Elaboration of the exam AE3-914 'Dynamics and Stability' of November 1, 2010.

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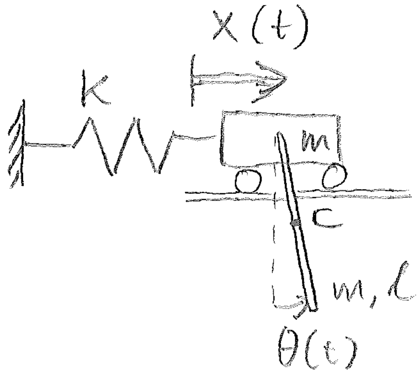
Question 1



a) The position of the centre of mass C of the pendulum is given by:

$$x_c = x + \tfrac{1}{2} l \cdot \sin\theta$$
$$y_c = \tfrac{1}{2} l \cos\theta$$

Accordingly, the velocity of point C is given by:

$$\dot{x}_c = \dot{x} + \tfrac{1}{2} l \cos\theta\, \dot{\theta}$$
$$\dot{y}_c = -\tfrac{1}{2} l \sin\theta\, \dot{\theta}$$

The kinetic energy of the whole system is given by

$$T = T_{car} + T_{pendulum}$$
$$= \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m (\dot{x}_c^2 + \dot{y}_c^2) + \tfrac{1}{2} I_c \dot{\theta}^2$$

$$\Rightarrow T = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m \left(\left(\dot{x} + \tfrac{1}{2} l \cos\theta \dot{\theta} \right)^2 + \left(-\tfrac{1}{2} l \sin\theta \dot{\theta} \right)^2 \right) + \tfrac{1}{2} \cdot \tfrac{1}{12} m l^2 \cdot \dot{\theta}^2$$

$$\Rightarrow T = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m \dot{x} l \cos\theta \dot{\theta} + \tfrac{1}{2} m \cdot \tfrac{1}{4} l^2 \cos^2\theta \dot{\theta}^2 + \tfrac{1}{2} m \cdot \tfrac{1}{4} l^2 \sin^2\theta \dot{\theta}^2 + \tfrac{1}{24} m l^2 \dot{\theta}^2$$

$$\Rightarrow \boxed{T = m \dot{x}^2 + \tfrac{1}{2} m \dot{x} l \cos\theta \dot{\theta} + \tfrac{1}{6} m l^2 \dot{\theta}^2}$$

b.) $V = V_{spring} + V_{gravity}$

$$\Rightarrow \boxed{V = \tfrac{1}{2} k x^2 - \tfrac{1}{2} m g l \cos\theta}$$

c.) $L = T - V$

$$\Rightarrow \boxed{\begin{aligned} L = {} & m \dot{x}^2 + \tfrac{1}{2} m \dot{x} l \cos\theta \dot{\theta} + \tfrac{1}{6} m l^2 \dot{\theta}^2 \\ & - \tfrac{1}{2} k x^2 + \tfrac{1}{2} m g l \cos\theta \end{aligned}}$$

$$\frac{\partial L}{\partial \dot{x}} = 2 m \dot{x} + \tfrac{1}{2} m l \cos\theta \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = 2 m \ddot{x} - \tfrac{1}{2} m l \sin\theta \dot{\theta}^2 + \tfrac{1}{2} m l \cos\theta \ddot{\theta}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m \dot{x} l \cos\theta + \frac{1}{3} m l^2 \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{1}{2} m \ddot{x} l \cos\theta - \frac{1}{2} m \dot{x} l \sin\theta\,\dot{\theta} + \frac{1}{3} m l^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{1}{2} m \dot{x} l \sin\theta\,\dot{\theta} - \frac{1}{2} m g l \sin\theta$$

$$\frac{\partial L}{\partial x} = -kx$$

Eqs. of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad (1)$$

$$\Rightarrow \boxed{2m\ddot{x} - \frac{1}{2} m l \sin\theta\,\dot{\theta}^2 + \frac{1}{2} m l \cos\theta\,\ddot{\theta} + kx = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad (2)$$

$$\Rightarrow \frac{1}{2} m \ddot{x} l \cos\theta - \frac{1}{2} m \dot{x} l \sin\theta\,\dot{\theta} + \frac{1}{3} m l^2 \ddot{\theta} + \frac{1}{2} m \dot{x} l \sin\theta\,\dot{\theta} + \frac{1}{2} m g l \sin\theta = 0$$

$$\Rightarrow \boxed{\frac{1}{3} m l^2 \ddot{\theta} + \frac{1}{2} m \ddot{x} l \cos\theta + \frac{1}{2} m g l \sin\theta = 0}$$

d.) Since $L \neq L(t)$, the Jacobi energy integral is an integral of motion.

$$h = \dot{x} \cdot \frac{\partial L}{\partial \dot{x}} + \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L = \text{constant}$$

$$\Rightarrow h = \dot{x}\left(2m\dot{x} + \frac{1}{2} m l \cos\theta \, \dot{\theta}\right)$$

$$+ \dot{\theta}\left(\frac{1}{2} m \dot{x} l \cos\theta + \frac{1}{3} m l^2 \dot{\theta}\right)$$

$$- m\dot{x}^2 - \frac{1}{2} m \dot{x} l \cos\theta \, \dot{\theta} - \frac{1}{6} m l^2 \dot{\theta}^2$$

$$+ \frac{1}{2} k x^2 - \frac{1}{2} m g l \cos\theta$$

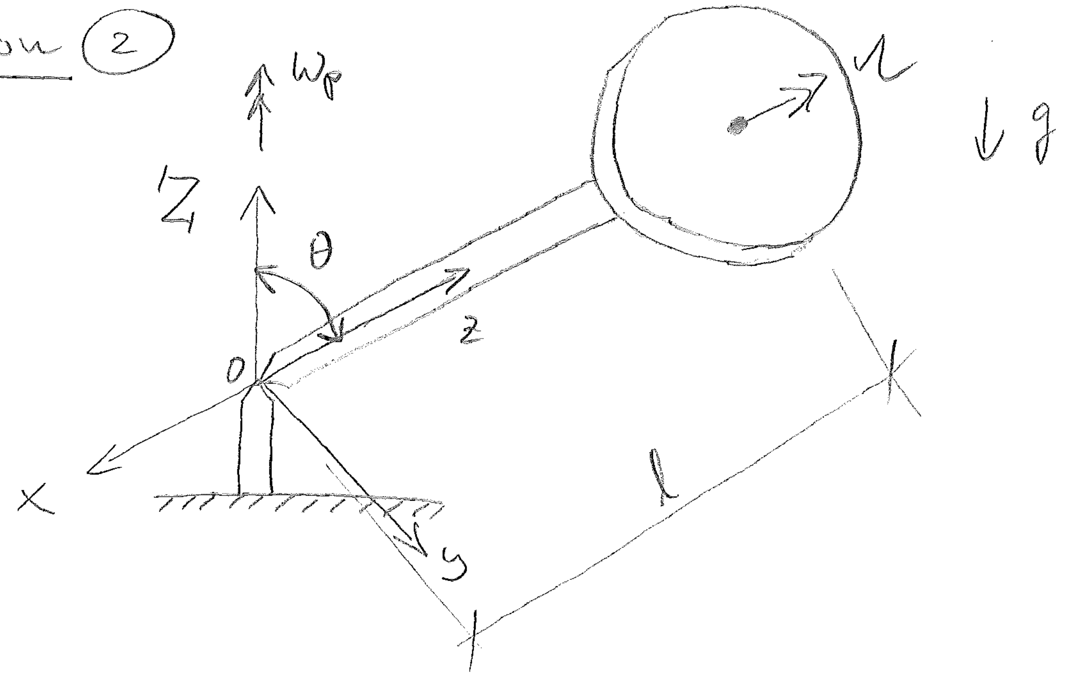
$$\Rightarrow h = \overbrace{m\dot{x}^2 + \frac{1}{2} m \dot{x} \dot{\theta} l \cos\theta + \frac{1}{6} m l^2 \dot{\theta}^2}^{T}$$

$$\underbrace{+ \frac{1}{2} k x^2 - \frac{1}{2} m g l \cos\theta}_{V}$$

$$\Rightarrow h = T + V = \text{constant.}$$

The Jacobi energy integral equals the total energy $E = T + V$, so the total energy is conserved.

Question 2



a) $\bar{\omega} = \omega_p \bar{K} + \Omega \bar{k}$ (1) (θ is constant, so $\dot{\theta} = 0$).

$\bar{K} = \cos\theta \, \bar{k} - \sin\theta \, \bar{j}$

$$\Rightarrow \boxed{\bar{\omega} = -\omega_p \sin\theta \, \bar{j} + (\omega_p \cos\theta + \Omega) \bar{k}}$$

b) $\bar{\alpha} = \dfrac{d\bar{\omega}}{dt}$

Using Eq. (1), this gives:

$$\bar{\alpha} = \dot{\omega}_p \bar{K} + \dot{\Omega} \bar{k} + \Omega \dot{\bar{k}}$$

(since $\bar{K}$ is related to an intertial frame of reference, $\dot{\bar{K}} = 0$). Because ω_p and Ω are constant $\dot{\omega}_p = 0$ and $\dot{\Omega} = 0$.

Accordingly, the expression for the angular acceleration can be further developed as

$$\bar{\alpha} = \Omega \dot{\bar{k}}$$
$$= \Omega \, (\bar{\omega}^* \times \bar{k})$$

where $\bar{\omega}^* = -\omega_p \sin\theta \, \bar{j} + \omega_p \cos\theta \, \bar{k}$

(which is the rotation of the frame of reference x-y-z).

$$\Rightarrow \bar{\alpha} = \Omega \, (-\omega_p \sin\theta \, \bar{j} + \omega_p \cos\theta \, \bar{k}) \times \bar{k}$$

$$\Rightarrow \boxed{\bar{\alpha} = -\Omega \, \omega_p \sin\theta \, \bar{i}}$$

c.) $\bar{L}_0 = \omega_1 I_1 \bar{i} + \omega_2 I_2 \bar{j} + \omega_3 I_3 \bar{k}$

with $I_2 = I + m l^2$ (Steiner)

and $I_3 = J$

this becomes

$$\bar{L}_0 = 0 - \omega_p \sin\theta \left(I + m l^2\right) \bar{j} + (\omega_p \cos\theta + \Omega) J \, \bar{k}$$

$$\Rightarrow \boxed{\bar{L}_0 = -\omega_p \sin\theta \, (I + m l^2) \, \bar{j} + (\omega_p \cos\theta + \Omega) J \, \bar{k}}$$

d.) The equations of motion about point O can be derived as (7)

$$\frac{d\bar{L}_o}{dt} = \Sigma \bar{M}_o \qquad (1)$$

Using the expression for $\bar{L}_o$ given on the previous page leads to

$$\frac{d\bar{L}_o}{dt} = \bar{\omega}^* \times \bar{L}_o \qquad (\text{since } \dot{\omega}_p = \dot{\Omega} = \dot{\theta} = 0)$$

$$\Rightarrow \frac{d\bar{L}_o}{dt} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & -\omega_p \sin\theta & \omega_p \cos\theta \\ 0 & -\omega_p \sin\theta \, (I + ml^2) & [(\omega_p \cos\theta + \Omega)J] \end{vmatrix}$$

$$\Rightarrow \frac{d\bar{L}_o}{dt} = \bar{i} \cdot \Big(-\omega_p \sin\theta \, (\omega_p \cos\theta + \Omega) J + \omega_p^2 \sin\theta \cos\theta \, (I + ml^2) \Big)$$

Furthermore, $\Sigma \bar{M}_o = -mgl \sin\theta \, \bar{i}$

Inserting these two relations into (1) gives

$$-\omega_p \sin\theta \, (\omega_p \cos\theta + \Omega) J$$
$$+ \omega_p^2 \sin\theta \cos\theta \, (I + ml^2) \quad = \quad -mgl \sin\theta$$

$$\Rightarrow \quad -J\omega_p (\omega_p \cos\theta + \Omega) + (I + ml^2)\, \omega_p^2 \cos\theta$$
$$= -mgl$$

$$\Rightarrow \boxed{(I + ml^2 - J)\, \omega_p^2 \cos\theta - J\omega_p \Omega + mgl = 0}$$

e.) From the above 2nd order polynomial in terms of ω_p, the following 2 roots can be computed:

$$\boxed{\omega_p = \frac{J\Omega \pm \sqrt{(J\Omega)^2 - 4 * (I + ml^2 - J)\cos\theta \, mgl}}{2(I + ml^2 - J)\cos\theta}}$$

which represent slow and fast steady precession.

Question (3)

Examine the stability of all equilibrium points in the domain $x \in [0, 2\pi]$ for a dynamical system with the equation of motion given by: $\boxed{\ddot{x} + 3\dot{x} + 4\cos x = 0}$ (1)

For an equilibrium point we have:
$\ddot{x} = \dot{x} = 0$. Hence, with the above equation of motion we thus have to determine the solutions of:

$$4\cos x = 0$$

$$\Longrightarrow \quad x = \frac{\pi}{2} \quad \vee \quad x = \frac{3\pi}{2}.$$

Writing the velocity as $y = \dot{x}$, the above eq. of motion can be decomposed into two 1st order differential equations:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -3y - 4\cos x \end{bmatrix}$$

Linearizing this set of differential equations leads to

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{dy}{dx} & \frac{dy}{dy} \\ \frac{d(-3y-4\cos x)}{dx} & \frac{d(-3y-4\cos x)}{dy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4\sin x & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Evaluating the above set of diff. eqs. about the 1st equilibrium point $x = \frac{\pi}{2}$ gives

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Writing the above system of eqs. as $\dot{\bar{x}} = \bar{A}\,\bar{x}$, and seeking for solutions $\bar{x} = \bar{c}\,e^{\lambda t}$, leads to the equation

$$\left| \bar{\bar{A}} - \lambda \bar{\bar{I}} \right| = 0$$

or

$$\begin{vmatrix} -\lambda & 1 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 + 3\lambda - 4 = 0$$

$$\lambda = 1 \quad \vee \quad \lambda = -4$$

Since one of the roots is positive, the equilibrium point $x = \frac{\pi}{2}$ is <u>unstable</u>.

Using a similar procedure for the 2nd equilibrium point $x = \frac{3\pi}{2}$, leads to

$$\begin{vmatrix} -\lambda & 1 \\ -4 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 4 = 0$$

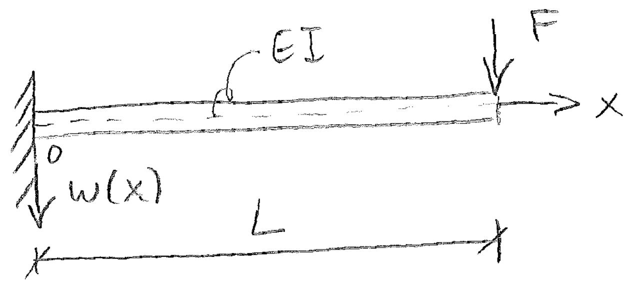
$$\Rightarrow \quad \lambda_{1,2} = \frac{-3 \pm \sqrt{9-16}}{2}$$

$$\Rightarrow \lambda_{1,2} = -\frac{3}{2} \pm \frac{i}{2}\sqrt{7}$$

$\lambda_{1,2} \in \mathbb{C}$ with $Re(\lambda) < 0$,

So the response is harmonically decaying, and thus <u>stable</u> at $x = \frac{3\pi}{2}$.

Question (4)



a) Formulate the generalised potential energy.

The virtual work generated by the external load F is

$$\delta W = F \cdot \delta w(L).$$

For conservative systems we may write

$$\delta W = -\delta V^{loading}$$

with $V^{loading}$ the potential energy caused by the loading; i.e.,

$$V^{loading} = -F\, w(L).$$

Adding this potential energy to the strain energy leads to the generalised potential energy as :

$$V = V^e + V^{loading}$$

$$\Rightarrow \boxed{V = \frac{1}{2} \int_0^L EI \, w_{xx}^2 \, dx - F w(L)}$$

b.) Find the Euler-Lagrange equation for the elastic deformation, and the corresponding boundary conditions.

For a static problem the potential energy must be stationary :

$$\delta V = 0$$

$$\Rightarrow \delta V = \int_0^L EI \, w_{xx} \, \delta w_{xx} \, dx - F \, \delta w(L)$$

$$= 0$$

Integration by parts leads to

$$\delta V = EI \, w_{xx} \, \delta u_x \Big|_0^L - \int_0^L EI \, w_{xxx} \, \delta w_x \, dx$$

$$- F \, \delta w(L) = 0 .$$

Again performing integration by parts gives:

$$\delta V = EI\, w_{xx}\, \delta w_x \Big|_0^L - \left\{ EI\, w_{xxx}\, \delta w \Big|_0^L - \int_0^L EI\, w_{xxxx}\, \delta w\, dx \right\} - F\, \delta w(L) = 0$$

Rewriting this expression leads to

$$\delta V = \underbrace{EI\, w_{xx}\, \delta w_x \Big|_0^L}_{1} - \underbrace{(EI\, w_{xxx} + F)\, \delta w \Big|_{x=L}}_{2} + \underbrace{EI\, w_{xxx}\, \delta w \Big|_{x=0}}_{3} + \underbrace{\int_0^L EI\, w_{xxxx}\, \delta w\, dx}_{4} = 0$$

The Euler-Lagrange equation follows from 4 $= 0$, i.e.,

$$\boxed{EI\, w_{xxxx} = 0 \qquad \text{for } 0 < x < L}$$

The <u>natural</u> boundary conditions follow from 1 $= 0$, and 2 $= 0$, as:

1 $= 0$: $EI\, w_{xx}\, \delta w_x \Big|_{x=L} = 0 \Rightarrow \boxed{EI\, w_{xx}(L) = 0}$

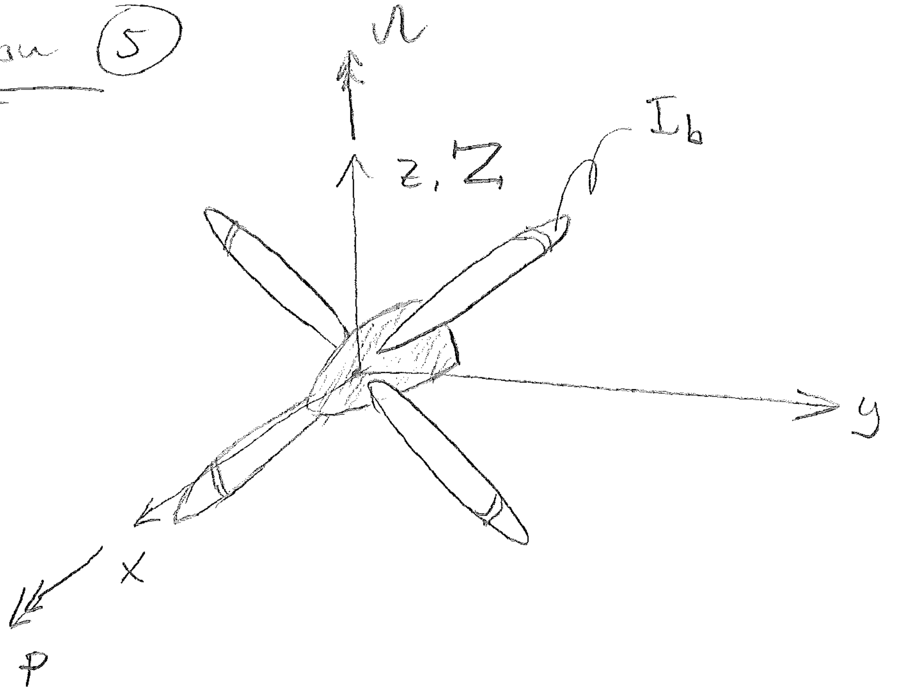
2 $= 0$: $(EI\, w_{xxx} + F)\, \delta w \Big|_{x=L} = 0 \Rightarrow \boxed{EI\, w_{xxx}(L) = -F}$

The essential boundary conditions follow from ① = 0 and ③ = 0 as:

① = 0 : $EI\, w_{xx}\, \delta w_x \Big|_{x=0} = 0 \Rightarrow \boxed{\delta w_x(0) = 0 \text{ or } w_x(0) = 0}$

③ = 0 : $EI\, w_{xxx}\, \delta w \Big|_{x=0} = 0 \Rightarrow \boxed{\delta w(0) = 0 \text{ or } w(0) = 0}$

Question ⑤



a) The mass moments of inertia in the x-y-z frame of reference are:

$$I_{xx} = I_1 = 4 \cdot I_b$$

In the y-z plane there are 4 symmetry axes. Hence, all directions are principal in that plane, i.e.,

$I_{yy} = I_{zz} = I_2 = I_3$ (1)

In addition, since the propeller motion is confined to the y-z plane, we may write

$$I_{xx} = \int_m (y^2 + z^2)\, dm = \int_m y^2 dm + \int_m z^2 dm$$

$$\Rightarrow \quad I_{xx} = I_{yy} + I_{zz} \quad (2)$$

Combining (1) and (2) yields:

$$I_{yy} = I_{zz} = \frac{I_{xx}}{2} \quad \text{or} \quad I_2 = I_3 = \frac{I_1}{2}$$

With $I_1 = 4 I_b$, we thus get $I_2 = 2 I_b$, and $I_3 = 2 I_b$.

b.) The angular velocity of the propeller is given by

$$\bar{\omega} = p\,\bar{i} + \Omega\,\bar{K}$$
$$= p\,\bar{i} + \Omega\,\bar{k}$$

Hence, the angular momentum becomes

$$\bar{L}_0 = I_1 \cdot p\,\bar{i} + I_3 \cdot \Omega\,\bar{K}$$

$$\Rightarrow \bar{L}_o = 4 I_b \cdot p \bar{i} + 2 I_b \cdot \Omega \bar{K}$$

accordingly,

$$\frac{d\bar{L}_o}{dt} = 4 I_b p \mathring{\bar{i}} \qquad (\text{since } \mathring{p} = \mathring{\Omega} = 0 \text{ and } \dot{\bar{K}} = 0)$$

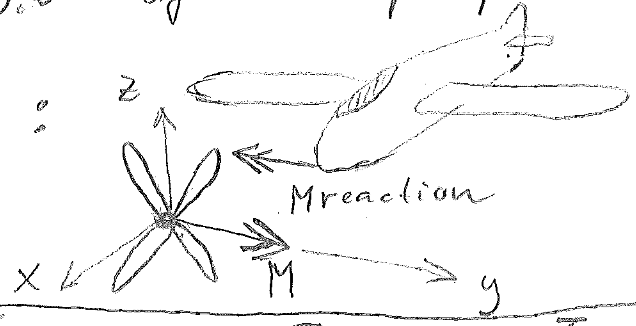
$$\Rightarrow \frac{d\bar{L}_o}{dt} = 4 I_b \cdot p (\bar{\omega}^* \times \bar{i})$$

where $\bar{\omega}^* = \Omega \bar{k}$ is the rotation of the x-y-z frame of reference.

$$\Rightarrow \frac{d\bar{L}_o}{dt} = 4 I_b p (\Omega \bar{k} \times \bar{i})$$
$$= +4 I_b p \Omega \bar{j}$$

Since $\frac{d\bar{L}_o}{dt} = \sum \bar{M}_o$, the moment on the propeller is $\bar{M} = 4 I_b p \Omega \bar{j}$

From the F.B.D.'s of the propeller and the airplane:



we conclude that $\boxed{\bar{M}_{reaction} = -\bar{M} = -4 I_b p \Omega \bar{j}}$

Alternative elaboration: Use the Euler equations for computing the moment $\bar{M}_{reaction}$.

$$I_1 \alpha_1 - (I_2 - I_3)\omega_2\omega_3 = M_1$$

$$I_2 \alpha_2 - (I_3 - I_1)\omega_1\omega_3 = M_2 \qquad (1)$$

$$I_3 \alpha_3 - (I_1 - I_2)\omega_1\omega_2 = M_3$$

The Euler equations refer to a frame of reference connected to the propellers (and not to the nose). However, this gives the same result, since All directions in the y-z-plane are principal directions. Hence, we may use the frame of reference x-y-z.

Since $\bar{\omega} = p\bar{i} + \Omega\bar{K}$,
$= p\bar{i} + \Omega\bar{k}$

$$\bar{\alpha} = \frac{d\bar{\omega}}{dt} = p\dot{\bar{i}} \qquad (\text{since } \dot{p} = 0,\ \dot{\Omega} = 0 \text{ and } \dot{\bar{K}} = 0)$$

$$\bar{\alpha} = p(\bar{\omega} \times \bar{i})$$

$$\Rightarrow \bar{\alpha} = p((p\bar{i} + \Omega\bar{k}) \times \bar{i})$$

$$\Rightarrow \bar{\alpha} = p\Omega\bar{j}$$

Substituting the components of $\bar{\omega}$ and $\bar{\alpha}$ in the Euler relations (1) gives:

$$4I_b \cdot 0 - (2I_b - 2I_b) \cdot 0 \cdot \Omega = M_1$$

$$2I_b \cdot p\Omega - (2I_b - 4I_b) \cdot p\Omega = M_2$$

$$2I_b \cdot 0 - (4I_b - 2I_b) \cdot p \cdot 0 = M_3$$

$$\Rightarrow \quad M_1 = 0$$

$$\Rightarrow \quad 4I_b p\Omega = M_2$$

$$\Rightarrow \quad M_3 = 0.$$

So, $\bar{M} = 4I_b p\Omega \cdot \bar{j}$

$$\boxed{\bar{M}_{reaction} = -\bar{M} = -4I_b p\Omega \, \bar{j}}$$

(as deduced from the F.B.D.'s of the propeller and the Airplane, see pg (17))