opgave no.		
		naam
		studienummer
		vak AE3-914 | code
		datum 4-4-7 | fac.

Gebruik voor elke opgave een afzonderlijk vel papier !

1

a. $T = \frac{1}{2} m (\dot r^2 + r^2 \dot\theta^2) + \frac{1}{2} m \dot r^2$

$V = -mg(L - r) = mgr \underbrace{- mgL}_{\text{constant}}$

$L = T - V = \boxed{m\dot r^2 + \frac{1}{2} m r^2 \dot\theta^2 - mgr}$

b. $\frac{\partial L}{\partial \dot r} = 2m\dot r$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = 2m\ddot r$

$\frac{\partial L}{\partial r} = mr\dot\theta^2 - mg$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0$

$\boxed{2\ddot r - r\dot\theta^2 + g = 0}$

$\frac{\partial L}{\partial \dot\theta} = m r^2 \dot\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = m(2r\dot r\dot\theta + r^2\ddot\theta)$

$\frac{\partial L}{\partial \theta} = 0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$

$\boxed{2\dot r\dot\theta + r\ddot\theta = 0}$

c.

$$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \theta \text{ is ignorable}$$

As θ is ignorable $\frac{\partial L}{\partial \dot{\theta}}$ is an integral of motion

$$\frac{\partial L}{\partial \dot{\theta}} = \boxed{m r^2 \dot{\theta} = C_\theta}$$

As $L \neq L(t)$, i.e., $\frac{\partial L}{\partial t} = 0$, h is an integral of motion:

$$h = \frac{\partial L}{\partial \dot{r}} \dot{r} + \frac{\partial L}{\partial \dot{\theta}} \dot{\theta} - L = 2 m \dot{r}^2 + m r^2 \dot{\theta}^2$$

$$- m \dot{r}^2 - \frac{1}{2} m r^2 \dot{\theta}^2 + m g r$$

$$= \boxed{m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + m g r = C_h}$$

d.

$$R = C_\theta \dot{\theta} - L \qquad \dot{\theta} = \frac{C_\theta}{m r^2}$$

$$R = \frac{C_\theta^2}{m r^2} - m \dot{r}^2 - \frac{1}{2} m r^2 \frac{C_\theta^2}{m^2 r^4} + m g r$$

$$= \boxed{\frac{1}{2} \frac{C_\theta^2}{m r^2} - m \dot{r}^2 + m g r}$$

$R = R(r, \dot{r})$; $R \neq R(t) \Rightarrow$ h based on R is an integral of motion depending on $r, \dot{r}$ only:

$$h = R - \frac{\partial R}{\partial \dot{r}} \dot{r} =$$

$$= \frac{C_\theta^2}{2mr^2} - m\dot{r}^2 + mgr + 2m\dot{r}^2$$

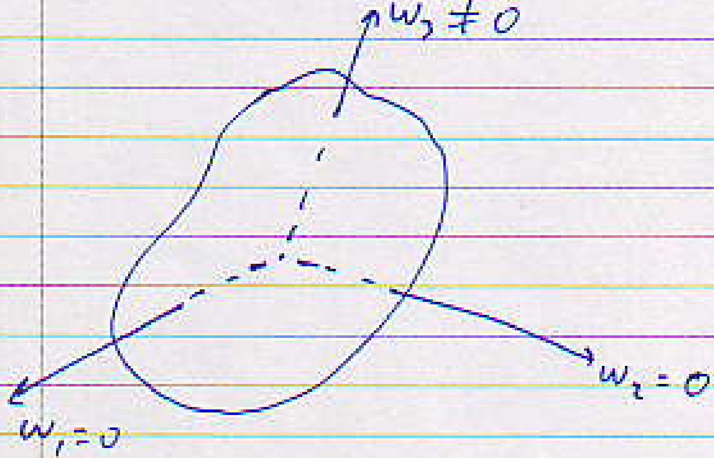
$$= \boxed{m\dot{r}^2 + \frac{C_\theta^2}{2mr^2} + mgr = C_h}$$

2. Euler equations:

$$M_1 = I_1 \dot{\omega}_1 - (I_2 - I_3)\,\omega_2\,\omega_3$$

$$M_2 = I_2 \dot{\omega}_2 - (I_3 - I_1)\,\omega_1\,\omega_3$$

$$M_3 = I_3 \dot{\omega}_3 - (I_1 - I_2)\,\omega_1\,\omega_2$$



$\underline{M} = \underline{0}$

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\,\omega_2\,\omega_3$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\,\omega_1\,\omega_3$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\,\omega_1\,\omega_2$$

Rotation about third axis $\Rightarrow$ $\omega_1 = \omega_2 = 0$

and consequently $\dot{\omega}_3 = 0$

$$
\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{I_2 - I_3}{I_1} w_3 \\ \frac{I_3 - I_1}{I_2} w_3 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
$$

This is a linear dynamic system in w_1 and w_2. Rotation about w_3 is equivalent to an equilibrium point. Indeed:

$$
w_1 = w_2 = 0 \Rightarrow \dot{w}_1 = \dot{w}_2 = 0
$$

Stability of this equilibrium can be assessed through the eigenvalues of the system matrix:

$$
\begin{vmatrix} -\lambda & \frac{I_2 - I_3}{I_1} w_3 \\ \frac{I_3 - I_1}{I_2} w_3 & -\lambda \end{vmatrix} = \lambda^2 - \frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2} w_3^2 = 0
$$

For stability $\lambda \in \mathbb{C} - \mathbb{R} \Rightarrow \frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2} < 0$

$$
\Rightarrow \begin{cases} I_2 < I_3 ;\ I_3 > I_1 \Rightarrow I_3 > I_1, I_2 \\ I_2 > I_3 ;\ I_3 < I_1 \Rightarrow I_3 < I_1, I_2 \end{cases}
$$

3.

a.

$$\delta u(0) = \delta u(L) = 0 \Rightarrow$$

$$\left.\begin{array}{l} u(0) = 0 \\ u(L) = 0 \end{array}\right\} \text{ are essential B.C.}$$

b.

$$V(u) = \int_0^L \phi(u'', u', u)\, dx$$

$$\delta V = \int_0^L \left(\frac{\partial \phi}{\partial u''} \delta u'' + \frac{\partial \phi}{\partial u'} \delta u' + \frac{\partial \phi}{\partial u} \delta u \right) dx$$

$$= \frac{\partial \phi}{\partial u''} \delta u' \Big|_0^L + \int_0^L \left(\frac{\partial \phi}{\partial u'} - \frac{d}{dx}\left(\frac{\partial \phi}{\partial u''} \right) \right) \delta u' \, dx$$

$$+ \int_0^L \frac{\partial \phi}{\partial u} \delta u \, dx$$

$$= \frac{\partial \phi}{\partial u''} \delta u' \Big|_0^L + \left[\frac{\partial \phi}{\partial u'} - \frac{d}{dx}\left(\frac{\partial \phi}{\partial u''} \right) \right] \delta u \Big|_0^L$$

$$+ \int_0^L \left[\frac{d^2}{dx^2}\left(\frac{\partial \phi}{\partial u''} \right) - \frac{d}{dx}\left(\frac{\partial \phi}{\partial u'} \right) + \frac{\partial \phi}{\partial u} \right] \delta u \, dx$$

$$
= \overbrace{\left.\frac{\partial \phi}{\partial u''}\right|_{x=L} \delta u'(L)}^{1} - \overbrace{\left.\frac{\partial \phi}{\partial u''}\right|_{x=0} \delta u'(0)}^{2}
$$

$$
+ \overbrace{\left.\left[\frac{\partial \phi}{\partial u'} - \frac{d}{dx}\left(\frac{\partial \phi}{\partial u''}\right)\right]\right|_{x=L} \delta u(L)}^{3} - \overbrace{\left.\left[\frac{\partial \phi}{\partial u'} - \frac{d}{dx}\left(\frac{\partial \phi}{\partial u''}\right)\right]\right|_{x=0} \delta u(0)}^{4}
$$

$$
+ \underbrace{\int_0^L \left(\frac{d^2}{dx^2}\left(\frac{\partial \phi}{\partial u''}\right) - \frac{d}{dx}\left(\frac{\partial \phi}{\partial u'}\right) + \frac{\partial \phi}{\partial u}\right) \delta u \, dx}_{5} = 0
$$

$1 = 0 \Rightarrow \left.\dfrac{\partial \phi}{\partial u''}\right|_{x=L} = 0$ Natural B.C. $\boxed{u''(L) = 0}$

$2 = 0 \Rightarrow \left.\dfrac{\partial \phi}{\partial u''}\right|_{x=0} = 0$ Natural B.C. $\boxed{u''(0) = 0}$

$3 = 0$ always, because $\delta u(L) = 0$, Essential B.C.

$4 = 0$ always, because $\delta u(0) = 0$, Essential B.C.

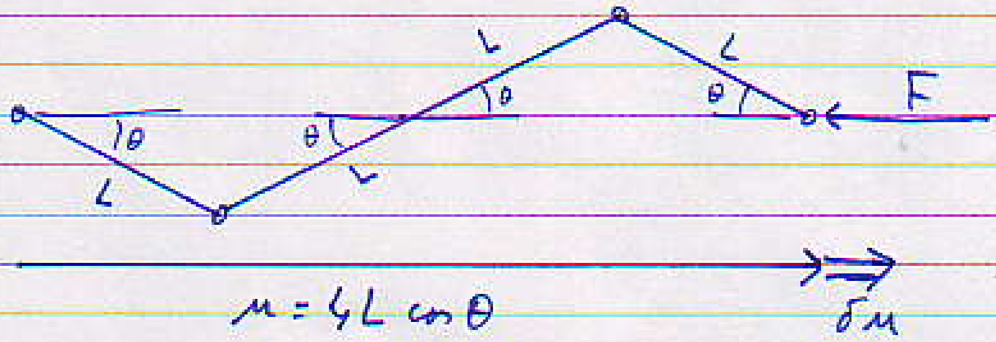
$5 = 0 \Rightarrow \dfrac{d^2}{dx^2}\left(\dfrac{\partial \phi}{\partial u''}\right) - \dfrac{d}{dx}\left(\dfrac{\partial \phi}{\partial u'}\right) + \dfrac{\partial \phi}{\partial u} = 0$

$$
\boxed{EI\, u'''' + F u'' + q = 0}
$$

Euler - Lagrange equation

4

a.



$$u = 4L\cos\theta$$

$$\delta W = -F\,\delta u$$

$$\delta u = \frac{\partial u}{\partial \theta}\delta\theta = -4L\sin\theta\,\delta\theta$$

$$\delta W = -F(-4L\sin\theta\,\delta\theta) = \underbrace{4FL\sin\theta}_{Q_\theta}\,\delta\theta$$

$$\boxed{Q_\theta = 4FL\sin\theta}$$

This generalised force is a <u>moment</u>

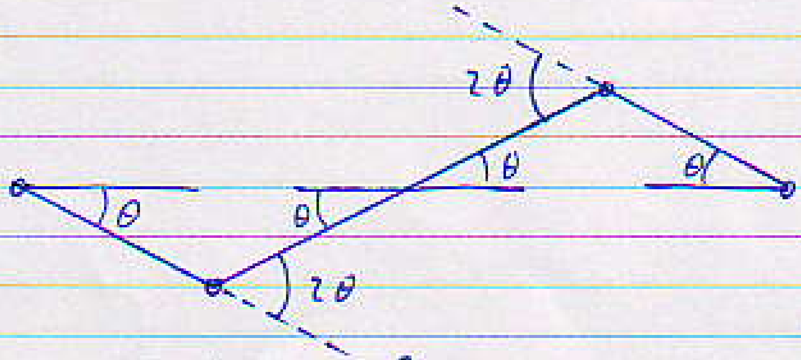
b.

$$-\frac{\partial V_{gen}}{\partial \theta} = Q_\theta$$

$$V_{gen} = \int -Q_\theta\, d\theta = \int -4FL\sin\theta\, d\theta$$

$$= 4FL\cos\theta$$

c.



$$V_{el} = 2 \cdot \frac{1}{2} k \cdot (2\theta)^2 = 4k\theta^2$$

$$V = V_{el} + V_{gen} = 4k\theta^2 + 4FL\cos\theta$$

$$V' = 8k\theta - 4FL\sin\theta$$

$V'(0) = 0 \rightarrow$ so $\theta = 0$ is an equilibrium point

$$V'' = 8k - 4FL\cos\theta$$

$V''(0) = 8k - 4FL > 0$ for stable equilibrium

$$\boxed{F < \frac{2k}{L}}$$