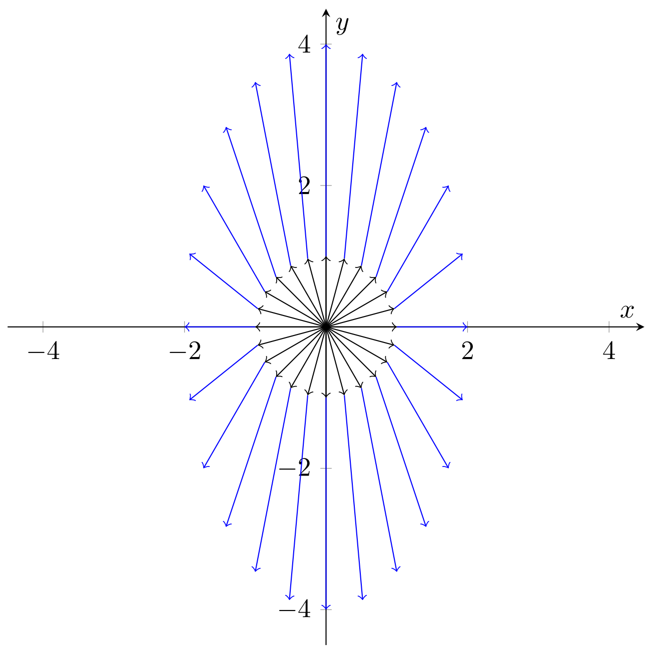
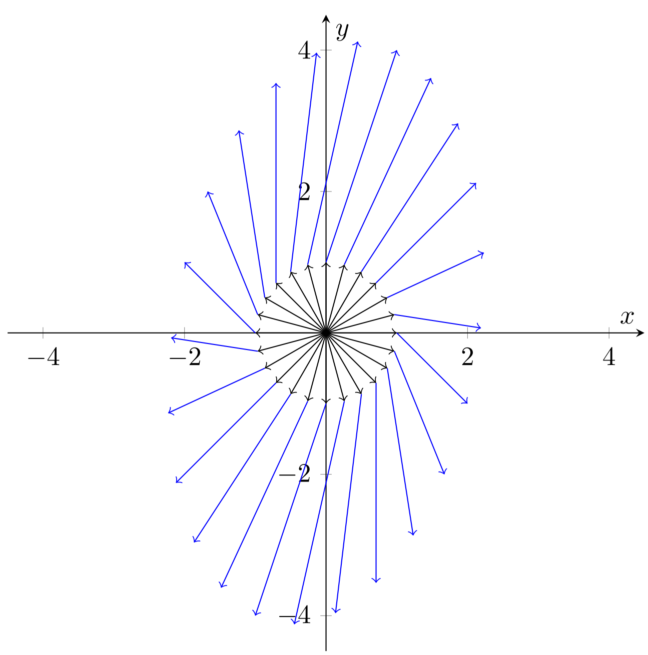
Pen and Paper Exercises - eigenvalues and eigenvectors

1. In each of the following pictures, unit vectors $\mathbf{x}$ in $\mathbb{R}^2$ and their images $A\mathbf{x}$ under the action of a 2×2 matrix A are drawn head-to- tail. Estimate the eigenvectors and eigenvalues of A from each "eigenpicture".

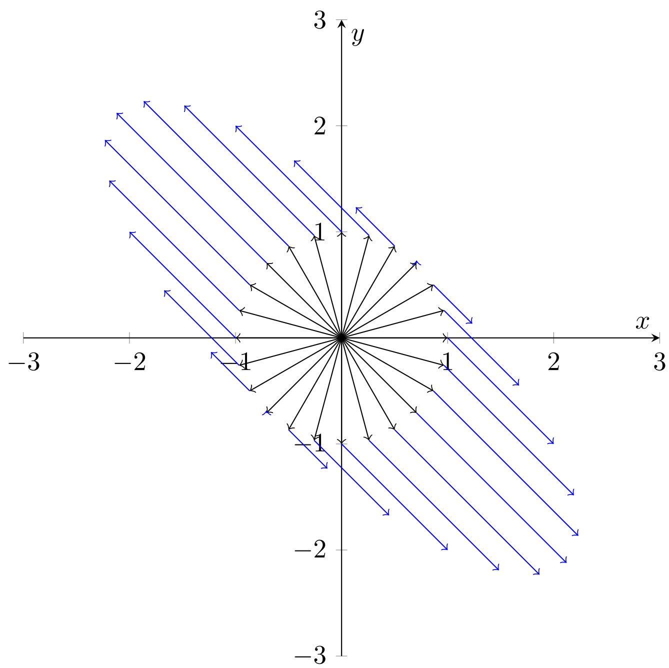
   (a)



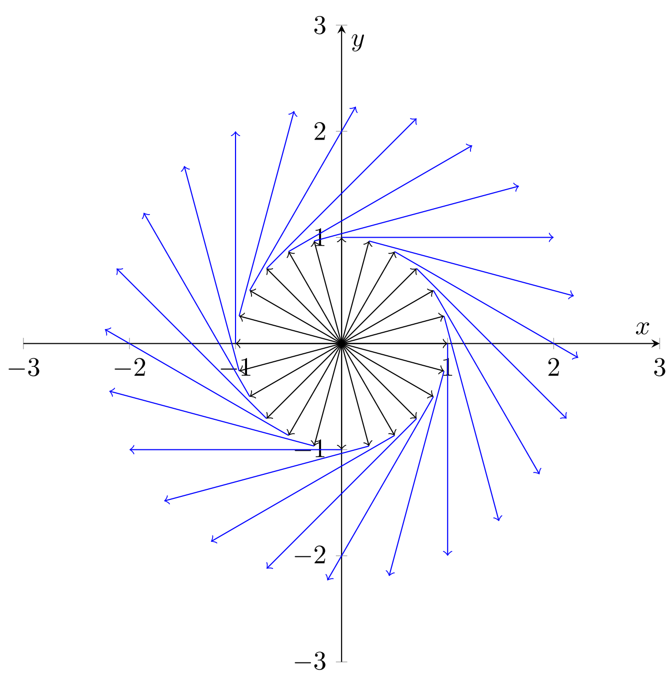
   (b)



   (c)



(d)



2. Let A be a 2×2 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ corresponding to eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$. Furtermore the vector $\mathbf{x} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ is given.

   (a) Find $A^5\mathbf{x}$ without determining A explicitly.

   (b) Find $A^k\mathbf{x}$ without determining A explicitly. What happens for large values of k?

3. Diagonalize the matrix A and use this diagonalization to compute A^k.

   (a) $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$, k arbitrary

   (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$, $k = 20$

4. Find all (real) values of k for which A is (real) diagonalizable.

   (a) $A = \begin{bmatrix} 1 & 1 \\ 0 & k \end{bmatrix}$

   (b) $A = \begin{bmatrix} k & 1 \\ -1 & 0 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & k \\ 0 & 0 & 1 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 1 & 1 & k \\ 2 & 2 & 2k \\ 3 & 3 & 3k \end{bmatrix}$

5. Prove the following statements using the the relevant definitions, or disprove the statement using an appropriate counterexample.

   (a) If A is row equivalent to B, then A and B have the same eigenvalues.

   (b) If A is diagonalizable and invertible, them A^{-1} is diagonalizable.

   (c) If A is diagonalizable and similar to B, then B is diagonalizable.

   (d) If A and B are similar, then the eigenvalues of A and B are the same.

   (e) If A and B are similar, then the geometric multiplicities of the eigenvalues of A and B are the same.