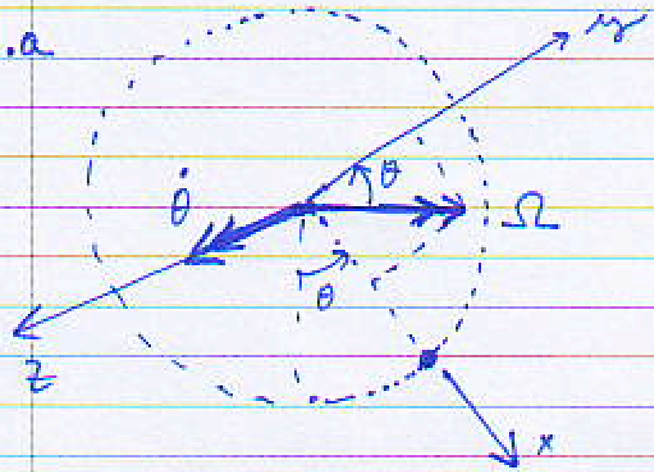
opgave no.	

naam	
studienummer	
vak AE3 - 914	code
datum 13 - 4 - 6	fac.

Gebruik voor elke opgave een afzonderlijk vel papier !

1.a



$$\underline{r}_{rel} = R\,\underline{i}$$

$$\underline{\omega} = \Omega \sin\theta\,\underline{i} + \Omega\cos\theta\,\underline{j} + \dot{\theta}\,\underline{k}$$

$$\underline{v}_{rel} = \underline{0}$$

$$\underline{v}_{xyz} = \underline{0}$$

$$\underline{v} = \underline{v}_{xyz} + \underline{\omega} \times \underline{r}_{rel} + \underline{v}_{rel} =$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \Omega\sin\theta & \Omega\cos\theta & \dot{\theta} \\ R & 0 & 0 \end{vmatrix} = R\dot{\theta}\,\underline{j} - R\Omega\cos\theta\,\underline{k}$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m R^2 (\dot{\theta}^2 + \Omega^2 \cos^2\theta)$$

$$V = \frac{p Q \cos\theta}{4\pi\varepsilon_0 R^2}$$

$$L = T - V$$

$$= \frac{1}{2} m R^2 (\dot{\theta}^2 + \Omega^2\cos^2\theta) - \frac{p Q \cos\theta}{4\pi\varepsilon_0 R^2}$$

1.b

$$\frac{\partial L}{\partial \dot\theta} = mR^2\dot\theta; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = mR^2\ddot\theta$$

$$\frac{\partial L}{\partial \theta} = -mR^2\Omega^2\cos\theta\sin\theta + \frac{pQ\sin\theta}{4\pi\varepsilon_0 R^2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\boxed{mR^2\ddot\theta + mR^2\Omega^2\sin\theta\cos\theta - \frac{pQ\sin\theta}{4\pi\varepsilon_0 R^2} = 0}$$

1.c

$$\ddot\theta + \Omega^2\sin\theta\cos\theta - \frac{pQ\sin\theta}{4\pi\varepsilon_0 m R^4} = 0$$

$\theta = 0 \Rightarrow \ddot\theta = 0$ (indeed, equilibrium point)

Linearisation Method

$$\left.\begin{aligned}\dot\theta &= \varphi\\ \dot\varphi &= -\Omega^2\sin\theta\cos\theta + \frac{pQ\sin\theta}{4\pi\varepsilon_0 m R^4}\end{aligned}\right\}$$

At $\theta=0$, linearisation provides:

$$\begin{bmatrix}\dot\theta\\ \dot\varphi\end{bmatrix} = \left.\begin{bmatrix}0 & 1\\ -\Omega^2(\cos^2\theta-\sin^2\theta)+\frac{pQ\cos\theta}{4\pi\varepsilon_0 m R^4} & 0\end{bmatrix}\right|_{\theta=0}\begin{bmatrix}\theta\\ \varphi\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\Omega^2 + \dfrac{pQ}{4\pi\varepsilon_a m R^4} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \varphi \end{bmatrix}$$

Eigenvalues of Jacobian:

$$\begin{vmatrix} -\lambda & 1 \\ -\Omega^2 + \dfrac{pQ}{4\pi\varepsilon_a m R^4} & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 = \frac{pQ}{4\pi\varepsilon_a m R^4} - \Omega^2$$

For stability $\lambda \in \mathbb{C} - \mathbb{R}$ or $\mathrm{Re}(\lambda) < 0$.

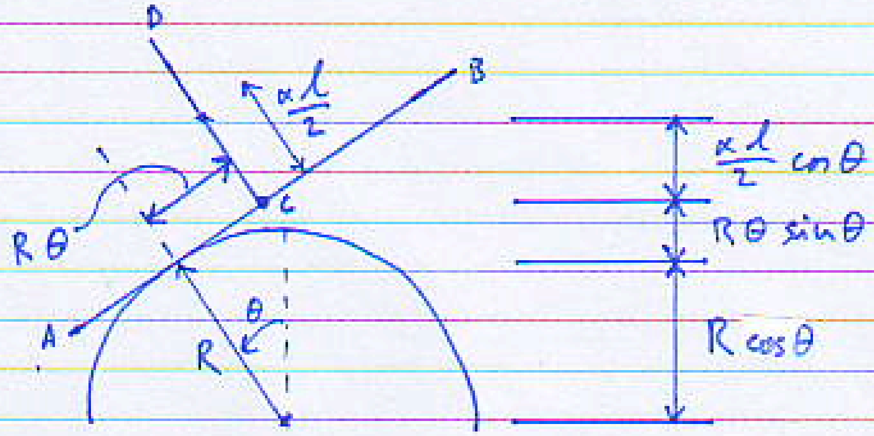
In this case: $\dfrac{pQ}{4\pi\varepsilon_a m R^4} - \Omega^2 < 0$ for stability

if $Q < 0$, stability condition is always fulfilled

if $Q > 0$, then the condition on Ω is

$$\Omega^2 > \frac{pQ}{4\pi\varepsilon_a m R^4}$$

2.a



$$V_{AB} = m_{AB}\, g\, h_{AB} = \rho A l g \left(R\cos\theta + R\theta\sin\theta \right)$$

$$V_{CD} = m_{CD}\, g\, h_{CD} = \rho A \alpha l g \left(R\cos\theta + R\theta\sin\theta + \frac{\alpha l}{2}\cos\theta \right)$$

$$V = V_{AB} + V_{CD} =$$

$$= \rho A l g \left((1+\alpha) R (\cos\theta + \theta\sin\theta) + \frac{\alpha^2 l}{2}\cos\theta \right)$$

2.b $R = 1\,m$, $A = 4 \cdot 10^{-4}\,m^2$, $\rho = 2500\,kg/m^3$, $l = 2\,m$, $g = 10\,m/s^2$

$$V = 20\left((1+\alpha+\alpha^2)\cos\theta + (1+\alpha)\theta\sin\theta \right)$$

$$V' = 20\left(-(1+\alpha+\alpha^2)\sin\theta + (1+\alpha)(\sin\theta + \theta\cos\theta) \right)$$

$V'(0) = 0 \Rightarrow \theta = 0$ indeed equilibrium

$$V'' = 20\left(-(1+\alpha+\alpha^2)\cos\theta + (1+\alpha)(\cos\theta + \cos\theta - \theta\sin\theta) \right)$$

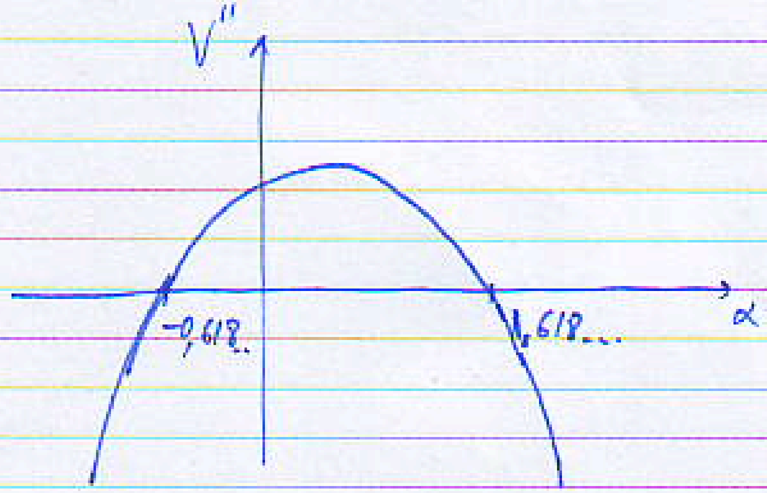
$$V''(0) = 20\left(-(1+\alpha+\alpha^2) + 2(1+\alpha) \right) = 20(1+\alpha-\alpha^2)$$

$V''(0) > 0$ for stability

$1 + \alpha - \alpha^2 > 0$

$$1 + \alpha - \alpha^2 = 0 \Rightarrow \alpha = \frac{-1 \pm \sqrt{1+4}}{-2} = \begin{cases} 1{,}618\ldots \\ -0{,}618\ldots \end{cases}$$

V'' v.s. α :



If $\alpha < 0$ the mass in V_{c0} would be negative, so the only acceptable interval for α is

$$0 \leq \alpha \leq 1{,}618\ldots$$

By the way, 1,618... is the golden section,

$$\Phi = \frac{1+\sqrt{5}}{2}$$

3.a

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + \dot{s}^2) + \beta(r)\dot{s} + \gamma(r) - V(r)$$

s is ignorable, so

$$\frac{\partial L}{\partial \dot{s}} = \boxed{m\dot{s} + \beta(r)} = C_s$$ is an integral of motion

$L \neq L(t)$ thus

$$h = \frac{\partial L}{\partial \dot{r}}\dot{r} + \frac{\partial L}{\partial \dot{s}}\dot{s} - L =$$

$$m\dot{r}^2 + m\dot{s}^2 + \beta(r)\dot{s} - \frac{1}{2} m (\dot{r}^2 + \dot{s}^2) - \beta(r)\dot{s} - \gamma(r) + V(r)$$

$$= \boxed{\frac{1}{2} m (\dot{r}^2 + \dot{s}^2) - \gamma(r) + V(r)}$$ is an integral of motion

3.b

$$R = C_s \dot{s} - L \qquad \dot{s} = \frac{C_s - \beta(r)}{m}$$

$$= C_s \cdot \frac{C_s - \beta(r)}{m} - \frac{1}{2} m \dot{r}^2 - \frac{1}{2} m \left(\frac{C_s - \beta(r)}{m}\right)^2$$

$$- \beta(r) \frac{C_s - \beta(r)}{m} - \gamma(r) + V(r)$$

$$= \boxed{\frac{[C_s - \beta(r)]^2}{2m} - \frac{1}{2} m \dot{r}^2 - \gamma(r) + V(r)}$$

3.c

$$h = R - \frac{\partial R}{\partial \dot{r}} \dot{r}$$

$$= \frac{(C_s - \beta(r))^2}{2m} - \frac{1}{2} m \dot{r}^2 - \Upsilon(r) + V(r) + m \dot{r}^2$$

$$= \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{(C_s - \beta(r))^2}{2m} - \Upsilon(r) + V(r)}_{V^{eff}}$$

$$\boxed{V^{eff} = \frac{(C_s - \beta(r))^2}{2m} - \Upsilon(r) + V(r)}$$

4.a

$$\delta u(0) = \delta u(L) = 0$$

$$\boxed{u(0) = 0; \quad u(L) = 0}$$

Essential boundary conditions

4.b Ritz's method:

$$u(x) = \alpha\, h_1(x) + \beta\, h_2(x) + \gamma\, h_3(x)$$

$$= \alpha + \beta x + \gamma x^2$$

Use boundary conditions:

$$u(0) = 0 \implies \alpha = 0$$

$$u(L) = 0 \implies \beta = -\gamma L$$

$u(x)$ is reduced to $u(x) = -\gamma L x + \gamma x^2$

Substitute in functional:

$$u'(x) = -\gamma L + 2\gamma x$$

$$u''(x) = 2\gamma$$

$$V(u(x)) = \Phi(\gamma) =$$

$$= \int_0^L \left[\frac{1}{2} EI\,(2\gamma)^2 - \frac{1}{2} F\,(2\gamma x - \gamma L)^2 + q\,(\gamma x^2 - \gamma L x)\right] dx$$

$$= \int_0^L \left[2EI\gamma^2 - \frac{1}{2}F\left(4\gamma^2 x^2 - 4\gamma^2 x L + \gamma^2 L^2\right) + q\left(\gamma x^2 - \gamma L x\right)\right] dx$$

$$= 2EI\gamma^2 L - \frac{2F\gamma^2 L^3}{3} + F\gamma^2 L^3 - \frac{1}{2}F\gamma^2 L^3 + q\gamma\frac{L^3}{3} - q\gamma\frac{L^3}{2}$$

$$= \left(2EI\gamma^2 - \frac{F\gamma^2 L^2}{6} - \frac{q\gamma L^2}{6}\right)L$$

Extremal if:

$$\frac{\partial \phi}{\partial \gamma} = 0 \Rightarrow$$

$$\left[4EI\gamma - \left(\frac{F\gamma}{3} + \frac{q}{6}\right)L^2\right]L = 0$$

$$\gamma\left(4EI - \frac{FL^2}{3}\right) = \frac{qL^2}{6}$$

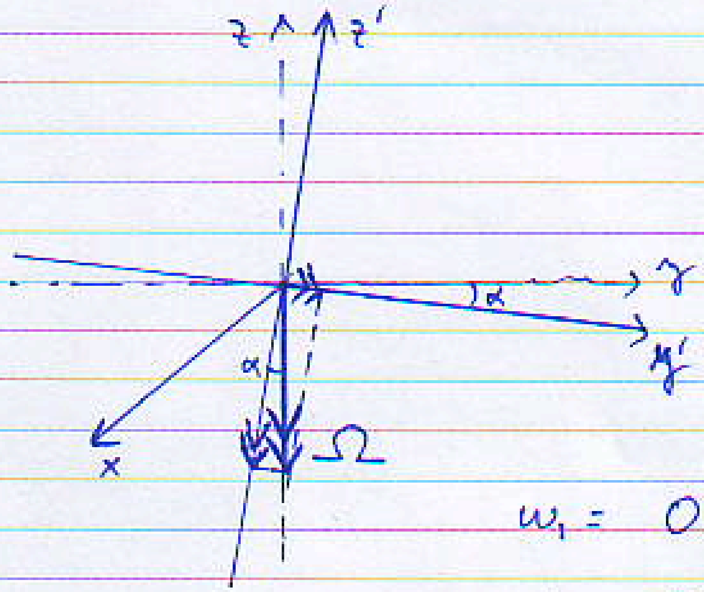
$$\gamma = \frac{qL^2}{24EI - 2FL^2}$$

$$u(x) = -\gamma L x + \gamma x^2$$

$$\boxed{= \frac{qL^2 x(x-L)}{24EI - 2FL^2}}$$

5

$$\underline{\underline{I}} = \begin{bmatrix} \frac{1}{4}mr^2 & 0 & 0 \\ 0 & \frac{1}{4}mr^2 & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix}$$



$\omega_1 = 0$

$\omega_2 = \Omega \sin\alpha$

$\omega_3 = -\Omega \cos\alpha$

$\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$

$$M_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = -\left(-\frac{1}{4}mr^2\right)(-1)\,\Omega^2 \sin\alpha\cos\alpha$$

$$M_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = 0$$

$$M_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0$$

Moment components on disc:

$M_1 = -\frac{1}{8}mr^2\Omega^2 \sin(2\alpha)$; $M_2 = 0$; $M_3 = 0$

Moment components on shaft:

$M_1 = +\frac{1}{8}mr^2\Omega^2 \sin(2\alpha)$; $M_2 = 0$; $M_3 = 0$