Chapter 8

Shock wave analysis

Summary of the chapter. Flow-speed-density states change over time and space. When these changes of state occur, a boundary is established that demarks the time-space domain of one flow state from another. In rough terms, this boundary is referred to as a shock wave. In some situations, the shock wave can be rather smooth, for instance when a platoon of high-speed vehicles catches up with a sligthly slower moving vehicle. In other situations, the shock wave can be a very significant change in flow states, for instance when high-speed vehicles approach a queue of stopped vehicles.

This chapter presents shock wave analysis. We introduce the concept of kinematic waves and shock waves, and present a number of examples of application of shock wave analysis.

List of symbols

k	veh/m	traffic density
q	veh/s	traffic volume
$Q(k)$	veh/s	equilibrium flow as function of density
v, u	m/s	traffic speed
ω	m/s	shock wave speed
c	m/s	kinematic wave speed
q_c	veh/s	capacity

8.1 Kinematic waves

A useful tool to describe dynamic phenomena in traffic flow is the *conservation of vehicle equation.* This equations reflects the fact that no vehicles are created or lost, i.e. that vehicles are conserved. If for a specific roadway sections, more vehicles are entering than leaving, then the number of vehicles in the section must increase: i.e. the density increases. Let $q = q(x,t)$ and $k = k(x,t)$ denote the traffic volume and density at instant t and location x. We assume that q and k are both smooth functions in time and space, and are defined from the smooth cumulative vehicle count $\tilde{N}(x,t)$. Then, the conservation of vehicle equation equals

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \tag{8.1}$$

We combine equation (8.1) with the expression of the fundamental diagram $q = Q(k)$, yielding the following partial derivative in the traffic density (see chapter 2)

$$\frac{\partial k(x,t)}{\partial t} + \frac{dQ}{dk}\frac{\partial k(x,t)}{\partial x} = 0 \tag{8.2}$$

Using this partial differential equation, we can describe the dynamics of the traffic flow under the assumption that the speed of the traffic reacts *instantaneously and locally to the traffic*

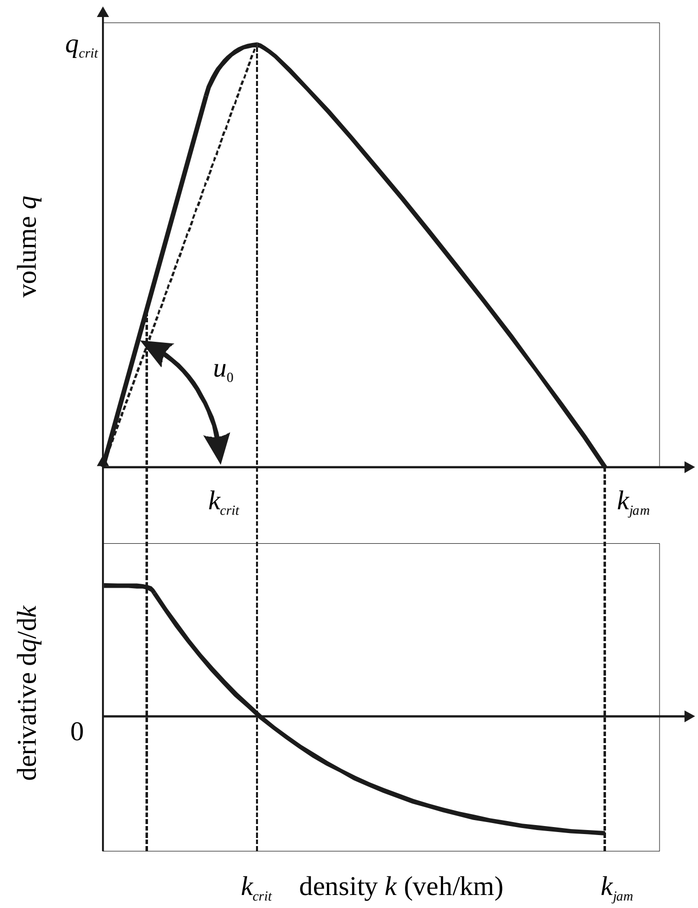

Figure 8.1: Fundamental diagram $Q(k)$ and speed of the kinematic wave $c = dQ/dk$ as a function of the density k

density, that is $u\left(x,t\right) = V\left(k\left(x,t\right)\right)$ or $q\left(x,t\right) = Q\left(k\left(x,t\right)\right)$[1]. The resulting model (conservation of vehicle equation and fundamental relation) constitutes one of the oldest macroscopic traffic flow models (kinematic wave model, LWR model), and was first studied by [34]. We can show that the partial differential equation (8.2) has the following general solution

$$k(x,t) = F(x - ct, t) \quad \text{where} \quad c = \frac{dQ}{dk} \text{ and } F \text{ an arbitrary function} \tag{8.3}$$

This means that a certain density k_0 propagates with speed $c = \frac{dQ}{dk}(k_0)$. Roughly speaking, c is the speed with which very small disturbances are propagating; these are called kinematic waves.

In chapter 4 several examples of fundamental diagrams have been discussed. Some of these, like Smulder's function or Daganzo's function, $Q(k) = u_0 k$ for small densities (see also Fig. 8.1). Then $\frac{dQ}{dk} = u_0$ for small densities, i.e. the speed of the traffic flow and the speed of the kinematic wave are equal. This is why this density region is referred to as the *stable area*. For larger densities, the difference between the speed of the traffic u and the kinematic wave speed c will increase. This region is referred to as the *unstable area*, since small disturbances in the density will grow into discontinuities. In the congested area, the kinematic wave speed c becomes negative implying that small disturbances in the flow will propagate upstream.

[1] Note that in chapter 5, it was shown how this may not be a realistic assumption.

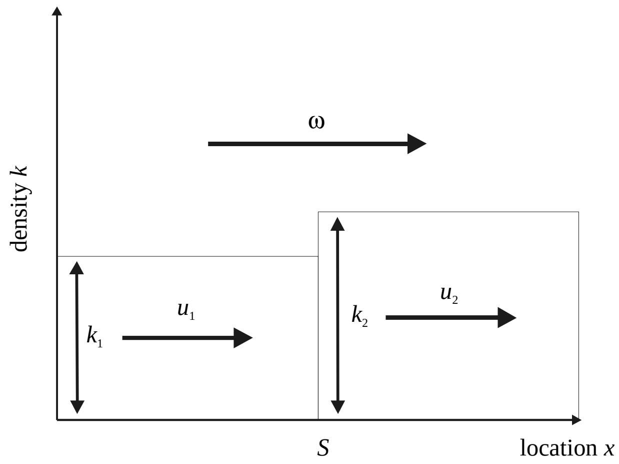

Figure 8.2: Derivation of shock wave speed

8.2 Shock wave speeds

The waves that occur in the traffic flow are categorised as either *kinematic waves* of *shock waves*. Shock waves are waves that originate from a sudden, substantial change in the state of the traffic flow. That is, a shock wave is defined by a discontinuity in the flow-density conditions in the time-space domain.

1. Fig. 8.2 schematises a shock wave. The shock is the boundary between two areas in for which different traffic conditions hold.

2. The state at the left area is (k_1, u_1); the state in the right area is (k_2, u_2). The line S indicates the shock wave with speed ω. The relative speeds with respect to line S for the left and the right area are respectively equal to $u_1 - \omega$ and $u_2 - \omega$.

If we consider the area left of line S, then the volume passing S equals

$$q_1 = k_1 (u_1 - \omega) \tag{8.4}$$

This holds equally for the right area, i.e.

$$q_2 = k_2 (u_2 - \omega) \tag{8.5}$$

Since the flow into the shock must equal the flow out of the shock, we can easily show that the wave speed ω equals

$$\omega = \frac{q_1 - q_2}{k_1 - k_2} \tag{8.6}$$

Note that for small changes in q and k, equation (8.6) becomes $\omega = \frac{dQ}{dk} = c$. In other words, the kinematic wave is the limiting case of a shock wave. Fig. 8.3 shows the following

1. Traffic flow in state (q_1, k_1) with speed $u_1 = \tan \phi_1$.

2. Traffic flow in state (q_2, k_2) with speed $u_2 = \tan \phi_2$.

3. The shock wave which occurs when traffic in state (q_1, k_1) transients into state (q_2, k_2). Between both states a shock wave occurs with speed $\omega = \tan \theta_1 = \frac{q_1 - q_2}{k_1 - k_2}$. Since $q_1 - q_2$ is positive and $k_1 - k_2$ is negative, the shock wave will move upstream.

4. The state (q_1, k_1) itself propagates with the speed of the accompanying kinematic wave, i.e. the tangent of $Q(k)$ at $k = k_1$; this holds equally for the state (q_2, k_2).

The following sections discuss several examples of applications of shock wave analysis, both to different situations, and using different forms of the fundamental diagram.

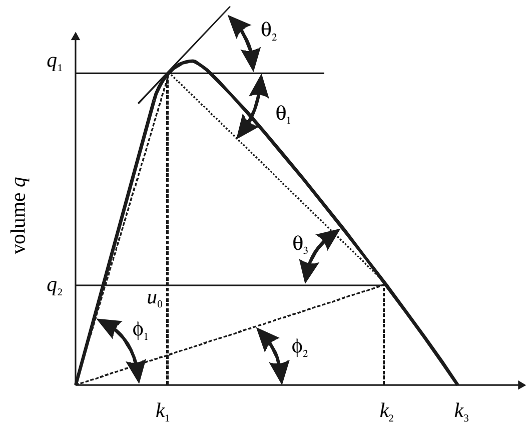

Figure 8.3: Shock waves in the $q - k$ plane

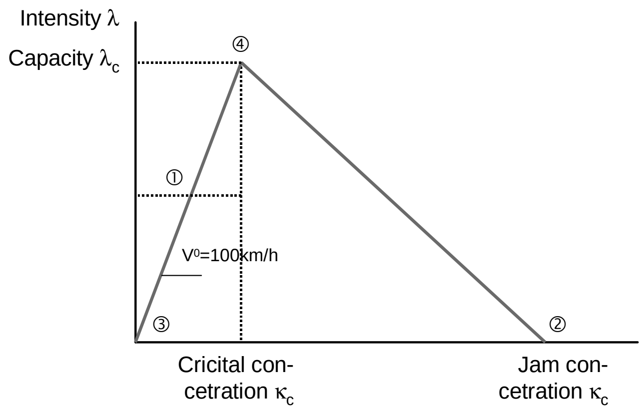

Figure 8.4: Schematised intensity - density (concentration) relation

8.3 Temporary blockade of a roadway

Blockades occur in practice in a number of circumstances, for instance when a bridge is open, during a severe incident blocking all the roadway lanes, or during the red-phase of a controlled intersection. Consider a two-lane road where the volume is equal to 2500 veh/h. The capacity q_c of the road equals 5000 veh/h. The jam-density k_j equals 250 veh/km. The blockade starts at $t = t_0$, and ends at $t = t_1$. At the time the blockade occurs, three shock waves occur (see Fig. 8.5):

1. The transition for freely flowing traffic (state (1)) to standing-still (state (2)). The speed of the shock wave equals $\omega = \frac{q_1}{q_1/u_1 - k_j} = \frac{2500}{25-250} = -11.1$. The wave is called the *stop wave*.

2. The head of the queue (state (2)) borders on an empty region (state (3)). Formally, this is a shock wave as well; clearly $\omega = 0km/h$.

3. Downstream of x_0 a region in which traffic is present that has just passed x_0 before the blockade occurred. This is also a shock wave with speed $w = u_0$.

   When the blockade is removed at $t = t_1$, two more shock waves emerge

4. The transition from queuing (state (2)) to the capacity state (4) (the so-called start-wave) with speed $\omega = \frac{q_4 - q_2}{k_4 - k_2} = \frac{q_c}{k_c - k_j} = -25km/h$, implying that the head of the queue propagates upstream with a speed of $25km/h$.

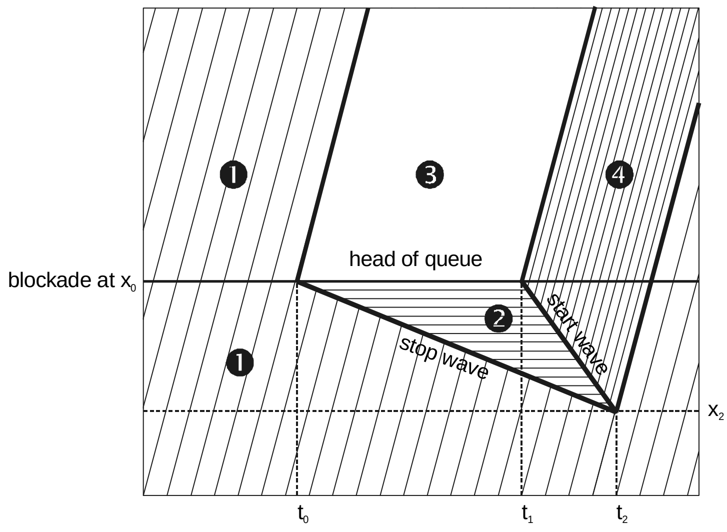

Figure 8.5: Trajectories of shock waves and vehicles. Note that the stop wave is often referred to as the backward forming shockwave; the start wave is also referred to as the backward recovery wave; furthermore the head of the queue is also called the frontal stationary wave

5. Downstream of x_0 exists a boundary between state (4) and the empty road. This shock wave has speed $\omega = u_0$.

Since the start wave has a higher speed that the stop wave, it will eventually catch up, say at time $t = t_2$. At that time, a new shock wave occurs, reflecting the transition from free-flow traffic (state (1)) to the capacity state (4). This wave has speed u_0.

8.4 Shock wave analysis for non-linear intensity-density curves

In the previous section, shock wave analysis was applied to determine traffic flow conditions for piecewise linear intensity-density curves. This section discusses application of shock wave analysis for general $Q(k)$ relations, i.e. relations that are not necessarily piecewise linear. This is done by revisiting the application examples discussed in the previous section.

8.4.1 Temporary blockade revisited

Let us first reconsider the temporary roadway blockade example, using the $Q(k)$ relation depicted in Fig. 8.6. Here, the derivative $Q'(k)$ is a monotonic decreasing function of the density k, with $Q'(k) < 0$. As before, the blockade of the roadway starts at $t = t_0$ and ends at $t = t_1$. At the time the blockade occurs, the shock wave theory will again predict the occurance of three shock waves (see Fig. 8.6), similar to the case of piecewise linear $Q(k)$. In line with [36], the shock wave theory predicts that at time $t = t_1$, traffic will start moving along the roadway, fully using the available capacity q_c. This causes a transition from state 3 (empty roadway) to state 4. Note that the speed at which shock between states 3 and 4 moves equals the capacity speed u_c, which is in this case less than the free flow speed u_0. The start wave and the stop wave propagate in a similar fashion as before. However, at the point where the start wave and the stop wave intersection (i.e. the queue has dissolved), the shock wave separating states 4 and 1 has a different character: its speed is less than the speed of the vehicles in state 1. As a result, the vehicles in state 1 will eventually catch up with the shock wave and driving into the state 4 region.

Clearly, this result is not very realistic: the shock wave theory predicts that the speed of the first vehicle in the queue equals the speed at capacity u_c, while in fact the first vehicle will

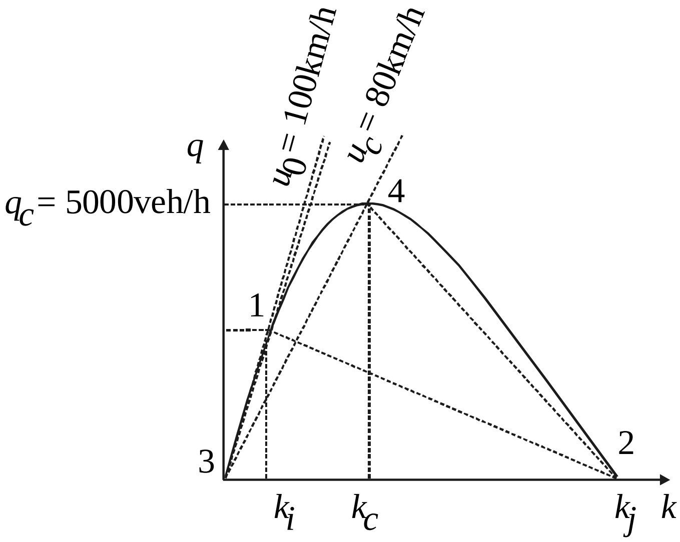

Figure 8.6: Intensity - density relation.

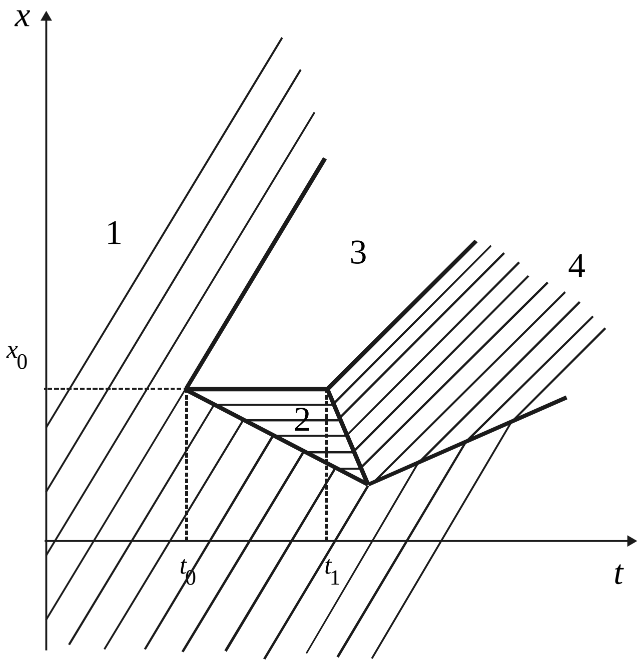

Figure 8.7: Shockwaves for non-linear $Q(k)$ for temporary blockade example

accelerate towards the free speed u_0. This is caused by the fact that shock wave theory does not describe the acceleration - nor the decelaration - of the first vehicle in the queue. As a result of the reduced speed of the first vehicle, the shock that is caused by the temporary blockade will not dissolve over time. Rather, state 4 will disperse further as time goes by.

8.5 Moving bottlenecks

In this section we consider a more complicated example, namely a moving bottle-neck (b-n) that is present on a road over a given distance and/or during a given period. The usefulness of this example is that it implies other relevant cases by assigning values zero to the speed and/or the capacity of the bottle-neck. Practical examples of a moving bottle-neck are:

1. A slow moving vehicle, e.g. an agricultural tractor with a speed of 20 km/h, on a two-lane road. The capacity of the b-n is determined by the overtaking opportunities, hence by the opposing flow and the overtaking sight distance. Those two factors can lead to a more or less constant capacity of the b-n.

2. A platoon of trucks on a long grade on a motorway. In such conditions trucks form a slow platoon and more or less block the right hand lane, causing a substantial capacity reduction.

3. Actions of protesting farmers or truck drivers that from a temporary slow platoon on one or more lanes of a motorway.

To simplify the case three assumptions will be made:

1. Demand intensity is constant and larger than the capacity of the b-n. If the demand is lower, the b-n only leads to a speed reduction with relatively little delay.

2. Daganzo's fundamental diagram is used.

3. The length of the b-n can be neglected; it is set at zero.

8.5.1 Approach 1

Figure 8.8 depicts the phenomena in the $q - k$ plane and the $x - t$ plane. The capacity of the roadway is 4500 veh/h; speed at capacity $u_c = 90km/h$; jam density $k_j = 250veh/km$. The moving b-n has a speed $\hat{v}$ (20 km/h) and covers a distance of 4 km. The b-n has a capacity of 1800 veh/h, a capacity speed of 60 km/h and $k_j = 125veh/km$.

In general terms the b-n will lead to *congestion upstream and free flow downstream with an intensity equal to the capacity of the b-n.* After the b-n is removed the congestion will decay until free flow is restored over the total affected road section. We start looking to the phenomena in the $q - k$ plane. It is obvious that in the b-n the (traffic flow) state is the cap. state of the b-n; state **2**. At the upstream end of the b-n a transition from a congested state **4** to the state **3** must occur and move with speed $\hat{v}$. At the downstream end of the b-n a transition from state **4** to free flow **1** occurs and this transition must also have speed $\hat{v}$. This implies that in the $q - k$ plane the *shock waves are represented by the line going through the capacity point of the b-n with a slope equal to* $\hat{v}$. Consequently the state in the congestion upstream the b-n is state **3** and the state downstream the b-n is state **4**.

Note: Consequently the intensity of the free flow downstream the b-n is not equal to the capacity of the b-n, as was loosely stated earlier, but it is less. Only if the speed of the b-n is zero that is the case.

Free flow state **1** and congested state **3** determine the speed of the shock wave at the tail of the queue. Given this description in the $q - k$ plane the geometrical description in the $x - t$ plane follows easily and the calculations are straightforward.

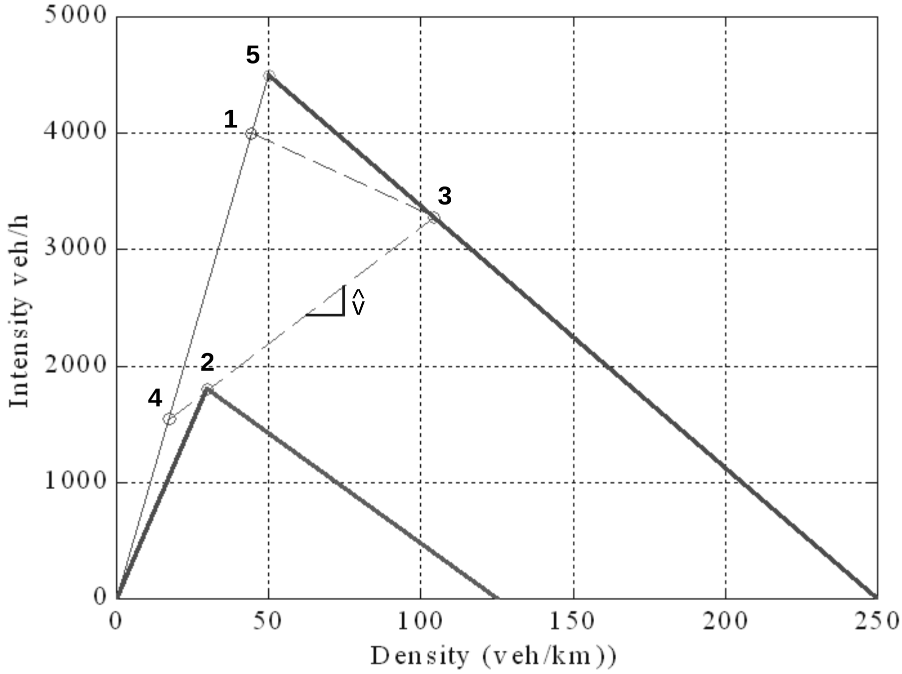

Figure 8.8: Effects of moving bottleneck in the flow-density plane.

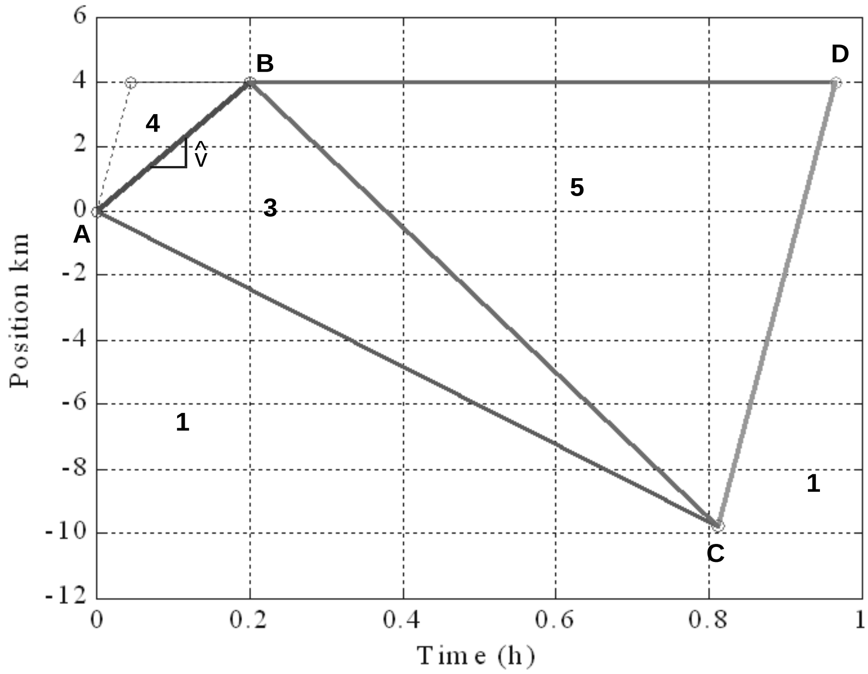

Figure 8.9: Results of moving bottleneck in $x - t$ plane.

The origin in Fig. 8.9 is where the b-n starts. It moves with speed $\hat{v}$ until it leaves the road again. The upstream border of the queue is a shock wave with speed ω_{13}. At a given moment and position the moving b-n leaves the road (point **B**). The congested traffic, state **3**, then transforms to the capacity state **5** and a shock wave between state **3** and state **5** starts and makes the size of congestion shrink. Shock waves S_{13} and S_{53} meet at point **C**, and then starts a shock wave representing the transition from the undisturbed state **1** to capacity state **5** with speed ω_{AE}. When this shock wave reaches the position of point **3** the disturbance is virtually over. Line **AB** is very special, since it represents state **2**; just upstream of a shockwave with speed ω_{32} is present and just downstream a shockwave with the same speed ω_{24}.

Note: downstream point **B**, Daganzo's diagram works out less realistically. The three states 4, 5, and 1 propagate all with the same speed and do not mix. In reality, $u_4 > u_1 > u_5$ and drivers will accelerate to areas with a lower density.

8.5.2 Approach 2

An alternative approach exist to determine the effects of a moving bottleneck. Let us again consider the situation above, where a moving bottleneck enters the roadway and drives with a constant speed $\hat{v}$.The trick is to reformulate the problem with all relevant variables (location, flows, speeds, etc.) defined relative to a frame of reference moving along with the moving bottleneck. This is usefull because the moving observers will notice that the relative flows passing them also obey some $q-k$ relation at each point (albeit different from the origing $q-k$ relation) *and that the bottleneck appears stationary.* Thus the solution in the new frame of reference can be obtained as discussed earlier without any changes, and from this solution one can then retrieve the desired 'static' solution.

The simple mathematical formalism follows. Using primes to denote the variables used in the moving frame of reference we definde

$$x' = x - \hat{v}t, \quad t' = t, \quad k' = k, \text{ and } q' = q - k\hat{v} \tag{8.7}$$

Note that q' is the relative flow measured by the moving observer, which can be either negative or positive. It should be obvious that the relative flows must be conserved and thus

$$\frac{\partial k'}{\partial t'} + \frac{\partial q'}{\partial x'} = 0 \tag{8.8}$$

Of course, this equation can also be obtained by changing variables in $\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0$. Most importantly, the original relation between q and k is transformed into a relation $Q'(k', x')$ that is independent of t

$$q' = q - k\hat{v} = Q(k, x - \hat{v}t) - k\hat{v} = Q'\left(k', x'\right) \tag{8.9}$$

Thus, the problem has been reduced to the stationary bottleneck case with Eq. (8.8). This new relation essentially lowers the traffic speed and the wave speed for every k by and amount of $\hat{v}$. To retrieve the solution in the original frame of reference, we simply need to adjusr vertically the edges of the polygons in the (x', t') solution using $x = x' + \hat{v}t'$ while keeping the same density inside of each polygon (since $k = k'$).

8.6 Shock wave classification

This chapter has presented shock wave analysis. By means of several examples and exercises we have seen different types of shock waves. Fig. 8.10 classifies the various types of shock waves. The high density area is shown in the centre of the figure, while low densities are located on the outside of the individual shock waves.

A *frontal stationary shock wave* must always be present at a bottle-neck location and indicates the location where traffic demand exceeds capacity. This may be due to recurrent situations where each workday the normal demands exceed the normal capacities, or be due to non

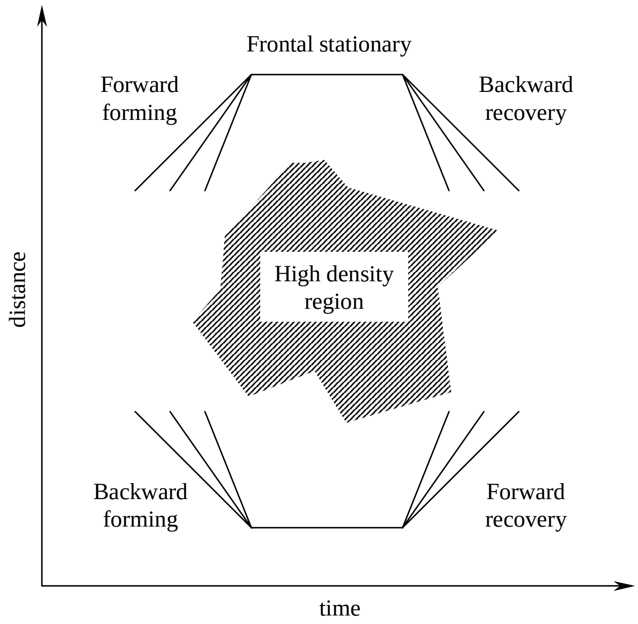

Figure 8.10: Classification of shock waves

recurrent situations where the normal demand exceeds reduced capacity (caused by an acci-dent or an incident) which may occur at any location and at any time. The term 'frontal' im-plies that it is at the front (or downstream edge) of the congested region with lower densities farther downstream and higher densities upstream. The term 'stationary' means that the shock wave is fixed by location.

Backward forming shock waves must always be present if congestion occurs and indicates the area in the time-space domain where excess demands are stored. The term 'backward' implies that over time, the shock wave is moving backward or upstream in the opposite direc-tion of traffic. The term 'forming' implies that over time, the congestion is (gradually) ex-tending to sections farther downstream. The time-space domain to the left of this shock wave has lower densities, and to the right the density levels are higher. There are examples of backward forming shock waves in the examples and exercises presented in the previous sec-tions.

The *forward recovery shock wave* is the next most commonly encountered type of shock-wave and occurs when there has been congestion, but demands are decreasing below the bottle-neck capacity and the length of the congestion is being reduced. The term 'forward' means that over time the shock wave is moving forward or downstream in the same direction as the traffic. The term 'recovery' implies that over time, free-flow conditions are gradually occurring on sections farther downstream.

The *rear stationary shock wave* may be encountered when the arriving traffic demand is equal to the flow in the congested region for some period of time. The *backward recovery shock wave* is encountered when congestion has occurred but then due to increased bottle-neck capacity the discharge rate exceeds the flow rate within the congested region. The last type of shock wave is the *forward forming shock wave*, which is not very common. It may occur when the capacity decreases slowly in the upstream direction, for instance in case trucks are slowed down due to the grade of the roadway.