so to speak the number of vehicles a MO counts, just as a standing observer. The difference is that vehicles passing the observer in the negative direction (vehicles the MO overtakes) get a minus sign.*

The MO drives in direction 2 and observes n_2, m_2, t_2

- q_i = intensity in direction i; $i = 1, 2$
- k_i = density in direction i

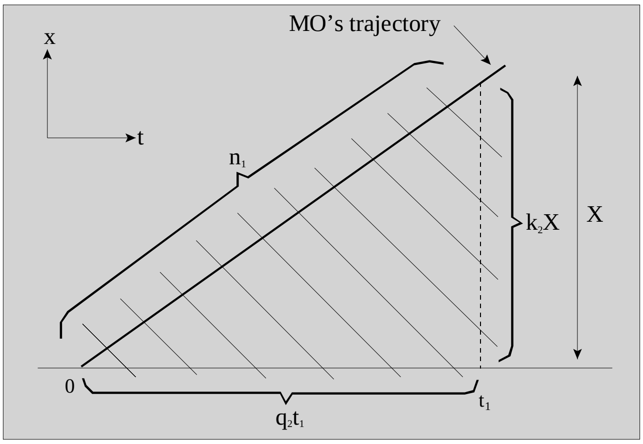

Figure 2.29: Determination number of oncoming vehicles

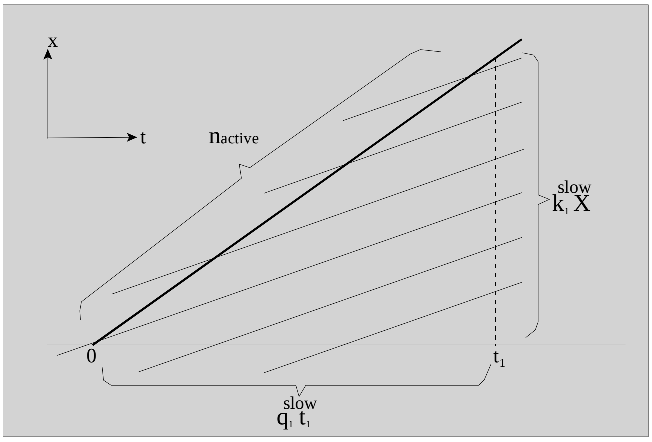

Figure 2.30: Determination of n_{active}

- u_i = mean speed of direction i
- X = section length

Derivation. Number of opposing vehicles

$$n_1 = q_2 t_1 + k_2 X \tag{2.88}$$

And for direction 2

$$n_2 = q_1 t_2 + k_1 X \tag{2.89}$$

To derive the number of overtakings, divide the stream into a part slow than the MO and a part faster than the MO. From Fig. 2.30 follows

$$n_{active} = k_1^{slow} X - q_1^{slow} t_1 \tag{2.90}$$

and Fig. 2.31

$$n_{passive} = q_1^{fast} t_1 - k_1^{fast} X \tag{2.91}$$

$$\begin{aligned} m_1 &= n_{passive} - n_{active} = \left(q_1^{fast} + q_1^{slow}\right) t_1 - \left(k_1^{fast} + k_1^{slow}\right) X && (2.92) \\ &= q_1 t_1 - k_1 X && (2.93) \end{aligned}$$

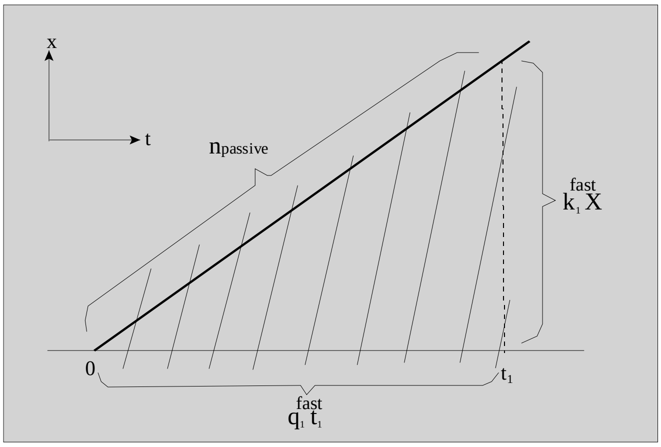

Figure 2.31: Determination of $n_{passive}$

For direction 2 we have the corresponding formula of (2.93)

$$m_2 = q_2 t_2 - k_2 X \tag{2.94}$$

We have 4 equations for the unknowns q_1, q_2, k_1 and k_2. In matrix form

$$\begin{pmatrix} 0 & t_1 & 0 & X \\ t_2 & 0 & X & 0 \\ t_1 & 0 & -X & 0 \\ 0 & t_2 & 0 & -X \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \\ m_1 \\ m_2 \end{pmatrix} \tag{2.95}$$

The equations can be solved easily. Sum row 1 and row 4 of matrix (2.95) $\Rightarrow (t_1 + t_2)\, q_2 = n_1 + m_2$

$$q_2 = \frac{n_1 + m_2}{t_1 + t_2} \tag{2.96}$$

Symmetry $\Rightarrow$

$$q_1 = \frac{n_2 + m_1}{t_1 + t_2} \tag{2.97}$$

Now take row (3) from matrix (2.95) $\Rightarrow$

$$k_1 = \frac{q_1 t_1 - m_1}{X} \tag{2.98}$$

Usually there is more interest in mean speed than in density

$$u_1 = \frac{q_1}{k_1} = \frac{q_1 X}{q_1 t_1 - m_1} = \frac{X}{t_1 - m_1/q_1} \tag{2.99}$$

From the latter equation it can be seen that if the MO drives with the average speed of the stream, i.e. $m_1 = 0$, then u_1 reduces to X/t_1, i.e. the mean speed of the MO. If $m_1 > 0$, i.e. MO is slower than the stream, then $u_1 > X/t_1$.

Symmetry $\Rightarrow$

$$u_2 = \frac{X}{t_2 - m_2/q_2} \tag{2.100}$$

Example 22 *Students of the IHE practice the MO-method annually on the Erasmusweg in Den Haag. This road section has two lanes in each direction and is rather narrow with vehicles parked on the roadway. The road section considered has a length of 1.8 km and contains a signalised intersection where intensity changes little. Table 2.4 shows the variation of results of four MO-teams, with and without the time the MO stopped at the intersection included. Fig. 2.32 shows a pretty large variation of the calculated mean speed over 48 MO measurements. In general the MO method requires many repetitions of the trip. The main advantage of the method is that no special equipment is needed. It can be a suitable method for pilot studies.*

MO-team	With stopped time			Without stopped time		
	Q (veh/h)	U (km/h)	K (veh/km)	Q (veh/h)	U (km/h)	K (veh/km)
1	713	44	17	835	51	17
2	630	45	14	715	51	14
3	652	44	15	754	51	15
4	746	49	15	822	53	16

Table 2.4: Results MO-method

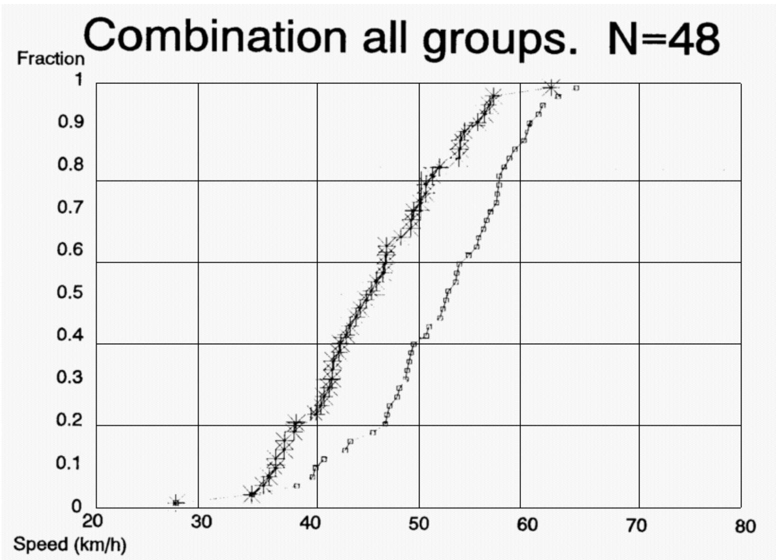

Figure 2.32: Distribution of mean speeds from 48 moving observer trips. Left: 'with stopped time'; Right: 'without stopped time'

2.10.1 Speed distribution as seen by a moving observer

Suppose we have a one directional traffic flow in a homogeneous and stationary state with space speed probability density function (p.d.f.) $f_M(v)$. A moving observer (MO) drives in this traffic flow with a constant speed v_0 and observes the absolute value of the speed of all vehicles that overtake, either active or passive. An example could be a police car measuring these speeds automatically with a radar speedometer. In fact such a speedometer measures the relative speed but we assume that the speed of the police car is added to it.

The question is: how does the p.d.f. of these observed speeds relate to the space p.d.f. of speeds. One point is obvious: the p.d.f. is zero at $v = v_0$ because a vehicle does not encounter vehicles with the same speed, but the rest of the p.d.f. has to be derived.

Proof. Consider a 'class' of vehicles with speed $v < v_0$, with density $k(v)$, and intensity $q(v)$. During period T, covering a distance $X = v_0T$, the MO makes n_1 active overtakings (see Fig. 2.30).

$$n_1(v) = k(v)X - q(v)T = k(v)v_0T - k(v)vT = k(v)T(v_0 - v) \text{ for } v < v_0 \tag{2.101}$$

With respect to vehicles with a speed $v > v_0$, the MO makes n_2 passive overtakings (see Fig. 2.31).

$$n_2(v) = q(v)T - k(v)X = k(v)vT - k(v)v_0T = k(v)T(v - v_0) \text{ for } v > v_0 \tag{2.102}$$

Eqns.(2.101) and (2.102) can be combined into:

$$n(v) = k(v)T|v - v_0| \tag{2.103}$$

The vehicle density as function of speed equals the total vehicle density times the p.d.f. of space speeds ⇒

$$n(v) = kf_M(v)T|v - v_0| \tag{2.104}$$

Hence a MO 'sees' vehicles with speed v with an intensity:

$$n(v)/T = kfM(v)|v - v_0| \tag{2.105}$$

The p.d.f. of speeds the MO sees, is the term (2.105) divided by the total intensity, which is the integral of (2.105) over speed:

$$g(v) = \frac{f_M(v)|v - v_0|}{\int f_M(v)|v - v_0|dv} \tag{2.106}$$

■

Example 23 *Fig. 2.33 depicts: 1) $f_M(v)$, a Gaussian p.d.f. with mean of 115 km/h and standard deviation of 15 km/h; (the values are from a measurement at a Dutch freeway with a speed limit of 120 km/h); 2) the corresponding $f_L(v)$; which is not a Gaussian p.d.f. but deviates only little from it at the free flow situation; 3) the p.d.f. as seen from a MO with a speed of 120 km/h. From this figure can be understood that fast drivers often claim that so many drivers drive even faster. The vehicles that overtake them have a mean speed of 136 km/h, whereas the mean speed of all speeds > 120 km/h is 130 km/h.*

2.10.2 Fraction of drivers that is speeding

Just like the p.d.f., the fraction of drivers that is speeding, i.e., has a speed higher than the speed limit, depends on the method of observation

- At a spot:

$$F_L(v) = \int_{v=v_{\text{lim}}}^{\infty} f_L(v)dv \tag{2.107}$$

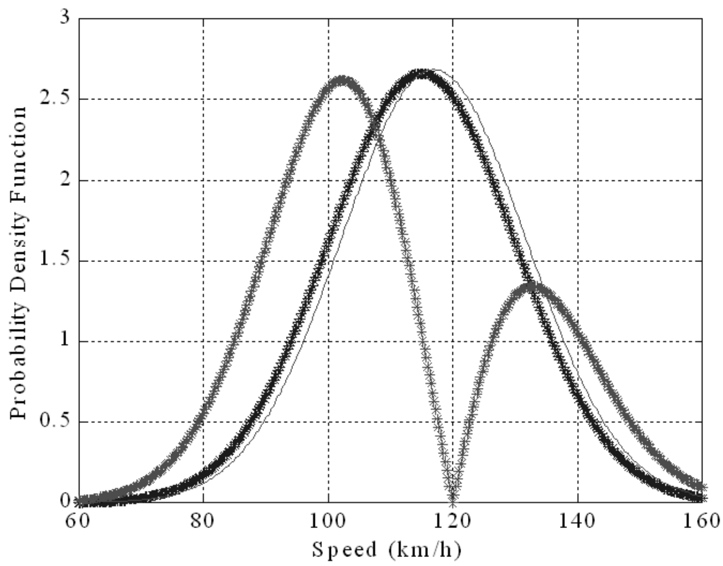

Figure 2.33: Probability density function for speed: local, instantaneous and as seen by moving observer.

- From an aerial photo (instantaneous observation):

$$F_M(v) = \int_{v=v_{\lim}}^{\infty} f_M(v)dv \tag{2.108}$$

- From a MO:

$$F_{MO}(v) = \int_{v=v_{\lim}}^{\infty} g(v)dv \tag{2.109}$$

Example 24 *Assuming a speed limit of 120 km/h, u_M = 115 km/h and STD = 15 km/h (as in the example discussed before) outcomes are: $F_L = 0.40$ and $F_M = 0.34$. Thus we see again that a sample of speeds at a spot has relatively more high speeds than a sample at a moment. The fraction FMO is a function of the speed; at speed 0 it equals the local value and at high speeds it approaches the space value (and reaches it at speed infinity). In between the function reaches a maximum value (0.64 in the example). Hence, naive policeman might think the fraction of speed offenders is very high.*

- The fraction F_q = (relative intensity) / (absolute intensity); it reaches a minimum if v_0 equals the space mean speed of the flow.
- The fraction $F_{MO}F_q$ i.e. the number of offenders a MO sees, divided by the absolute intensity. This function shows clearly that from a MO one sees a high fraction of offenders but the absolute number is relatively small. In fact the highest number of speed tickets (per time) can be obtained with a stationary observation method.

2.11 Summary of main definitions and terminology

Tab. 2.5 gives an overview of the definitions of intensity, density and mean speed. Tab. 2.6 shows an overview of the Dutch, German and English terminology used in literature.

	Spot observations	Aerial photos	Definitions Edie	Continuous variables
Variable	Location (x); period (T)	Section (X); moment (t)	Section (X); period (t)	Location (x); moment (t)
Intensity q (veh/h)	$q = \frac{n}{T}$	$q = u_M$	$q = \frac{\sum_i x_i}{XT}$	$q = \frac{\partial \tilde{N}(x,t)}{\partial t}$
Density k (veh/km)	$k = \frac{q}{u_M}$	$k = \frac{m}{X}$	$k = \frac{\sum_i t_i}{XT}$	$k = -\frac{\partial \tilde{N}(x,t)}{\partial x}$
Mean speed u_M (km/h)	$u_M = \frac{n}{\sum_i v_i}$	$u_M = \frac{\sum_i v_i}{m}$	$u_M = \frac{q}{k}$	$u(x,t) = \frac{q(x,t)}{k(x,t)}$

Table 2.5: Overview definitions of intensity, density and mean speed

Terms in notes	Dutch	German	English
intensity[3]	intensiteit	Stärke für $T = 0$; Intensität für $T > 0$	flow (UK); volume (USA); rate of flow for $T < 1h$ (USA)
density	dichtheid	Dichte für $X = 0$; Konzentration für $X > 0$	density (concentration)
local mean speed	gemiddelde lokale snelheid	Mitterwert lokaler Geschwindigkeiten	time-mean speed or average spot speed
instantaneous mean speed	gemiddelde momentane snelheid	Mittelwert momentaner Geschwindigkeiten	space-mean speed
stationary	stationair	stationär über Zeit	stationary, steady, time-homogeneous
homogeneous	homogeen	stationär über Weg	homogeneous

Table 2.6: Overview of terminology used in literature