Equation Handout

WB2301 System Identification & Parameter Estimation

Alfred C Schouten

version: February 24, 2010

1 Time domain

1.1 Time signals

$$u(i), y(i); \quad i \in [1, 2, 3 \ldots, N]$$
$$\Delta t = \frac{1}{f_s} = \frac{T}{N}$$

1.2 Cross-correlation function

$$\hat{\phi}_{uy}(\tau) = \frac{1}{N}\sum_{i=\tau}^{N} u(i-\tau)y(i); \quad |\tau| < N \tag{1}$$
$$= \texttt{xcorr(y,u,'biased')} \tag{2}$$

The cross-product function is scaled by N. Division by N gives the 'baised' estimator which is preferred over the 'unbiased' estimator (division by $N - |\tau|$).

1.3 Cross-covariance function

$$C_{uy}(\tau) = \frac{1}{N}\sum_{i=\tau}^{N} \left(u(i-\tau) - \mu_u\right)\left(y(i) - \mu_y\right); \quad |\tau| < N \tag{3}$$
$$= \texttt{xcov(y,u,'biased')} \tag{4}$$
$$C_{uu}(0) = {\sigma_u}^2 \tag{5}$$

The righthand-side of (3) can be expanded as:

$$\frac{1}{N}\sum_{i=\tau}^{N}(u(i-\tau)y(i) - \frac{1}{N}\sum_{i=\tau}^{N} u(i-\tau)\mu_y - \frac{1}{N}\sum_{i=\tau}^{N} y(i)\mu_u + \frac{1}{N}\sum_{i=\tau}^{N} \mu_u\mu_y \tag{6}$$

The first term of (6) equals $\hat{\phi}_{uy}$ (Eq. 1). For a large number of summation elements ($N - \tau \approx N$), each of the last three terms converge to $\mu_u\mu_y$ and Eq. (7) can be used. Note that in case an 'unbiased' estimator is used the fourth term of (6) equals $\mu_u\mu_y$ straightforward.

$$C_{uy}(\tau) = \hat{\phi}_{uy}(\tau) - \mu_u\mu_y \qquad N - \tau \approx N \tag{7}$$

1.4 Cross-correlation function

$$\begin{aligned} r_{uy}(\tau) &= \frac{1}{N}\sum_{i=\tau}^{N}\frac{(u(i-\tau)-\mu_u)}{\sigma_u}\frac{(y(i)-\mu_y)}{\sigma_y}; \quad |\tau|<N \\ &= \frac{C_{uy}(\tau)}{\sigma_u\sigma_y} \\ &= \texttt{xcov(y,u,'coeff')} \end{aligned}$$

2 Frequency domain

2.1 Discrete Fourier Transform (DFT)

$$\begin{aligned} U(f) &= DFT\{u(i)\} = \sum_{i=1}^{N} u(i)e^{-j2\pi\frac{fi}{N}}; f = 0,\ldots,N-1 \\ &= \texttt{fft(u)} \\ u(i) &= DFT^{-1}\{U(f)\} = \frac{1}{N}\sum_{f=0}^{N-1} U(f)e^{j2\pi\frac{fi}{N}}; i = 1,\ldots,N \\ &= \texttt{ifft(U)} \end{aligned}$$

2.2 Frequency signals

$$\begin{aligned} &U(f), Y(f); \quad f \in [0,1,2,\ldots,N-1] \\ &\Delta f = \frac{1}{N\Delta t} = \frac{1}{T} \\ &U(-f) = U^*(f) \end{aligned}$$

Note that the $U(f)$ is mirrored and only contains relevant information for $f \in \left[0,1,2,\ldots,\frac{N}{2}\right]$.

2.3 Spectral densities via cross-covariance function

$$S_{uy} = DFT\{C_{uy}\}$$

2.4 Spectral densities via Fourier transformed signals

$$\begin{aligned} S_{uu}(f) &= \frac{1}{N}U(f)U(-f) = \frac{1}{N}U(f)U^*(f) = \frac{1}{N}|U(f)|^2 \\ &= \frac{1}{N}|\texttt{fft(u)}|^2 \\ S_{uy}(f) &= \frac{1}{N}Y(f)U(-f) = \frac{1}{N}Y(f)U^*(f) \end{aligned}$$

2.5 Parseval's Theorem

$$RMS_u = \sqrt{\frac{\sum u^2}{N}}$$

Parseval's theorem gives the relation between the Root-Mean-Squared (RMS) of a signal and the integral of the auto spectral density:

$$\begin{aligned}
{RMS_u}^2 &= \frac{1}{N}\sum_{f=0}^{N-1} S_{uu}(f) \\
\frac{\sum u^2}{N} &= \frac{1}{N}\sum_{f=0}^{N-1}\frac{1}{N}|U(f)|^2 = \frac{1}{N^2}\sum_{f=0}^{N-1}|U(f)|^2 \\
\frac{\Sigma u^2}{N} &= \frac{1}{N^2}\sum |\texttt{fft(u)}|.^2
\end{aligned}$$

Note that in case of a zero-mean signal the RMS value (approximately) equals the standard deviation, i.e. the integral of the auto-spectral density equals the signal variance:

$$C_{uu}(0) = {\sigma_u}^2 = \frac{1}{N}\sum_{f=0}^{N-1} S_{uu}(f) = \frac{2}{N}\sum_{f=0}^{\frac{N}{2}-1} S_{uu}(f); \quad \texttt{if } \mu_u = 0$$

3 Identification of transfer functions

3.1 Open loop: time domain approach

$$\begin{aligned}
&y(t) = n(t) + \int g(t')u(t-t')dt' \\
&y(t+\tau)u(t) = n(t+\tau)u(t) + \int g(t')u(t+\tau-t')u(t)dt' \\
&C_{uy}(\tau) = C_{nu}(\tau) + \int g(t')C_{uu}(\tau)dt'
\end{aligned}$$

if $u(t)$ is white noise ($C_{uu}(\tau) = {\sigma_u}^2 \texttt{ if } \tau = 0 \texttt{ and } C_{uu}(\tau) = 0 \texttt{ if } \tau \neq 0$) then holds:

$$C_{yu}(\tau) = C_{nu}(\tau) + g(\tau){\sigma_u}^2$$

3.2 Open loop: frequency domain approach

Indirect approach (Fourier transform of covariances)

$$\begin{aligned}
&C_{uy}(\tau) = C_{nu}(\tau) + \int g(t')C_{uu}(\tau)dt' \\
&S_{uy}(f) = S_{nu}(f) + G(f)S_{uu}(f)
\end{aligned}$$

Direct appraoch (Fourier transform of signals)

$$
\begin{aligned}
&Y(f) = G(f)U(f) + N(f) \\
&Y(f)U^*(f) = G(f)U(f)U^*(f) + N(f)U^*(f) \\
&S_{uy} = G(f)S_{uu}(f) + S_{nu}(f)
\end{aligned}
$$

if u and n are not correlated ($S_{nu} = 0$):

$$
H(f) = \frac{S_{uy}(f)}{S_{uu}(f)}; \quad \texttt{if}\ S_{nu}(f) = 0\ \forall\ f
$$