Chapter 3

Microscopic flow characteristics

Contents of the chapter. This chapter describes several (distribution) models that describe (the relation between) different microscopic traffic variables (such as time headways, distance headways, etc.). We briefly discuss stochastic arrival processes, headway distributions, and individual speed distributions.

List of symbols

k	veh	arrivals
μ	-	mean
σ	-	standard deviation
h	s	time headway
q	veh/s	traffic intensity
$P(h)$	-	probability distribution function
$p(h)$	-	probability density function of time headway
ϕ	-	fraction of constrained vehicles in composite headway models

3.1 Arrival processes

Intensity varies more, also if the traffic flow is stationary, if the period over which it is observed is smaller. This is due too the fact that the passing of vehicles at a cross-section is to a certain extent a matter of chance. When using shorter observation periods, the smoothing of these random fluctuations reduces. For some practical problems it is useful to know the probability distribution of the number of vehicles that arrive in a short time interval.

Example 25 *The length of an extra lane for left-turning vehicles at an intersection must be chosen so large that in most cases there will be enough space for all vehicles, to prevent blocking of through going vehicles. In such a case it is not a good design practice to take account of the mean value or the 50 percentile. It is better to choose e.g. a 95 percentile, implying that only in 1 out of 20 cases the length is not sufficient.*

Several models describe the distribution of the number of vehicles arriving in a given, relatively short, period. We will discuss three of them.

Poisson-process

If drivers have a lot of freedom they will behave independently of each other. This implies that the passing of a cross-section is a pure random phenomenon. In general this will be the case if there is relatively little traffic present (a small intensity and density) and if there are no upstream 'disturbances', such as signalised intersections, that result in a special ordering of the

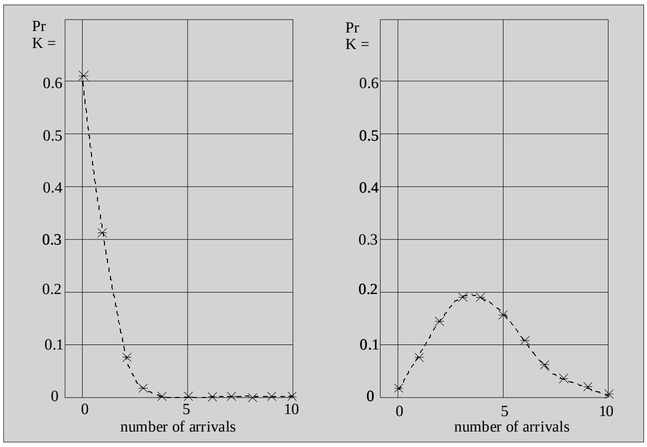

Figure 3.1: Probability function of Poisson with $\Delta t = 20s$; left $q = 90veh/h$ ($\mu = 0.5$); right $q = 720veh/h$ ($\mu = 4$).

vehicles in the stream. The conditions mentioned lead to the so called *Poisson-process.* The probability function of the number of arrivals k is given by:

$$\Pr\{K = k\} = \frac{\mu^k e^{-\mu}}{k!} \quad \text{for} \quad k = 0, 1, 2, ... \tag{3.1}$$

This probability function has only one parameter, the mean μ. Note that the parameter μ need not to be an integer. Fig. 3.1 shows an example of the Poisson probability function at two intensities.

Example 26 *Intensity = 400 veh/h and one wants to know the number of arrivals in a period of 30 s. Then we have: μ = (400/3600)·30 = 3.33 veh.*

A special property of the Poisson-distribution is that the *variance equals the mean.* This property can be used to test in a simple way if the Poisson-process is a suitable model: from a series of observations, one can estimate the mean and the variance. If the variance over the mean does not differ too much from 1, then it is likely that Poisson is an adequate model.

Binomial-process

When the intensity of a traffic stream increases, more and more vehicles form *platoons* (clusters, groups), and the Poisson-process is no longer valid. A model that is suitable for this situation is the model of a so-called *Binomial-process*, with probability function:

$$\Pr\{K = k\} = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \quad \text{for} \quad k = 0, 1, ..., n \tag{3.2}$$

Note that in this case, the variance over the mean is smaller than 1. The binomial distribution describes the number of 'successes' in n independent trials, at which the probability of success per trial equals p. Unfortunately this background does not help to understand why it fits the arrival process considered here.

Negative-Binomial-process

In discussing the Poisson-process it has been mentioned that it does not fit the situation *down-stream of a signalised intersection.* When this is the case, one can state that high and low intensities follow each other. The variance of the number of arrivals then becomes relatively

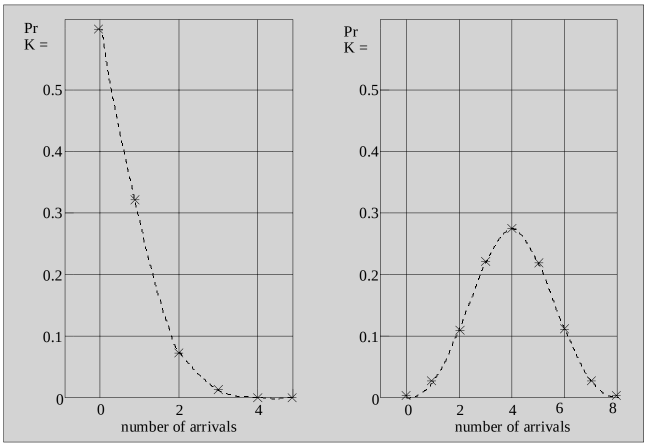

Figure 3.2: Binomial probability function at $\Delta t = 20s$; left $q = 90veh/h$ ($p = 0.1$, $n = 5$); right $q = 720veh/h$ ($p = 0.8,\ n = 8$).

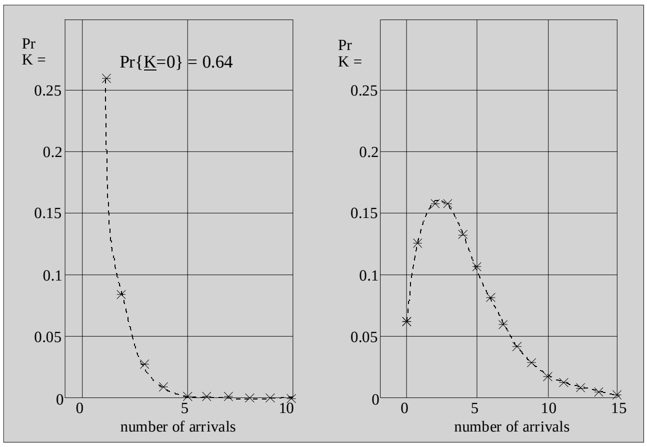

Figure 3.3:

large, leading to variance over mean being larger than 1. In that case the model of a *Negative-Binomial-process* is adequate :

$$\Pr\{K = k\} = \left(\begin{array}{c} k+n-1 \\ k \end{array} \right) p^n (1-p)^k \quad \text{for} \quad k = 0, 1, 2, ... \tag{3.3}$$

As with the Binomial-process a traffic flow interpretation of the Negative-Binomial-process is lacking.

Distribution	**Mean**	**Variance**	**St.dev**	**Relative St.dev.**
Poisson	μ	μ	$\sqrt{\mu}$	$1/\sqrt{\mu}$
Binomial	np	$np\,(1-p)$	$\sqrt{np\,(1-p)}$	$\sqrt{(1-p)\,/np}$
Negative binomial	$n\,(1-p)\,/p$	$n\,(1-p)\,/p^2$	$p\sqrt{n\,(1-p)}$	$1/\sqrt{n\,(1-p)}$

Table 3.1: Mean, variance, standard deviation, and, relative standard deviation of the three arrival distributions

3.1.1 Formulae for parameters, probability terms, and parameter estimation

Recursion formulae

These formulae allow to calculate in a simple way the probability of event $\{K = k\}$ from the probability of event $\{K = k - 1\}$.

- Poisson: $\Pr\{K = 0\} = e^{-\mu} \rightarrow \Pr\{K = k\} = \frac{\mu}{k} \Pr\{K = k - 1\}$ for $k = 1, 2, 3, ..$
- Binomial: $\Pr\{K = 0\} = (1 - p)^n \rightarrow \Pr\{K = k\} = \frac{p}{1-p} \frac{n-k+1}{k} \Pr\{K = k - 1\}$
- Neg. Bin.: $\Pr\{K = 0\} = p^n \rightarrow \Pr\{K = k\} = (1 - p) \frac{n+k-1}{k} \Pr\{K = k - 1\}$

Estimation of parameters

From observations one calculates the sample mean m and the sample variance s^2. From these two parameters follow the estimations of the parameters of the three probability functions:

- Poisson: $\hat{\mu} = m$
- Binomial: $\hat{p} = 1 - s^2/m$ and $\hat{n} = m^2/\left(m - s^2\right)$
- Neg. Bin.: $\hat{p} = m/s^2$ and $\hat{n} = m^2/\left(s^2 - m\right)$

3.1.2 Applications

Length of left-turn lane

One has to determine the length of a lane for left turning vehicles. In the peak hour the intensity of the left turning vehicles equals 360 veh/h and the period they are confronted with red light is 50 s. Suppose the goal is to guarantee that in 95 % of the cycles the length of the lane is sufficient. In 50 s will arrive on the average (50/3600) x 360 = 5 veh. It will make a difference which model one uses.

- Poisson: parameter $\mu = 5$ and all probabilities can be easily calculated.
- Binomial with $s^2 = 2.5$: $\hat{p} = (5 - 2.5)/5 = 0.5$ and $\hat{n} = 25/(5 - 2.5) = 10$
- Negative-Binomial with $s^2 = 7.5$: $\hat{p} = 5/7.5 = 0.667$ and $\hat{n} = 25/(7.5 - 5) = 10$

With these parameter values the probability functions and the distributions can be calculated; see Fig. 3.4.

From the distributions can be seen that the Binomial model has the least extreme values; it has a fixed upper limit of 10, which has a probability of only 0.001. Of the two other models the Negative-Binomial has the longest tail. From the graph can be read (rounded values) that the 95-percentile of Binomial equals 7; of Poisson it is 9 and of Negative Binomial it is 10. These differences are not large but on the other hand one extra car requires 7 to 8 extra m of space.

Probe-vehicles

Suppose one has to deduce the state of the traffic stream at a 2 km long road section from probe-vehicles that broadcast their position and mean speed over the last km. It is known that on the average 10 probe vehicles pass per hour over the section considered. The aim is to have fresh information about the traffic flow state every 6 minutes. The question is whether this is possible.

It is likely that probe vehicles behave independently, which implies the validity of the Poisson-distribution. Per 6 minutes the average number of probes equals: $(6/60) \cdot 10 = 1$

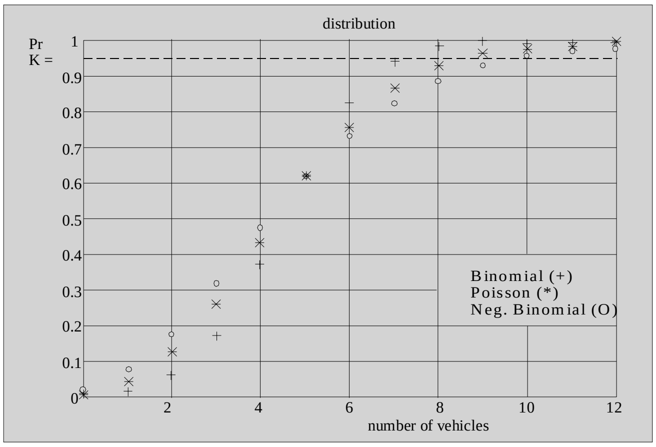

Figure 3.4: Distributions Binomial, Poisson and Negative-Binomial

probe. The probability of 0 probes in 6 minutes then equals: $e^{-1} = 0.37$ and this seems to be much too large for a reliable system.

The requirement is chosen less severe and one decides to update every 20 min. In that case

$$\hat{\mu} = (20/60) \cdot 10 = 3.33$$

and the probability of zero probes = exp[-3.33] = 0.036. This might be an acceptable probability of failure.

Remark 27 *In practice the most interesting point is probably whether the road section is congested. At congestion the probes will stay longer at the section and the frequency of their messages will increase.*

3.1.3 Choice of the appropriate model by using statistical testing

Earlier it has been mentioned that the quotient of sample variance and sample mean is a suitable criterion to decide which of the three distributions discussed, is a suitable model. Instead of using the rule-of-thumb, one may use statistical tests, such as the Chi-square test, to make a better founded choice.

3.2 Headway distributions

3.2.1 Distribution of headways and the Poisson arrival process

From the Poisson arrival process, one can derived a specific distribution of the headways: for a Poisson-process the number of arrivals k in an interval of length T has the probability function:

$$\Pr\{K = k\} = \frac{\mu^k e^{-k}}{k!} = \frac{(qT)^k e^{-qT}}{k!} \tag{3.4}$$

For $k = 0$ follows: $\Pr\{K = 0\} = e^{-qT}$. Remember: $0! = 1$ and $x^0 = 1$ for any x. $\Pr\{K = 0\}$ is the probability that zero vehicles arrive in a period T. This event can also be described as: the headway is larger than T. Consequently $\Pr\{K = 0\}$ equals the probability that a headway is larger than T. In different terms (replace period T by headway h)

$$\Pr\{H > h\} = S(h) = e^{-qh} \tag{3.5}$$

This is the so called *survival probability* or the *survival function S(h)*: the probability that a stochastic variable H is larger than a given value h. On the other hand, the complement is the

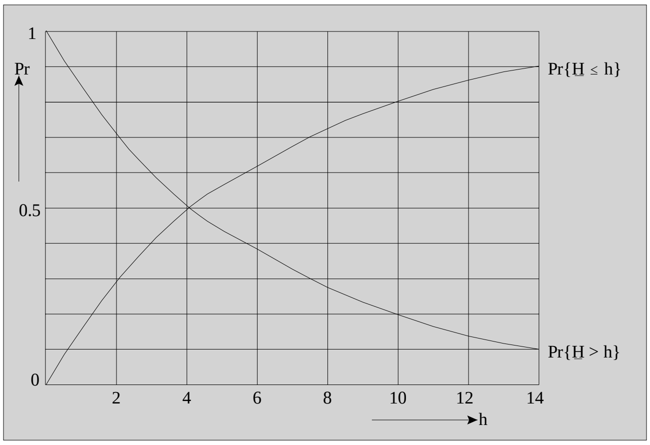

Figure 3.5: Survival function $S(h)$ and distribution function $P(h)$ of an exponential distribution

probability that a stochastic variable is smaller than (inclusive equal to) a given value, the so called distribution function. Consequently, the distribution function of headways corresponding to the Poisson-process is:

$$\Pr\{H \leq h\} = P(h) = 1 - e^{-qh} \tag{3.6}$$

This is the so called *exponential distribution function.* The survival function and the distribution function are depicted in Fig. 3.5 for an intensity $q = 600veh/h = 1/6veh/s$. Consequently the mean headway in this case is 6 s.

From the distribution function $P(h) = \Pr\{H \leq h\}$ follows the probability density function (p.d.f.) by differentiation:

$$p(h) = \frac{d}{dh}P(h) = qe^{-qh} \tag{3.7}$$

The mean value becomes (by definition):

$$\mu = \int_0^\infty hp(h)\,dh = \int_0^\infty hqe^{-qh}dh = \frac{1}{q} \tag{3.8}$$

We see that the mean (gross) headway μ equals the inverse of the intensity q. The variance σ^2 of the headways is:

$$\sigma^2 = \int_0^\infty (h-\mu)^2\,p(h)\,dh = \frac{1}{q^2} \tag{3.9}$$

Consequently the variation coefficient, i.e. the standard deviation divided by the mean, equals 1. Note the difference:

- Poisson distribution $\rightarrow$ Var/mean $= 1$
- Exponential distribution $\rightarrow$ St.dev./mean $= 1$

3.2.2 Use of the headway distribution for analysis of crossing a street

The sequence of headways is governed by a random process. When this process is analysed two points need to be well distinguished. Namely, this process can be considered:

- per event;
- as a process in time.

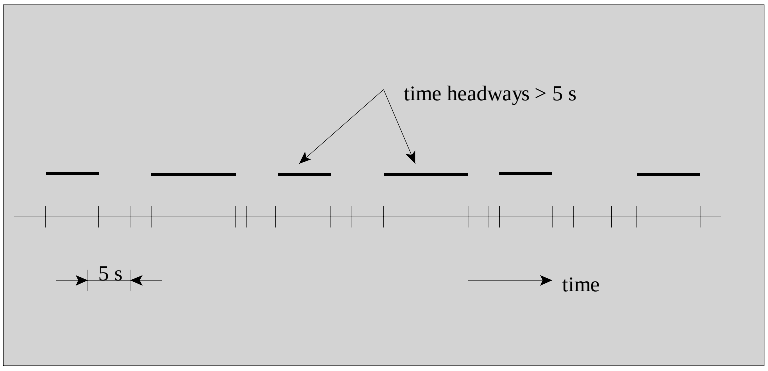

Figure 3.6: Realisation of sequence of gaps with gaps > 5s marked

Example 28 *Suppose only headways of 5 and 10 s are present, each with a probability of 0.5, and also assume that the headways are (stochastically) independent. This means that the events 'headway = 5 s' and 'headway = 10 s' occur on the average with the same frequency and in an arbitrary (random) sequence. Now consider a time axis on which the passage moments of vehicles are indicated, and by that also the gross headways. Then, on average, 5/15 = 1/3 of the time will be 'occupied' by headways of 5 s and 2/3 of the time by headways of 10 s. If one picks an arbitrary moment of the time axis, the probability to hit a 5 s headway is 1/3. This is not equal to the probability that a headway of 5 s occurs, which equals 0.5.*

Suppose a pedestrian wants to cross a road and needs a headway of at least x seconds. *Terminology*: in the context of crossing a street the headway is called a *gap*; the traffic flow offers gaps to the pedestrian, which he/she can either accept or reject (more about gap-acceptance will be discussed in Chapter 12).

The first idea could be that the crossing possibilities are determined by the *frequency of the gaps* larger than x s. However, it is more relevant to consider the *fraction of time* $G_1(x)$ for which the gaps are larger than x seconds? To analyse this, we start from a realisation of the gap offering process on a time axis (see Fig. 3.6). The required fraction G_1 for a given realisation of the process equals the sum of all gaps $> x$ divided by the sum of all gaps for (= period considered).

$$G_1(x) = \frac{\sum h_i | h_i > x}{\sum h_i} = \frac{\frac{1}{n}\sum h_i | h_i > x}{\frac{1}{n}\sum h_i} \tag{3.10}$$

In the nominator of Eq. (3.10) all gaps $> x$ s are summed and in the denominator all complete gaps are summed. If we consider many repetitions of this process, the required fraction $G_1(x)$ is a (mathematical) expectation:

$$G_1(x) = \frac{\int_x^\infty hp(h)\,dh}{\int_0^\infty hp(h)\,dh} = q\int_x^\infty hp(h)\,dh \tag{3.11}$$

The simplification in (3.11) is possible using: mean gap $= 1/q$. For an *exponential distribution* with probability density function $p(h) = e^{-qh}$, one can derive:

$$G_1(x) = e^{-qx}(1 + qx) \tag{3.12}$$

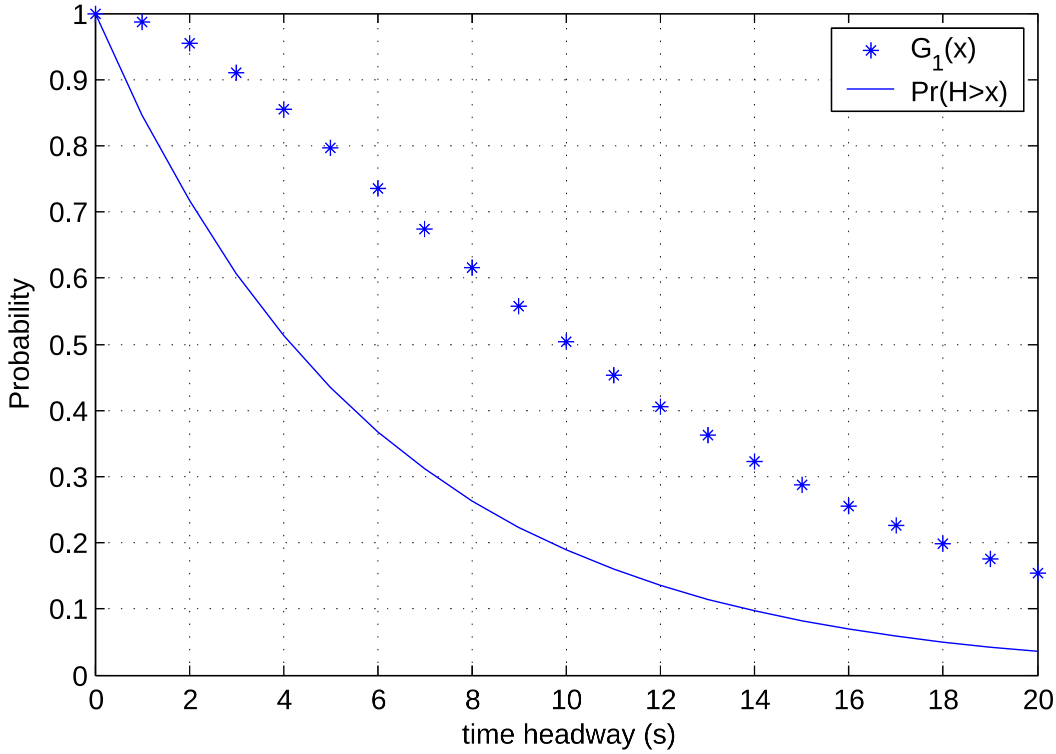

Figure 3.7: Survival function $S(x) = \Pr\{H > x\}$ and fraction of time that $H > x = G_1(x)$ for an exponential distribution of gaps at intensity $q = 600veh/h$.

Proof.

$$
\begin{aligned}
q\int_x^\infty hqe^{-qh}dh &= q^2\int_x^\infty he^{-qh}dh = q^2\int_x^\infty \frac{h}{-q}d\left(e^{-qh}\right) && (3.13)\\
&= -q\left(he^{-qh}\Big|_x^\infty - \int_x^\infty e^{-qh}dh\right) && (3.14)\\
&= -q\left(he^{-qh}\Big|_x^\infty + \frac{1}{q}e^{-qh}\Big|_x^\infty\right) && (3.15)\\
&= qxe^{-qx} + e^{-qx} = e^{-qx}(1+qx) && (3.16)
\end{aligned}
$$

■

From the illustration in Fig. 3.7 follows that the fraction of time a gap is larger than x, is larger than the frequency a gap larger than x occurs.

Fraction of time a 'rest gap' is larger than x

If we analyse the crossing process more precisely, it turns out that for the crossing pedestrian, it is not sufficient to arrive in a gap larger than x s. Rather, he/she should arrive *before the moment the period until the next vehicle arrives, equals x s.* In other words, the so called *restgap* should be larger than x s. To calculate the fraction of time that a restgap is larger than x s, denoted as $G_2(x)$, we consider again a time axis with a realisation of the gap process; see Fig. 3.8.

$$
\begin{aligned}
G_2(x) &= \frac{\sum (h_i - x)\,|h_i > x}{\sum h_i} = \frac{\frac{1}{n}\sum (h_i - x)\,|h_i > x}{\frac{1}{n}\sum h_i} && (3.17)\\
&= \frac{\int_x^\infty (h-x)\,p(h)\,dh}{\int_0^\infty (h-x)\,p(h)\,dh} = q\int_x^\infty (h-x)\,p(h)\,dh && (3.18)
\end{aligned}
$$

For an exponential gap distribution it can be derived that:

$$
G_2(x) = e^{-qx} \qquad (3.19)
$$

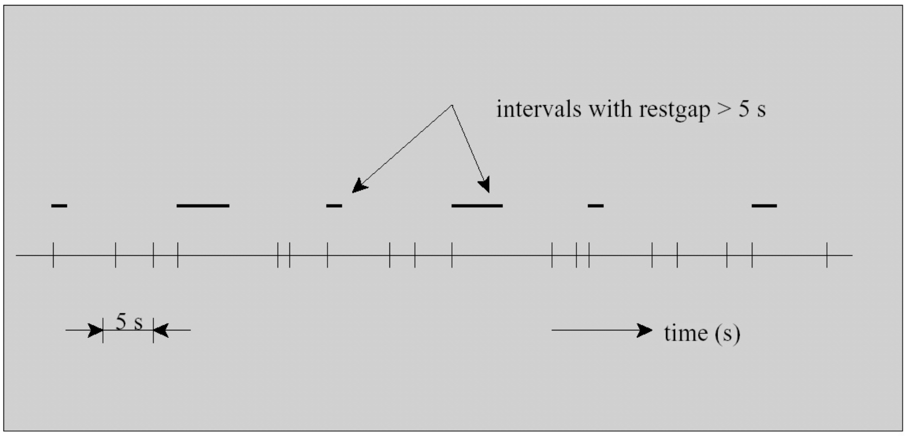

Figure 3.8: Realisation of sequence of gaps with intervals marked restgap > 5 s

Proof.

$$G_2(x) = q\int_x^\infty (h-x)\,qe^{-qh}dh = (\text{set } y = h-x)\ q\int_0^\infty yqe^{-q(y+x)}dy \tag{3.20}$$

$$= qe^{-qx}\int_0^\infty yqe^{-qy}dy = e^{-qx} \tag{3.21}$$

■

Remark 29 *Consequently the probability that a restgap is larger than x equals the probability that a headway, right away, is larger than x. This is a very special property of the exponential distribution, in fact a unique property, sometimes phrased as: 'the exponential process has no memory'.*

3.2.3 Use of headway distribution to calculate the waiting time or delay

If the crossing pedestrian arrives at a moment at which the restgap is too small, how long does he/she have to wait until the next vehicle arrives? Of course no more than x s, but more interesting is: how long will this take on average? This equals the mean of a drawing from a headway probability density $p(h)$ under the condition it is less than x:

$$\delta = \frac{\int_0^\infty hp(h)\,dh}{\int_0^x p(h)\,dh} \tag{3.22}$$

For an exponential distribution it can be derived that:

$$\delta = \frac{1}{q} - \frac{xe^{-qx}}{1-e^{-qx}} \tag{3.23}$$

Proof. Numerator:

$$\int_0^x hqe^{-qh}dh = (\text{partial integration}) \int_0^x -hd\left(e^{-qh}dh\right) \tag{3.24}$$

$$= -\left.he^{-qh}\right|_0^x + \int_0^x q\left(e^{-qh}\right)dh = -xe^{-qx} + \left(1-e^{-qx}\right)/q \tag{3.25}$$

Denominator:

$$\int_0^x qe^{-qh}dh = 1 - e^{-qx} \tag{3.26}$$

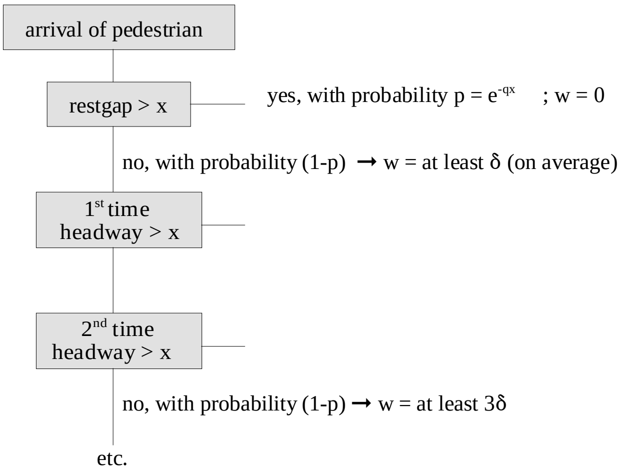

Figure 3.9: Flow diagram of waiting process

■

Next the pedestrian is offered the first full headway. If it is too small, then the waiting time is increased by δ; if it is large enough, then the waiting is over and the crossing can be carried out. The probabilities of both events are known. See Fig. 3.9 for a flow diagram of the process.

Every time a headway is too small, the waiting time is increased by (on average) δ, but the probability that this occurs becomes smaller and smaller; it decreases proportional to $(1-p)k$ where $k = 1, 2, 3, ...$ The expectation of the waiting time W, becomes (first term is p multiplied by zero):

$$E\left[W\right] = p \cdot 0 + (1-p)\, p\delta + (1-p)^2\, p2\delta + (1-p)^3\, p3\delta + ... \tag{3.27}$$

$$= p\delta \left[(1-p) + 2\,(1-p)^2 + 3\,(1-p)^3 + ...\right] \tag{3.28}$$

$$= \frac{\delta\,(1-p)}{p} \tag{3.29}$$

Proof. PM $r + r^2 + r^3 + ... = \frac{r}{1-r}$ (Geometric Series)
Differentiate and multiply by $r \rightarrow r\left(1 + 2r + 3r^2 + ...\right) = r\frac{d}{dr}\left(\frac{r}{1-r}\right)$
Thus $r + 2r + 3r + ... = \frac{r}{(1-r)^2}$ ■

For the exponential distribution we subsitute $p = e^{-qx}$, and δ according to Eq. (3.23). This leads to

$$E\left[W\right] = \left(e^{-qx} - qx - 1\right)/q \tag{3.30}$$

See Fig. 3.10 for an illustration with $E[W] = f(Q)$ for x = 4, 5, 6, 7 and 8 s.

3.2.4 Real data and the exponential distribution

In general the exponential distribution (ED) of the headways is a good description of reality at low intensities and unlimited or generous overtaking possibilities. If both conditions are not fulfilled, then there are interactions between the vehicles in the stream, leading to driving in platoons (also called clusters and groups).

In that case the ED is fitting reality badly; see Fig. 3.11. The minimum headways in a platoon are clearly larger than zero, whereas according to the ED the probability of extreme small headways is relatively large. The differences between the ED and reality have lead to the use of other headway models at higher intensities. We will first discuss some simple alternatives for the ED model in section 3.2.5. and after that more complex models, dividing the vehicles in two categories, in section 3.2.6.

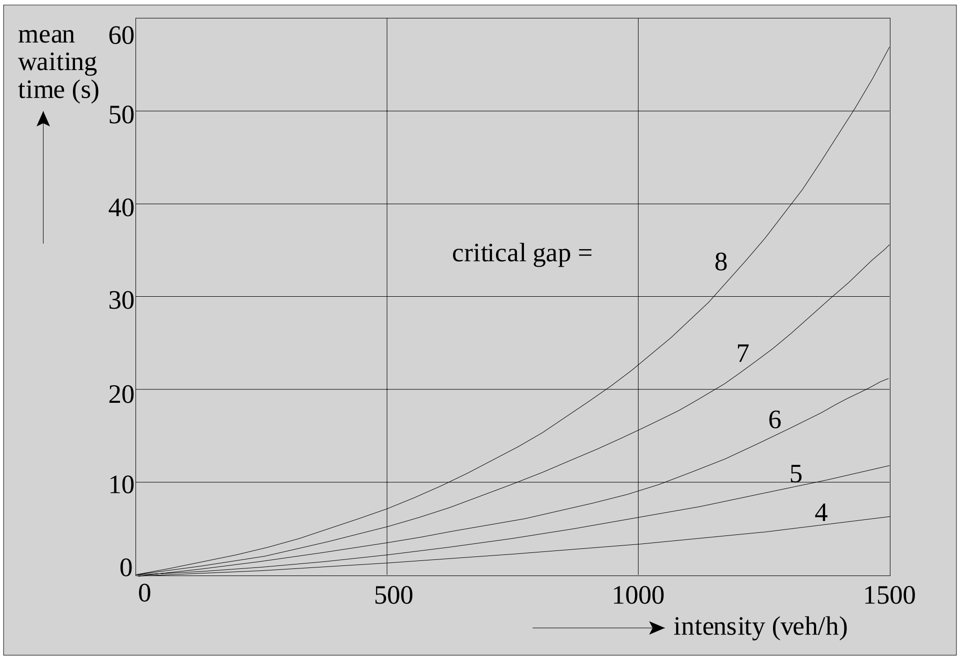

Figure 3.10: Mean delay as a function of the intensity at several critical gaps; gaps have an exponential distribution.

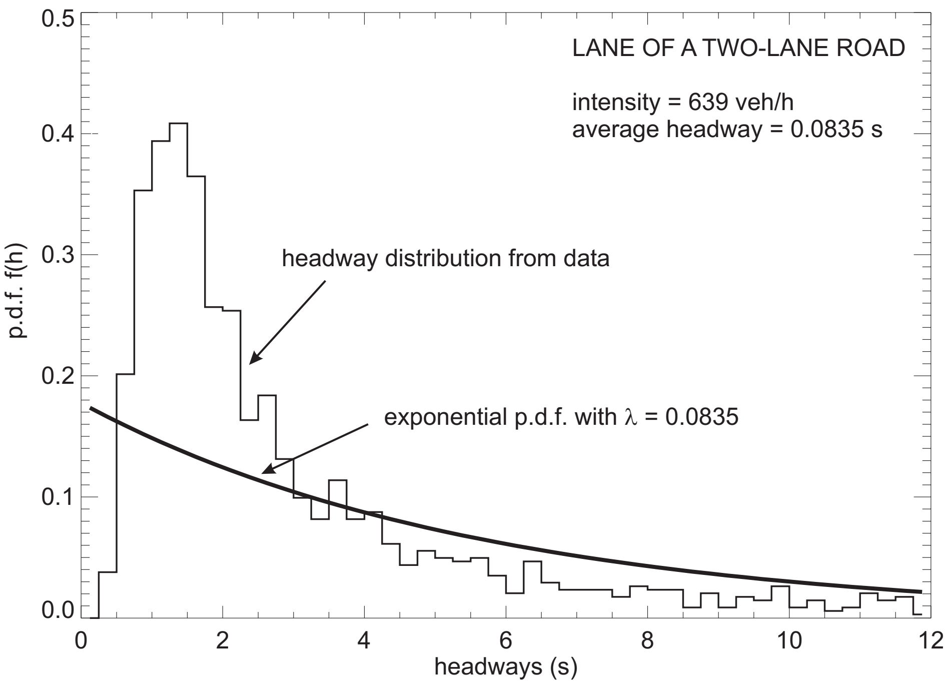

Figure 3.11: Histogram of observed headways compared to the exponential probability density function.

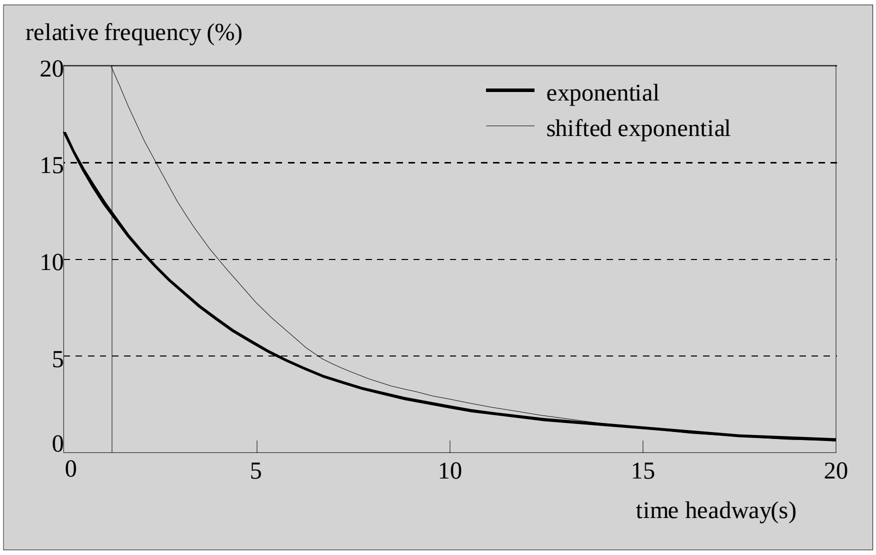

Figure 3.12: Non-shifted and shifted exponential probability density functions.

3.2.5 Alternatives for the exponential distribution

- Shifted exponential distribution

The shifted exponential distribution is characterised by a minimum headway h_m, leading to the distribution function:

$$\Pr\{H \leq h\} = 1 - e^{-\lambda(h-h_m)} \quad \text{with} \quad \lambda = \frac{q}{1-qh_m} \quad \text{for} \quad h \geq h_m \tag{3.31}$$

and the probability density function:

$$p(h) = 0 \quad \text{for} \quad h < h_m \tag{3.32}$$

$$p(h) = \lambda e^{-\lambda(h-h_m)} \quad \text{for} \quad h \geq h_m \tag{3.33}$$

The mean value is

$$\mu = h_m + \frac{1}{\lambda} \tag{3.34}$$

the variance equals:

$$\sigma^2 = \left(\frac{1}{\lambda}\right)^2 \tag{3.35}$$

and the variation coefficient

$$\frac{\sigma}{\mu} = \frac{1}{1+\lambda h_m} \tag{3.36}$$

which is always smaller than 1 (recall that for the ED, $\frac{\sigma}{\mu} = 1$) as long as $h_m > 0$. Fig. 3.12 depicts a non-shifted and a shifted probability density with the same mean value ($\mu = 6$ s) and a minimum headway h_m= 1 s.

In practice it is difficult to find a representative value for h_m ; moreover the abrupt transition at $h = h_m$ does not fit reality very well; see Fig. 3.11.

- Erlang distribution

A second alternative for the ED model is the Erlang distribution with a less abrupt function for small headways.

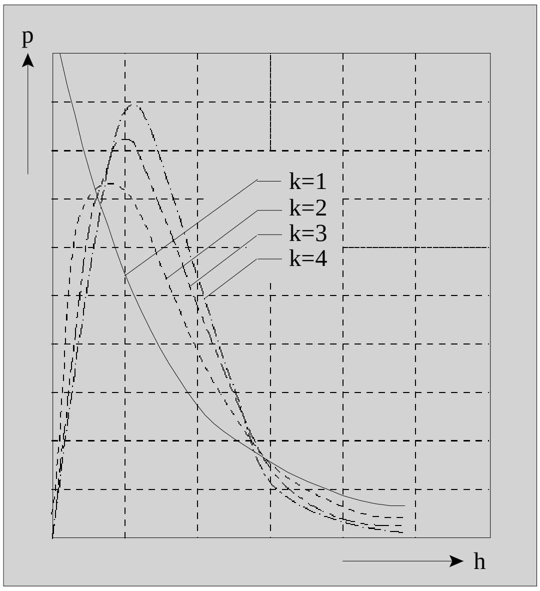

Figure 3.13: Erlang probability densities

The Erlang distribution function is defined by

$$\Pr\{H \leq h\} = 1 - e^{-kh/\mu} \sum_{i=0}^{k} \left(\frac{kh}{\mu}\right)^{i} \left(\frac{1}{i!}\right) \tag{3.37}$$

and the corresponding probability density:

$$p(h) = \frac{h^{k-1}}{(k-1)!} \left(\frac{k}{\mu}\right)^{k} e^{-kh/\mu} \tag{3.38}$$

Note that for $k = 1$ we have $p(h) = (1/\mu)e^{-h/\mu}$. Consequently the *exponential distribution is a special Erlang distribution* with parameter $k = 1$. For values of k larger than 1 the Erlang p.d.f. has a form that better suits histograms based on observed headways. This is illustrated by Fig. 3.13. which shows densities for $\mu = 6$ s and $k = 1$, 2, 3 and 4.The mean of an Erlang distribution $= \mu$, the variance equals μ^2/k and consequently the coefficient of variation equals $\frac{1}{\sqrt{k}}$ which is smaller or equal to 1.

- Lognormal distribution

A second alternative headway distribution is the lognormal distribution.

Definition 30 *A stochastic variable X has a lognormal distribution if the logarithm $ln(X)$ of the stochastic variable has a normal distribution.*

This can be applied as follows: if one has a set of observed headway h_i, then one can investigate whether $x_i = \log h_i$ has a normal distribution. If this is the case, then the headways themselves have a lognormal distribution. The p.d.f. of a lognormal distribution is:

$$p(h) = \frac{1}{h\sigma\sqrt{2\pi}} e^{-\frac{\ln^2\left(\frac{h}{\mu}\right)}{2\sigma^2}} \tag{3.39}$$

with mean μ^*

$$E(H) = \mu^* = \mu e^{\frac{1}{2}\sigma^2} \tag{3.40}$$

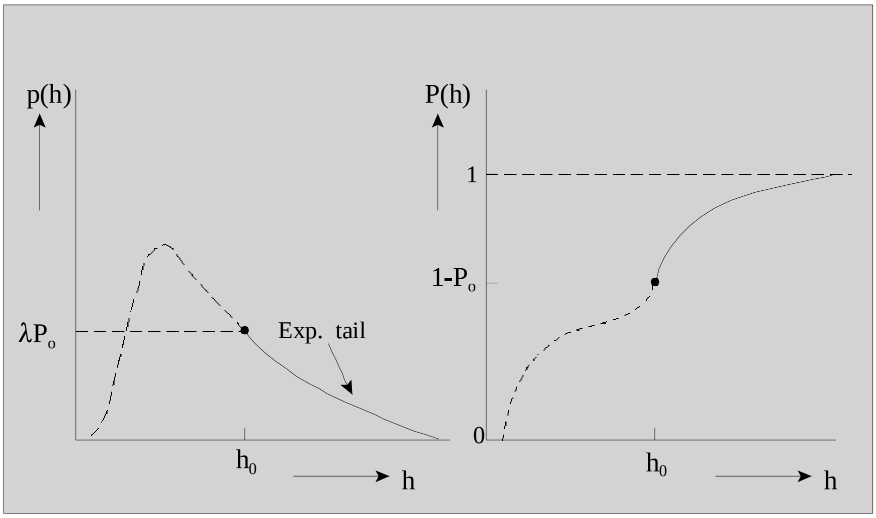

Figure 3.14: Probability density function and distribution of headways according to Exponential Tail Model

and variance $(\sigma^*)^2$

$$var\,(H) = (\sigma^*)^2 = \mu^2 e^{\sigma^2}\left(e^{\sigma^2} - 1\right) \tag{3.41}$$

The coefficient of variation thus equals

$$C_v = \frac{\sigma^*}{\mu^*} = \frac{\mu e^{\frac{1}{2}\sigma^2}\sqrt{\left(e^{\sigma^2} - 1\right)}}{\mu e^{\frac{1}{2}\sigma^2}} = \sqrt{e^{\sigma^2} - 1} \tag{3.42}$$

In contrast to the previous discussed distributions, the coefficient of variation of the lognormal distribution, can be smaller as well as larger than 1.

Remark 31 *The parameters of the p.d.f., μ and σ are not the mean and st. dev. of the lognormal variate* ***but of the corresponding normal variate****. If μ^* and $(\sigma^*)^2$ are given, μ and σ^2 follow from:*

$$\mu = \frac{\mu^*}{\sqrt{1 + C_v^2}} \quad \textit{and} \quad \sigma^2 = 2\ln\left(\sqrt{1 + C_v^2}\right) \tag{3.43}$$

In fact all alternatives for the exponential headway distribution discussed so far, are based on selecting a p.d.f. in which small headways have a small frequency and there is a long tail to the right.

Exponential tail model For estimating capacity we need a model for the p.d.f. of the empty zone. In contrast to that only large headways are relevant for the overtaking opportunities on two-lane roads. All headway less than, say 10 s, are too small for an overtaking and the precise distribution of those headways is not relevant. However, it is relevant how the frequency of large headways depends on intensity. In preparing backgrounds for new design guidelines the Transportation Laboratory of TU Delft has developed a headway model that neglects the small headways, the so called Exponential Tail Model (ETM). It has the following p.d.f.:

$$p\,(h) = \begin{cases} \lambda P_0 e^{-\lambda(h-h_0)} & h \geq h_0 \\ \text{not defined} & h < h_0 \end{cases} \tag{3.44}$$

with parameter P_0 is the probability that $h > h_0$. The survival function is:

$$\Pr\{H > h\} = P_0 e^{-\lambda(h-h_0)} \quad \text{for} \quad h > h_0 \tag{3.45}$$

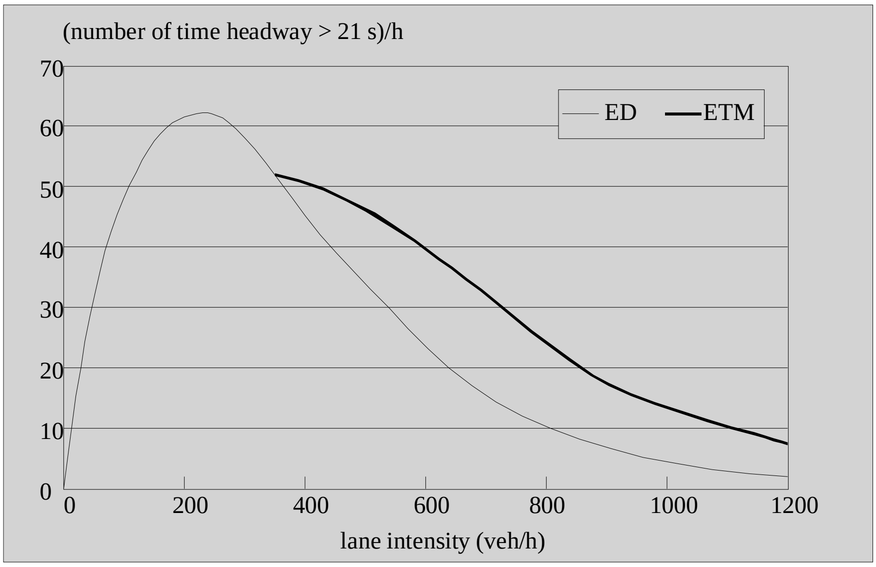

Figure 3.15: ‘Intensity’ of headways larger than 21 s as a function of lane intensity. according to exponential distributed headways (ED model) and according to exponential tail model (ETM).

The parameter h_0 has been set at 10 s, to be sure that the headways are from free drivers and consequently are very likely exponentially distributed. Based on observed headways at given intensities the two model parameters P_0 and λ can be estimated. In a second step of the analysis the relation between those parameters and intensity has been determined. The results were:

$$P_0 = -0.286 - 8.24q \tag{3.46}$$

$$\lambda = 0.0314 + 0.475q \tag{3.47}$$

Substitution of those two relations in eq. (3.45) leads to (for simplicity the logarithm of eq. (3.45) has been used):

$$\ln\left(\Pr\left\{H > h\right\}\right) = 0.028 - 3.49q - 0.0314h - 0.475qh \text{ (q in veh/s and h in s)} \tag{3.48}$$

Using this equation it is easy to calculate the probability of a certain (large) headway. In fact this is nearly as easy as when using the exponential distribution of headways, but it corresponds much better to reality.

See Fig. 3.15 for an illustration with a target headways of 21 s, which is a representative value for a gap that is acceptable for an overtaking at a two-lane road. It can be concluded that the new model offers much more overtaking opportunities than the exponential model.It should be added that the ETM is not a good description for intensities smaller than about 300 veh/h. However, for these intensities the simple exponential model is a good description. For both models it is a requirement that no substantial upstream disturbances (e.g. signalised intersections) have influenced the traffic stream.

3.2.6 Composite headway models: distributions with followers and free drivers

Comparisons of observed histograms of headways and the simple models discussed earlier have often lead to models badly fitting data. This has been an inspiration to develop models that have a stronger traffic behaviouristic background than the simple ones discussed above.

In so-called *composite headway models*, it is assumed that drivers that are obliged to follow the vehicle in front (because they cannot make an overtaking or a lane change), maintain a certain minimum headway (the so-called *empty zone* or *following headway*). They are in a constrained or following state. If they have a headway which is larger than their minimum, they are called free drivers. Driver-vehicle combinations are thus in either of two states: free or constrained. As a result, the p.d.f. $p(h)$ of the headways has two components: a fraction ϕ of constrained drivers with p.d.f. $p_{fol}(h)$ and a fraction $(1-\phi)$ with $p_{free}(h)$

$$p(h) = \phi p_{fol}(h) + (1-\phi)\, p_{free}(h) \tag{3.49}$$

The remaining problem is how to specify the p.d.f. of both free drivers and constrained drivers. Different approaches have been presented in the past. Let us discuss the most important ones.

Composite headway model of the Branston type

Several theoretical models have been used to determine expressions for the free and the constrained headway distributions. The approach discussed here is relatively straightforward and was first proposed by [27]. It is based on the idea that the total headway H can be written as the sum of two other random variates, the empty zone X and the free headway U. The distribution of the empty zone X is described by the p.d.f. $p_{fol}(x)$. The p.d.f. describing the free headway is given by the following expression

$$p_{free}(u) = \phi\delta(u) + (1-\phi)\,\lambda e^{-\lambda h} \tag{3.50}$$

where $\delta(u)$ is the δ-dirac function defined by $\int f(y)\,\delta(y-x)\,dy = f(x)$. The term $\lambda e^{-\lambda h}$ again denotes the exponential distribution, which is valid for independent arrivals (free flow). The rational behind this expression is that drivers which are constrained (with probability ϕ) have a free headway which is equal to zero. Since the total headway H is the sum of X and U, it's p.d.f. $p(h)$ can be determined by convolution

$$\begin{aligned} p(h) &= \int p_{fol}(s)\, p_{free}(h-s)\, ds && (3.51)\\ &= \int p_{fol}(s)\left(\phi\delta(h-s) + (1-\phi)\,\lambda e^{-\lambda(h-s)}\right) ds && (3.52)\\ &= \phi p_{fol}(h) + (1-\phi)\,\lambda e^{-\lambda h}\int_0^h p_{fol}(s)\, e^{-\lambda s} ds && (3.53)\end{aligned}$$

which thus implies

$$p_{fol}(h) = p_{fol}(h) \text{ and } p_{free}(h) = \lambda\int p_{fol}(h-s)\, e^{-\lambda h} ds \tag{3.54}$$

Model Cowan type III [13]

Cowan's model [13] assumes that all constrained drivers have the same headway h_0 and all free drivers have a headway that is distributed according to a shifted exponential distribution (shifted with h_0).

$$p(h) = \phi\delta(h-h_0) + (1-\phi)\, H(h-h_0)\,\lambda e^{-\lambda(h-h_0)} \tag{3.55}$$

$$P(h) = H(h-h_0)\left(\phi - (1-\phi)\left(1-e^{-\lambda(h-h_0)}\right)\right) \tag{3.56}$$

with delta-function δ and Heaviside-step-function (or unit step function) H; see Fig. 3.16

Although this model is certainly not completely realistic (constrained vehicles do not all have the same minimum headway), it turns out to be a good description for a class of practical problems. The value of the minimum headway h_0 is in the order of 2 to 3 s.

If intensity increases, the fraction of constrained vehicles ϕ has to increase too and in the limiting case, i.e. when intensity reaches capacity, ϕ should have the value 1. The capacity than equals $1/h_0$.

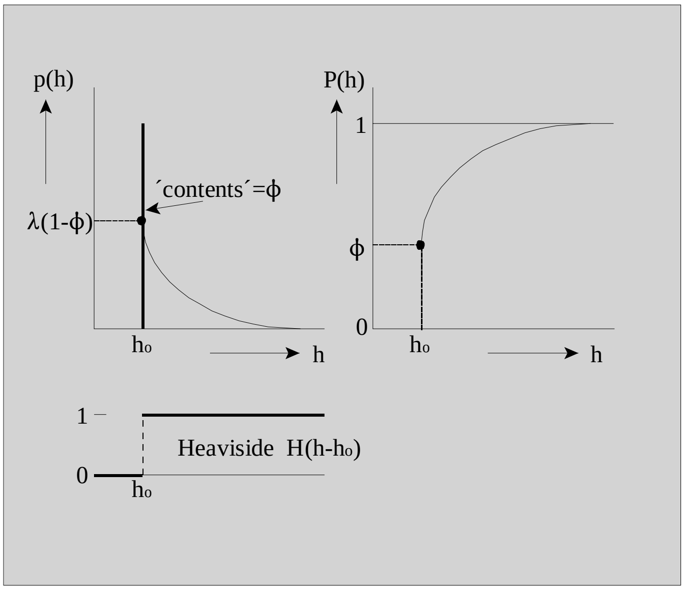

Figure 3.16: P.d.f. $p(h)$ and distribution function $P(h)$ of Cowan's model type III

If we assume further that with the increasing of ϕ, the parameter h_0 does not change, then it is possible to use this model for capacity estimation – as well as the other composite headway models presented in the remainder of this section. This could be carried out by estimating the parameter h_0 at traffic flow states at which capacity has not yet been reached.

Cowans model type III is not used for this purpose because it is too schematised for that procedure. We have only discussed this application to explain the principle of the capacity estimation method. The method is applied using the more detailed headway distribution model of Branston, discussed in the next session.

Model of Branston [7]

In Cowans model it was assumed that all constrained vehicles had the same headway. Branston assumes that the headways of constrained vehicles have their own p.d.f. and derives a model of which a main characteristic is that the p.d.f. of the free drivers is overlapping with the p.d.f. of the constrained drivers.

The model has a traffic flow background which was used in [7] to derive the model: consider a one-directional traffic stream on a roadway, which is so wide that practically speaking overtaking possibilities are unlimited. The headways, referring to the total roadway, then should have an exponential distribution.

Now consider this wide road narrowing to one lane; consequently overtaking possibilities become zero. Vehicles arriving at the transition sometimes have to wait until they can enter the one lane section. It is assumed that the *intensity is smaller than the capacity* of the one lane section, otherwise no equilibrium state is possible. For his situation one can derive (see [7]) a stochastic queueing model, that implies the following headway distribution:

$$p(h) = \phi p_{fol}(h) + (1-\phi)\lambda e^{-\lambda h} \int_0^h p_{fol}(\eta) e^{\lambda \eta} d\eta \tag{3.57}$$

with $p_{fol}(h)$ an arbitrary probability density function.

In this queueing model the parameter ϕ is not free but depends on $p_{fol}(h)$ and λ. Branstons trick has been to define ϕ as a free parameter, arguing that in this way a realistic headway distribution is described, because in reality overtaking possibilities are between zero and unlimited.

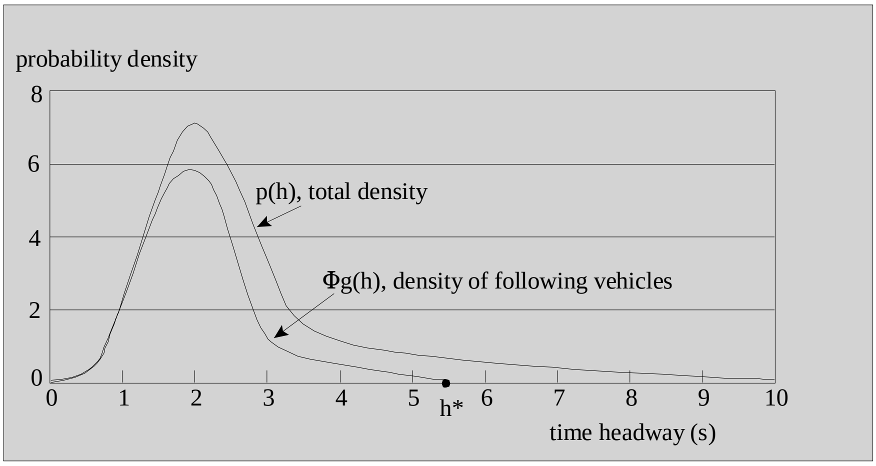

Figure 3.17: Branston's model for the probability density of headways

The p.d.f. of Branstons model has been depicted in Fig. 3.17. A special property of the model is that rather small headways can be free ones; this might occur in reality due to overtakings.

It is likely that $p_{fol}(h)$ has an upper limit, i.e. there are no constrained vehicles with a headway larger than, say, h^*; $p_{fol}(h) = 0$ for $h > h^*$. This implies that in that region the total p.d.f. has an exponential character, the so called exponential tail of the headway p.d.f. In fact parameter h^* is the most important parameter of the model because it is difficult to estimate.

Example 32 Estimating capacity with composite headway distribution models. *According to the principle explained in the preceding section the capacity can be estimated. To carry out this procedure the components of the headway model have to be estimated, based on a sample of observed headways. The procedure consists of two steps:*
Step 1. *Determining boundary value h^*. Parameter h^* is the headway where the exponential tail starts. To determine h^* the logarithm of the survival probabilities is plotted against the headway. As long as the tail is exponential, the result will be a straight line; because:*

$$P(h) = 1 - e^{-qh} \rightarrow S(h) = 1 - P(h) = e^{-qh} \rightarrow \log(S(h)) = -qh \tag{3.58}$$

Practical procedure: sort the headways in ascending order; result denoted as: $h_{(i)}$, $i = 1, 2, 3, ..., n$. Then $i/(n + l)$ is the estimated probability a headway is smaller than h_i and $1 - i/(n + 1)$ is the estimated probability a headway is larger than h_i; the survival probability. Consider: $y_{(i)}$ $= \ln[l - i/(n + l)]$ as a function of the headway h_i; see Fig. 3.18. The headway at which the function starts to deviate from a straight line is the boundary value h^*. Studies of the Transportation Laboratory of TU Delft, with data from two lane rural roads, have lead to values for h^* between 4 and 6 s. On motorways this value is smaller; drivers feel less constrained at the same small headway, because they have more lateral freedom to manoeuver.
Step 2: Estimation of the other parameters of Branston's model. *The other parameters ϕ, λ and $p_{fol}(h)$ can be estimated by new methods developed by [23] and [51]. Discussion of these methods falls outside the scope of this subject.*
Fig. 3.19 shows an example of the result of the method with data from a two-lane rural road. In general application of this method results in rather high capacity values. The conjecture is that drivers at an intensity of say, 1/2 capacity, have a different (a shorter) constrained headway than they have at capacity.

Buckley's model [10] One of the drawbacks of the Branston model is the fact that the total headway H is composed of the free headway U and the following headway X ($H = X + U$), where the former follows the exponential distribution. However, for independent arrivals - e.g. when

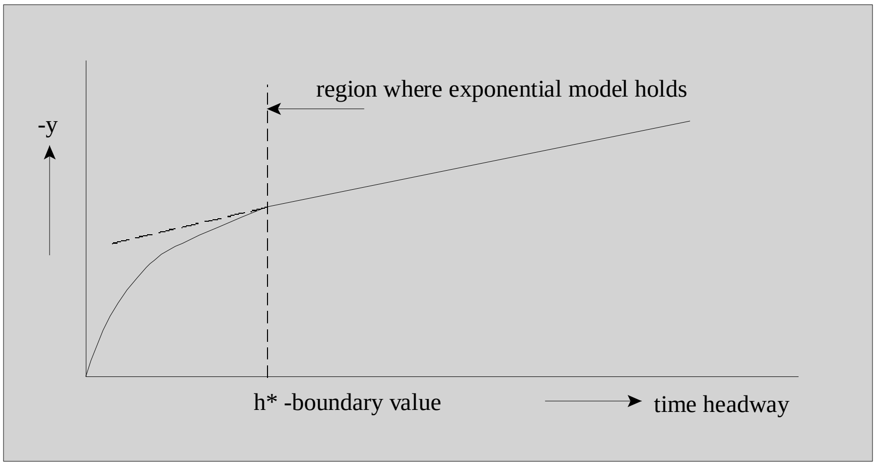

Figure 3.18: Method to determine headway value h^*

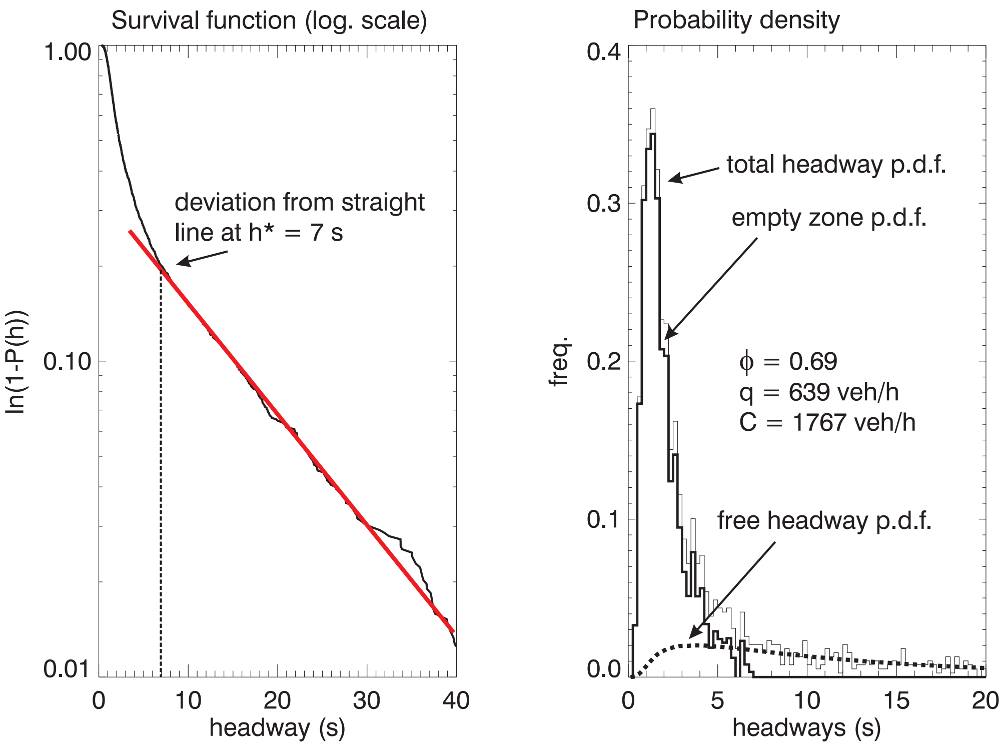

Figure 3.19: Results of application of capacity estimation.

traffic flow conditions are free - the *total headway* H would follow the exponential distribution, and drivers would not consider their empty zone. In the model of Buckley [10], this problem does not occur: drivers either follow at their respective empty zone X or at the free headway U, i.e. $H = \min\{U, X\}$. Let $p_{fol}(w)$ denote the probability density function of W. Let C denote the random variate describing the state of the vehicle. By convention, let $C = 1$ and $C = 2$ respectively denote unconstrained and constrained vehicles. The mathematical formulation of model is given by the following equation:

$$p(h) = \phi p_{fol}(h) + (1 - \phi)p_{free}(h) \tag{3.59}$$

where $p_{fol}(h)$ and $p_{free}(h)$ respectively denote the headway probability density functions of the followers (constrained vehicles) and the leaders (free driving vehicles); ϕ denotes the fraction of followers. Assuming no major downstream distributions, the headway distribution of the free drivers can be proven to be exponential in form. As a direct result, for sufficiently large headways we can write for $p(h)$

$$p(h) = (1 - \phi)p_{free}(h) = A\lambda \exp(-\lambda h) \text{ for } h > h^* \tag{3.60}$$

Here h^* is a threshold headway value corresponding to a separation beyond which there is no significant probability of interactions between vehicles, i.e. none of the vehicles is following; λ (arrival rate for free vehicles) and A (the so-called normalization constant, given by $A = \int_0^\infty \lambda \exp(-\lambda s)p_{fol}(s)ds$; see [11]) are parameters to be determined from headway data. For headway values $h < h^*$ the exponential form is no longer valid and must thus be corrected to take account of the following vehicles. This correction is effected *by removing from the exponential distribution the fraction of vehicles that have preferred following times larger than* t, since the assumption is that no vehicle will be found at less than its preferred following headway. This fraction $\pi(h)$ is given by

$$\pi(h) = \int_h^\infty p_{fol}(s)\,ds \tag{3.61}$$

Hence, we write

$$(1 - \phi)p_{free}(h) = A\lambda \exp(-\lambda h)[1 - \pi(h)] = A\lambda \exp(-\lambda h)\left[1 - \int_h^\infty p_{fol}(s)\,ds\right] \tag{3.62}$$

Using $p_{fol}(h) = \frac{p(h)-(1-\phi)p_{free}(h)}{\phi}$ we can easilty obtain for $p_{free}(h)$ the following *integral equation*

$$(1 - \phi)p_{free}(h) = \frac{A\lambda}{\phi} \exp(-\lambda h) \int_0^h [p(s) - (1 - \phi)p_{free}(s)]\,ds \tag{3.63}$$

The parameters A and λ can be evaluated from the observed headways in the range $h > h^*$, where Eq. (3.60) applies. Then the integral equation can be solved numerically subject to the constraint $\phi = \int_0^\infty g_1(s)\,ds$ to yield the quantity ϕ and the function $(1-\phi)p_{free}(h)$ in the range $h < h^*$.

3.3 Distance headway distributions

Most results discussed for time headways in the earlier sections are also valid, with some modifications, for distance headways. It is more easy to observe time headways than distance headways, for the same reason as it is more easy to observe intensity than density.

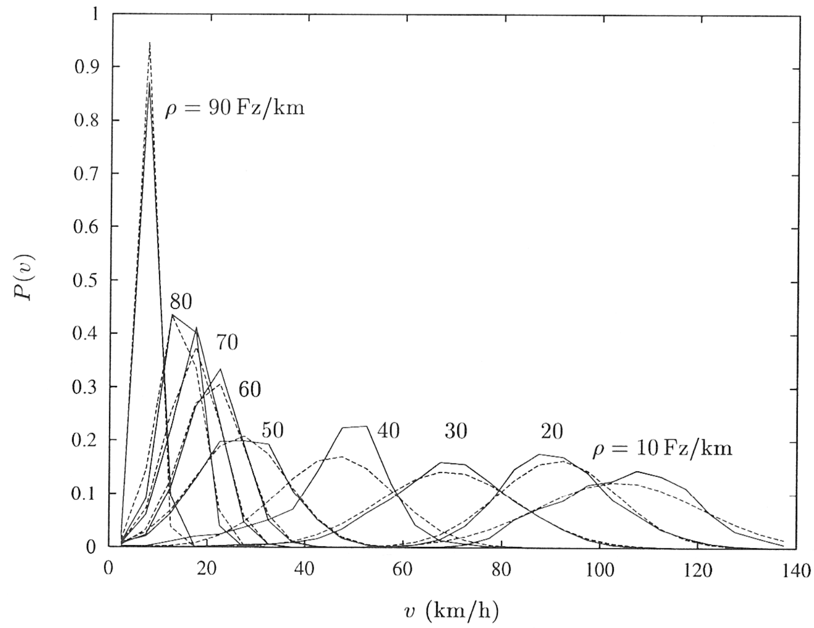

Figure 3.20: Probability density functions of local speeds collected at the A9 two-lane motorway in the Netherlands, for different density values. The distributions are compared to Normal probability density functions (dotted lines).

3.4 Individual vehicle speeds

Just as time and distance headways, speeds have a continuous distribution function. From observations and analysis it follows that speeds usually have a Normal (or Gaussian) distribution. That means the p.d.f., with parameters mean μ and standard deviation σ is:

$$p(v) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(v-\mu)^2}{2\sigma^2}} \tag{3.64}$$

If the histogram of observed speeds is clearly not symmetric, then the Lognormal distribution is usually a good alternative as a model for the speed distribution. As with the mean speed, that could be defined locally and instantaneously, one can consider the distribution of speeds locally and instantaneously.

3.5 Determination of number of overtakings

Vehicles on a road section usually have different speeds, which leads to faster ones catching up the slower ones, and a desire to carry out overtakings. One can calculate the number of desired overtakings from the quantity of traffic and the speed distribution. It depends on the overtaking possibilities to what extend desired overtakings can be carried out.

Consider a road section of length X during a period of length T. Assume the state of the traffic is homogeneous and stationary. Assume further that the instantaneous speed distribution is Normal with mean μ and standard deviation σ. Then the number of desired OT's is:

$$n = XTk^2\sigma/\sqrt{\pi} \tag{3.65}$$

The greater part of this equation is obvious. The number of OT's is:

- linear dependent on the time-road region one considers (term X multiplied by T);

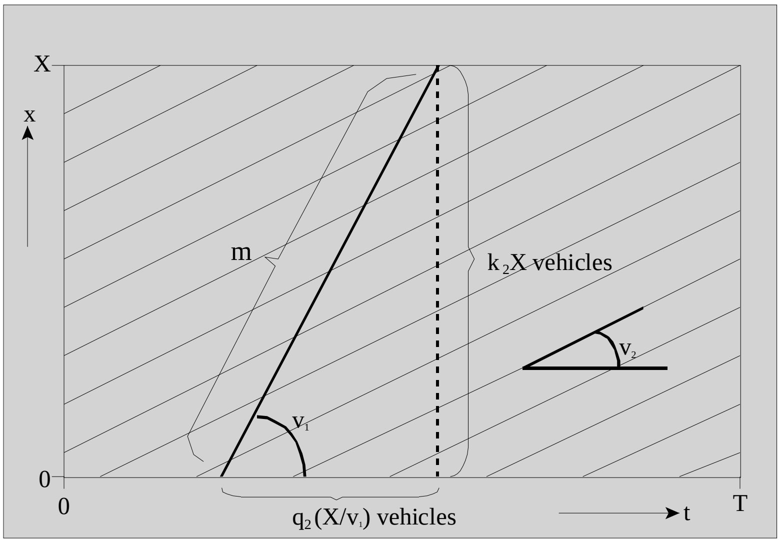

Figure 3.21: Schematised vehicle trajectories to determine number of overtakings

- increases with the quantity of traffic squared (term k^2);
- is larger if the speeds are more different (term σ).

Only the constant term $1/\sqrt{\pi}$ is not obvious.

Proof. Derivation of eq. (3.65). In a road-space region (X multiplied by T) a vehicle drives with speed v_1 through a vehicle stream with intensity q_2 and uniform speed v_2 ($v_1 > v_2$);see Fig. 3.21. At the road section the vehicle overtakes m vehicles. (we apply here the conservation of vehicle trajectories over a triangle: what comes in over the left and lower side must go out over the right side).

$$m = k_2 X - q_2 (X/v_1) \tag{3.66}$$

Suppose we have not one vehicle with speed v_1 but an intensity q_1 (with density k_1). During a period of T that makes $q_1 T$ vehicles, that each overtake m slower vehicles. The total number of OT's then becomes:

$$n = q_1 T(k_2 X - q_2(X/v_1)) \tag{3.67}$$

Putting the terms q_2 and X in eq. (3.67) outside the brackets, leads to:

$$n_1 = q_1 q_2 XT(k_2/q_2 - 1/v_1) = q_1 q_2 XT(1/v_2 - 1/v_1) \tag{3.68}$$

An alternative can be derived by replacing the intensities, substituting $q_i = k_i v_i$ $(i = 1, 2)$:

$$n_2 = k_1 v_1 T(k_2 X - k_2 v_2(X/v_1)) = XTk_1 k_2(v_1 - v_2) \tag{3.69}$$

Generalisation to continuous variables: In stead of considering two densities k_1 and k_2 with corresponding speeds v_1 and v_2, one can generalise towards many densities k_i with speeds v_i. Then one can sum over all classes i or take continuous variables and integrate. We then get a multiple integral:

$$n_2 = XT \int_{v_1=a}^{b} \int_{v_2=v_1}^{b} k^2 p_M(v_i)\, p_M(v_j)\,(v_i - v_j)\, dv_i dv_j \tag{3.70}$$

Substituting a Normal p.d.f. for $p_M(v)$, one can calculate equation (3.65). ■

The quality of operation on a separate cycle path is assumed to depend on the number of overtakings

On most cycle paths overtaking possibilities are rather abundant and it is assumed that the quality of operation is negatively influenced by the number of OT's a cyclist has to carry out (active OT) or has to undergo (passive OT).

Example 33 *Consider a one-way path of 1 km length during 1 hour. The bicycle (bic) intensity = 600 bic/h and the moped (mop) intensity = 150 mop/h. From observations it is known that speeds of bics and mops have, with a good approximation, a Normal distribution:* $u_{bic} = 19$ *km/h;* $\sigma_{bic} = 3$ *km/h;* $u_{mop} = 38$ *km/h;* $\sigma_{mop} = 5$ *km/h.*
Then can be calculated: OT[bic-bic] = (600/19)2 $3/\sqrt{\pi} = 1688$ *and OT[mop-mop] = (150/38)2* $5/\sqrt{\pi} = 44$*. The number of OT[mop-bic] can be calculated with the earlier derived eq. (3.69). It is valid when both speed distribution do not have overlap and in practice that is the case. OT[mop-bic] = (600/19) (150/38) (38 – 19) = 2341.*
The outcomes of the calculation show clearly that the mopeds are responsible for an enormous share in the OT's. The operational quality for the cyclists will increase much if mopeds do not use cyclepaths.

Remark 34 *The measure to forbid mopeds to use cycle paths inside built-up areas has been implemented in the Netherlands in 2001.*

3.6 Dependence of variates (headways, speeds)

When analysing stochastic variables (briefly variates) such as headways and speeds, usually their distribution is the first point of interest. A secondary point is the possible dependence of sequential values. This is relevant for the determination of confidence intervals for estimated parameters and also it is of importance when generating input streams for a simulation model.

For a sample of n independent drawings the following rule holds: the standard deviation of the mean is the standard deviation of the population divided by $\sqrt{n}$. However, if the elements of the sample are not independent but positively correlated, then this rule does not hold anymore and the st. dev. of the mean becomes larger. From practical studies it is known that sequential time headways of vehicles are usually independent or at most very weakly dependent. This is certainly not true for speeds, which is obvious if the headways are small. A vehicle in a platoon has a speed that is very dependent on the speed of its leader. This is illustrated in Fig. 3.22: inside the platoons the speeds vary less than between platoons.This effect has been used in a procedure to determine the headway that separates constrained and free driving. One considers the mean of the absolute relative speeds (relative speed is the speed of a vehicle considered minus the speed of its predecessor) and depicts it as a function of the headway. Usually then a boundary valuer is visible in the graph; see Fig. 3.23.

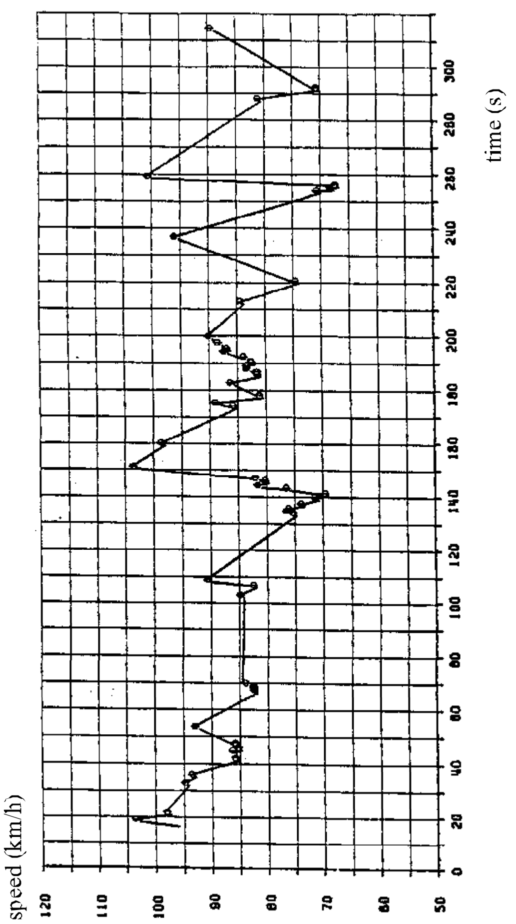

Figure 3.22: Time series of speeds observed at a cross-section to illustrate platooning; data from Dutch two lane road

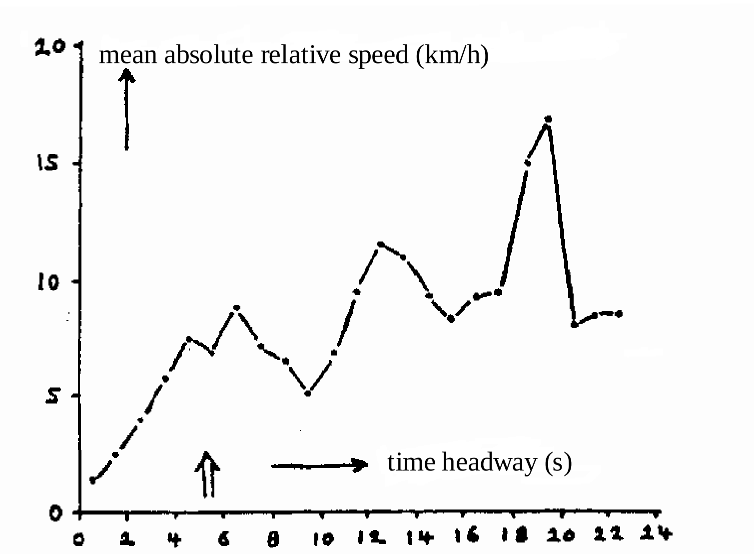

Figure 3.23: Results of processing headways and speeds of a cross-section to determine the boundary between constrained and free driving; data from a Dutch two-lane rural road; lane intensity = 680 veh/h; sample of 1200 vehicles. Outcome: boundary value ≈ 5 s