Bending Deflection – Macaulay Step Functions

AE1108-II: Aerospace Mechanics of Materials

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Recall Moment-Curvature Relationship

$$M = -EI\frac{d^2v}{dz^2}$$

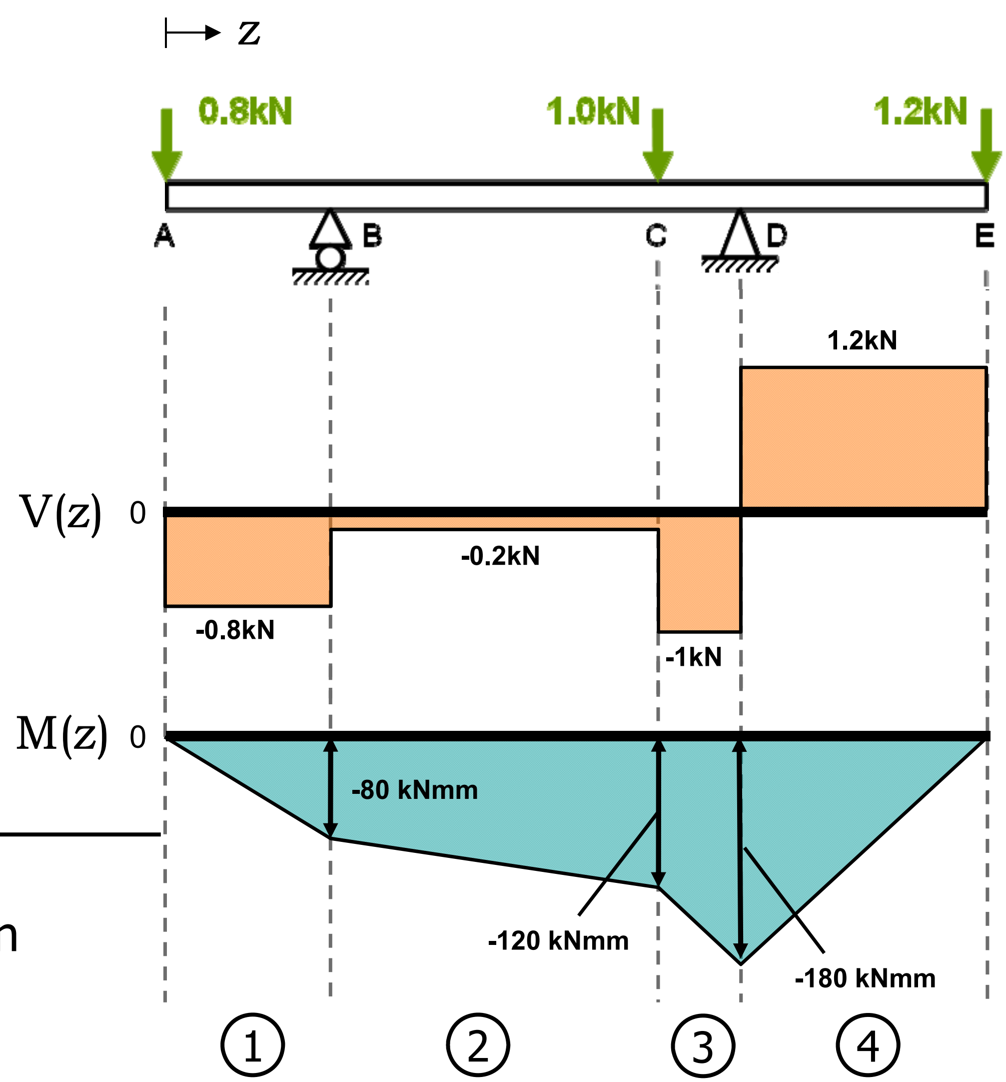
$\iint$

v (beam displacement)

What if the moment equation is not continuous?

4 moment equations due to discontinuities

$\Rightarrow$ 8 constants of integration



What if we had a mathematical switch that we could integrate?

We could formulate a discontinuous function as a continuous one!

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Macaulay's Step Function

$$f_n(z) = [z-a]^n \quad \left[\text{or } \langle z-a \rangle^n\right]$$

Such that when: $z < a$ $\quad f_n(z) = 0$

$z \geq a$ $\quad f_n(z) = (z-a)^n$

And:

$$\int [z-a]^n \, dz = \frac{[z-a]^{n+1}}{n+1} + A$$

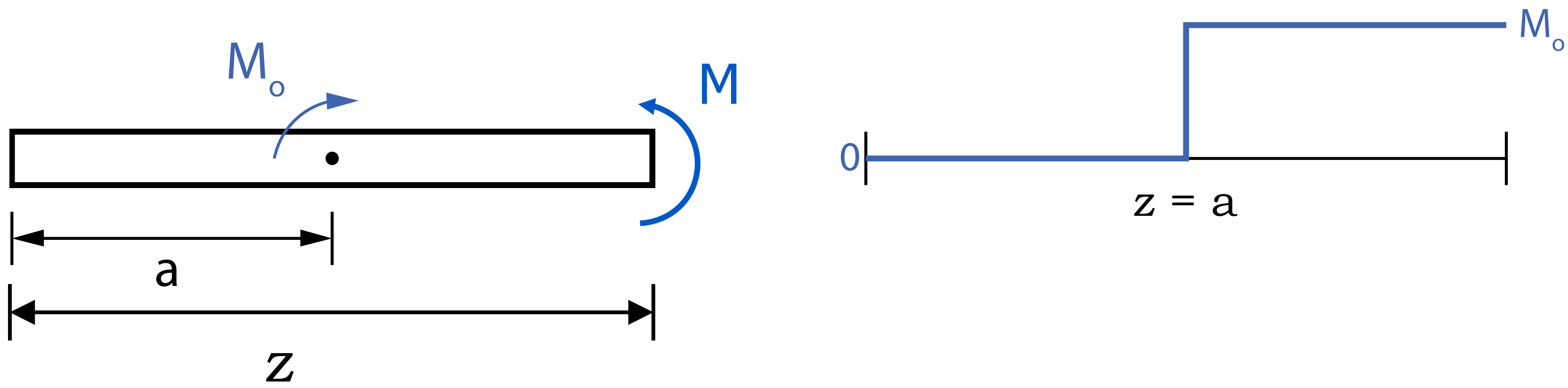
constant of integration

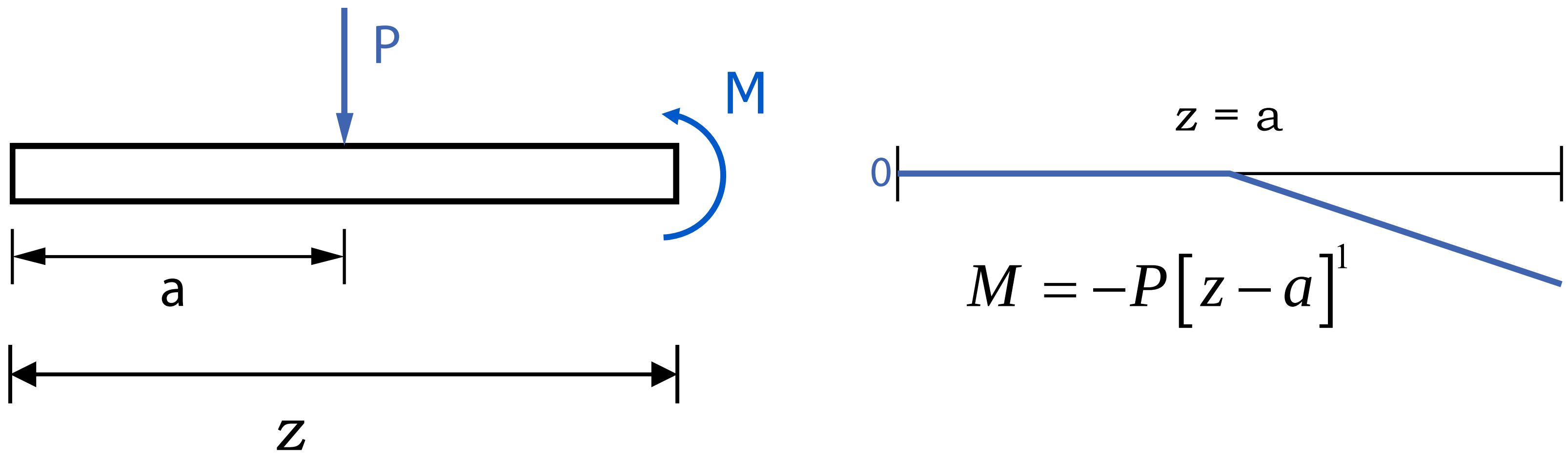
NOTE: $$\int [z-a]^n \, dz \neq \left[\frac{z^2}{2} - az\right] + A$$

Macaulay’s Step Function

Examples

$$M = M_0 \left[z - a \right]^0$$





$$M = -P \left[z - a \right]^1$$

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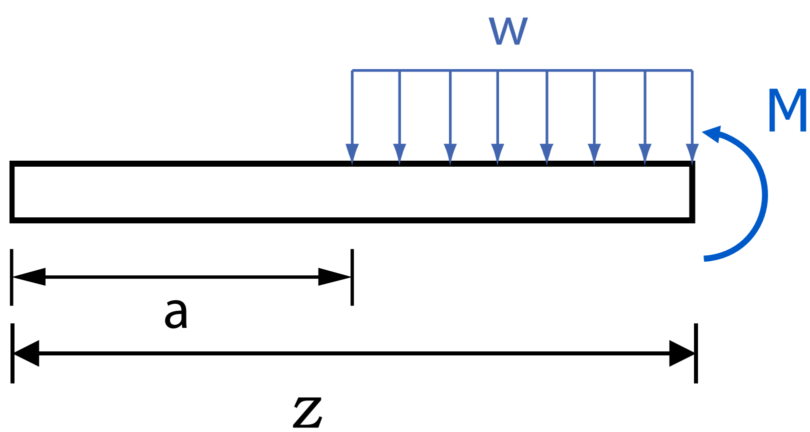
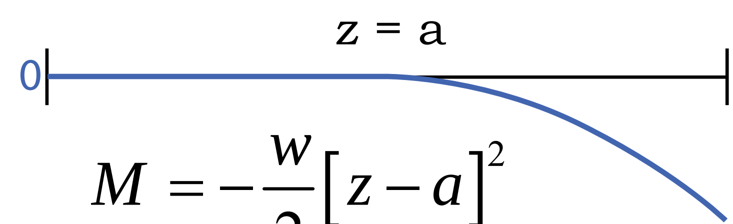
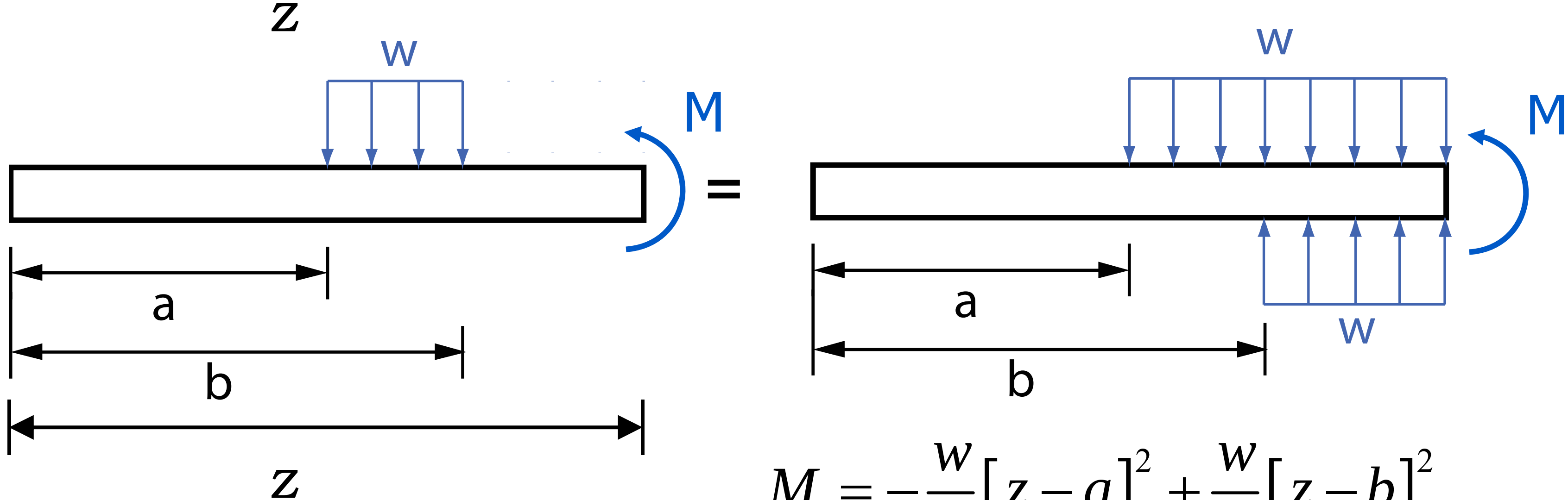
Macaulay's Step Function

Examples (cont)





$$M = -\frac{w}{2}\left[z-a\right]^2$$



$$M = -\frac{w}{2}\left[z-a\right]^2 + \frac{w}{2}\left[z-b\right]^2$$

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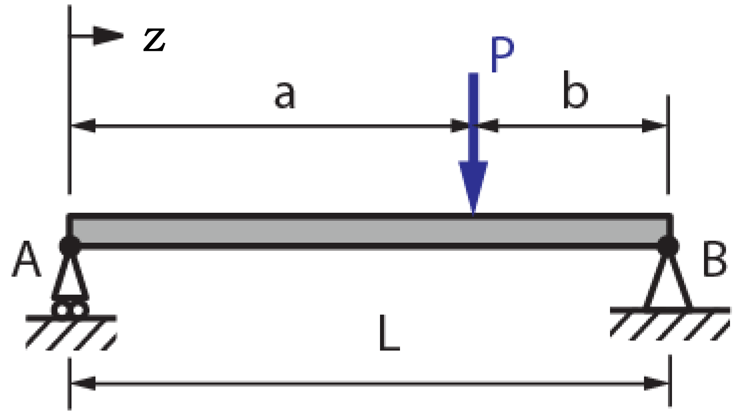
Example Problem

Using Macaulay's step functions, determine the deflection at L/2 (flexural rigidity = EI)



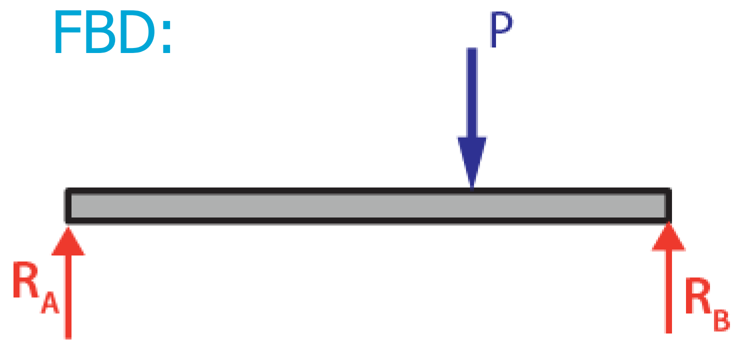
Equilibrium:

$$\sum F^{\uparrow} \Rightarrow R_A + R_B = P$$

$$\sum M_A \Rightarrow Pa = R_B L$$

$$\therefore R_A = \frac{Pb}{L}, R_B = \frac{Pa}{L}$$

FBD:



Macaulay Moment Function:

(always on) — 0 (always off)

$$M = R_A[z-0] - P[z-a] + R_B[z-L]$$

$$= \frac{Pb}{L}z - P[z-a]$$

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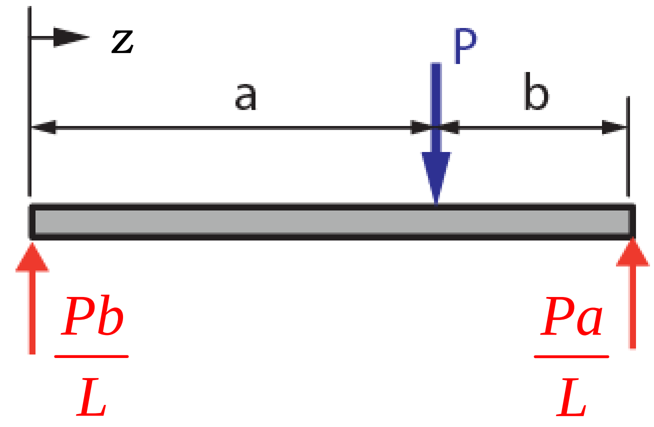
Example Problem (cont)



Moment-Curvature:

$$M = -EI\frac{d^2v}{dz^2} = \frac{Pb}{L}z - P[z-a]$$

$$\therefore EI\frac{dv}{dz} = -\frac{Pb}{2L}z^2 + \frac{P}{2}[z-a]^2 + A$$

$$\therefore EIv = -\frac{Pb}{6L}z^3 + \frac{P}{6}[z-a]^3 + Az + B$$

Boundary Conditions

$@\; z=0,\; v=0; \qquad \therefore B = 0$

$@\; z=L,\; v=0; \qquad \therefore 0 = -\frac{Pb}{6L}L^3 + \frac{P}{6}\underbrace{(L-a)}_{=\,b}{}^3 + AL$

$$\therefore A = \frac{Pb}{6L}\left(L^2 - b^2\right)$$

Example Problem (cont)

Displacement at L/2:

$$v = \frac{1}{EI}\left(-\frac{Pb}{6L} z^3 + \frac{P}{6}\left[z-a \right]^3 + \frac{Pb}{6L}\left(L^2 - b^2 \right) z \right)$$

$$v\left(\tfrac{L}{2}\right) = \frac{1}{EI}\left(-\frac{Pb}{6L}\left(\frac{L}{2}\right)^3 + \frac{P}{6}\left[\left(\frac{L}{2}\right) - a \right]^3 + \frac{Pb}{6L}\left(L^2 - b^2 \right)\left(\frac{L}{2}\right) \right)$$

$$= \frac{1}{EI}\left(\frac{Pb}{48}\left(3L^2 - 4b^2 \right) + \frac{P}{6}\left[\left(\frac{L}{2}\right) - a \right]^3 \right)$$

0 if a ≥ L/2

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