D&S April 6 2009 14:00-17:00

Problem I

a)

$$V = \frac{4Qq}{4\pi\varepsilon_0 s} + \frac{Qq}{4\pi\varepsilon_0 (d-s)}$$

$$V' = \frac{Qq}{4\pi\varepsilon_0}\left(-\frac{4}{s^2} + \frac{1}{(d-s)^2}\right)$$

Equilibrium $\Leftrightarrow$ $V' = 0$

$$-\frac{4}{s^2} + \frac{1}{(d-s)^2} = 0$$

$$-4(d-s)^2 + s^2 = 0$$

$$-4d^2 + 8ds - 4s^2 + s^2 = 0$$

$$3s^2 - 8ds + 4d^2 = 0$$

$$s = \frac{8d \pm \sqrt{64d^2 - 48d^2}}{6} = \begin{cases} \cancel{2d} & \text{NOT BETWEEN CHARGES} \\ \frac{2}{3}d \end{cases}$$

$$\boxed{s = \frac{2}{3}d}$$

b)

$$V'' = \frac{Qq}{4\pi\varepsilon_a}\left(\frac{8}{s^3} + \frac{2}{(d-s)^3}\right)$$

$$V''\left(\tfrac{2}{3}d\right) = \frac{Qq}{4\pi\varepsilon_a}\left(\frac{8}{\frac{8}{27}d^3} + \frac{2}{\frac{1}{27}d^3}\right)$$

$V''\left(\frac{2}{3}d\right) > 0$ for stability

As the term within brackets is always positive and so do Q and ε_a, the only possibility is

$$\boxed{q > 0}$$

so the moving charge needs to be positive for stability.

Problem II

a)

$$\underline{r}(t) = x(t)\,\underline{i} + y(t)\,\underline{j}$$

$$I(\underline{r}) = \int_0^{100} L(\underline{r})\,dt$$

$$L = T - V; \quad T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2); \quad V = mgy$$

$$\boxed{I(\underline{r}) = \int_0^{100} \left(\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy\right) dt}$$

b)

$$x(t) = a + bt + ct^2 \qquad y(t) = d + et + ft^2$$

$$x(0) = 0 \Rightarrow \boxed{a = 0}$$
$$y(0) = 0 \Rightarrow \boxed{d = 0}$$

$$x(100) = 100 \Rightarrow 100b + 10000c = 100$$

$$b = 1 - 100c$$

$$y(100) = 10000 \Rightarrow 100e + 10000f = 10000$$

$$e = 100 - 100f$$

$$x(t) = (1-100c)t + ct^2$$

$$y(t) = (100-100f)t + ft^2$$

$$\dot{x}(t) = (1-100c) + 2ct$$
$$\dot{y}(t) = (100-100f) + 2ft$$

$$I(\underline{r}) = \int_0^{100} m\left(\frac{1}{2}\left(((1-100c)+2ct)^2 + ((100-100f)+2ft)^2\right)\right.$$
$$\left. - 10((100-100f)t + ft^2)\right) dt$$

$$= m\int_0^{100}\left(\frac{1}{2}\left((1-100c)^2 + 2(1-100c)2ct + 4c^2t^2\right.\right.$$
$$\left. + (100-100f)^2 + 2(100-100f)2ft + 4f^2t^2\right)$$
$$\left. - 10\left((100-100f)t + ft^2\right)\right) dt$$

$$= m\left((1-100c)^2 50 + (1-100c)\,c\cdot 10000 + \frac{2}{3}c^2\cdot 1000000\right.$$
$$+ (100-100f)^2 50 + (100-100f)\,f\cdot 10000 + \frac{2}{3}f^2\cdot 1000000$$
$$\left. - 10\left((100-100f)5000 + \frac{1}{3}f\,1000000\right)\right)$$

$$= \Phi(c, f)$$

To find an extremal:

$$\frac{\partial \phi}{\partial c} = \frac{\partial \phi}{\partial f} = 0$$

$$\frac{\partial \phi}{\partial c} = 100(1-100c)\cdot(-100) - 100\cdot c\cdot 10000 + (1-100c)\cdot 10000$$
$$+ \frac{4}{3}c\cdot 1000000 = 0$$

$$-10000 + 10000c - 1000000c + 10000 - 1000000c$$
$$+ \frac{4000000}{3}c = 0$$

$$\boxed{c = 0}$$

$$\frac{\partial \phi}{\partial f} = 100(100-100f)(-100) - 100 f\cdot 10000 + (100-100f)\cdot 10000$$
$$+ \frac{4}{3}f\cdot 1000000 + 1000 \cdot 5000 - \frac{10000000}{3}$$

$$= -1000000 + 1000000f - 1000000f + 1000000 - 1000000f$$
$$+ \frac{4000000}{3}f + 5000000 - \frac{10000000}{3}$$
$$= \frac{1000000}{3}f + \frac{5000000}{3} = 0$$

$$\boxed{f = -5}$$

$$\boxed{\begin{aligned} b &= 1 - 100c = 1 \\ e &= 100 - 100f = 600 \end{aligned}}$$

So the solution is

$$\boxed{\begin{aligned} x(t) &= t \\ y(t) &= 600t - 5t^2 \end{aligned}}$$

c) The actual path of the projectile should be a parabola, which is a consequence of the kinematics of the problem.

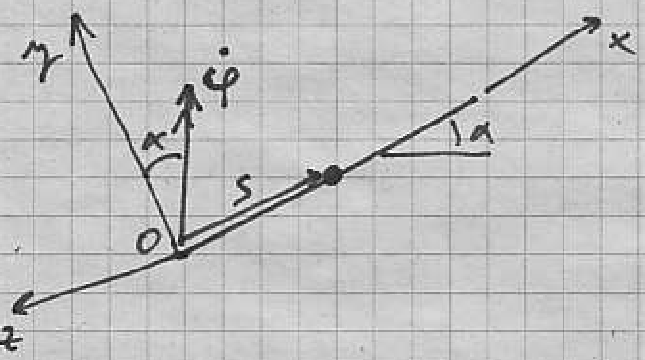
A second order expansion for Ritz method applied to Hamilton's principle provides consequently the <u>exact</u> answer.

This is further demonstrated by the obtained coefficients. Indeed, $c=0$ which corresponds to a linear dependence of the path on time in the x-direction.

Also the coefficient $f=-5$ corresponds to the term "$-\frac{1}{2}gt^2$" of the path in the y-direction.

Problem III

a)



$$\underline{r}_{rel} = s\,\underline{i} \qquad \underline{v}_{rel} = \dot s\,\underline{i}$$

$$\underline{\omega} = \dot\varphi \sin\alpha\,\underline{i} + \dot\varphi\cos\alpha\,\underline{j}$$

$$\underline{v} = \underline{v}_0 + \underline{\omega}\times\underline{r}_{rel} + \underline{v}_{rel}$$

$$= \underline{0} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dot\varphi\sin\alpha & \dot\varphi\cos\alpha & 0 \\ s & 0 & 0 \end{vmatrix} + \dot s\,\underline{i}$$

$$= \dot s\,\underline{i} - s\dot\varphi\cos\alpha\,\underline{k}$$

$$T = \frac{1}{2} m(\dot x^2 + \dot y^2) = \frac{1}{2} m(\dot s^2 + s^2\dot\varphi^2\cos^2\alpha)$$

$$V = mgs\sin\alpha$$

$$\boxed{L = T - V = \frac{1}{2} m(\dot s^2 + s^2\dot\varphi^2\cos^2\alpha) - mgs\sin\alpha}$$

b)

$L \neq L(\varphi) \Rightarrow \varphi$ is ignorable

$$C_\varphi = \frac{\partial L}{\partial\dot\varphi} = ms^2\dot\varphi\cos^2\alpha$$

This represents conservation of angular momentum c)

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = m\ddot{s}$$

$$\frac{\partial L}{\partial s} = m s \dot{\varphi}^2 \cos^2\alpha - mg \sin\alpha$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = \boxed{m\ddot{s} - ms\dot{\varphi}^2\cos^2\alpha + mg\sin\alpha = 0}$$

$$\dot{\varphi} = \frac{C_\varphi}{m s^2 \cos^2\alpha}$$

$$m\ddot{s} - m s \cos^2\alpha \frac{C_\varphi^2}{m^2 s^4 \cos^4\alpha} + mg\sin\alpha = 0$$

$$\boxed{\ddot{s} - \frac{C_\varphi^2}{m^2 s^3 \cos^2\alpha} + g\sin\alpha = 0}$$

d) $\dot{s} = \ddot{s} = 0$

$$\cancel{m} s \dot{\varphi}^2 \cos^2\alpha = \cancel{m} g \sin\alpha$$

$$\boxed{s\dot{\varphi}^2 = \frac{g\sin\alpha}{\cos^2\alpha}}$$

e)

$$\ddot{s} = \frac{C_\varphi^2}{m^2 s^3 \cos^2\alpha} - g \sin\alpha$$

$$\left.\begin{aligned} \dot{s} &= r \\ \dot{r} &= \frac{C_\varphi^2}{m^2 s^3 \cos^2\alpha} - g \sin\alpha \end{aligned}\right\}$$

Jacobian at steady motion:

$$\left.\begin{bmatrix} 0 & 1 \\ -\dfrac{3\, C_\varphi^2}{m^2 s^4 \cos^2\alpha} & 0 \end{bmatrix}\right|_{s\dot{\varphi}^2 = \frac{g \sin\alpha}{\cos^2\alpha}}$$

$$= \begin{bmatrix} 0 & 1 \\ -\dfrac{3\, \cancel{m^2} \cancel{s^4} \dot{\varphi}^2 \cos^{\cancel{4}}\alpha^{2}}{\cancel{m^2} \cancel{s^4} \cancel{\cos^2}\alpha} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\dfrac{3 g \sin\alpha}{s} & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{3g\sin\alpha}{s} & -\lambda \end{vmatrix} = \lambda^2 + \frac{3g\sin\alpha}{s} = 0$$

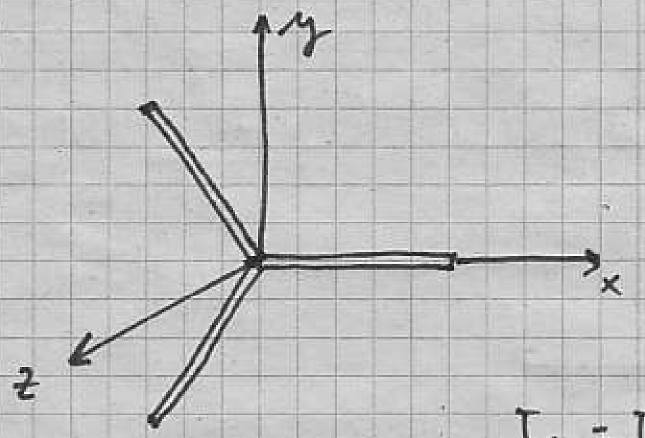
$$\lambda = \pm i\sqrt{\frac{3g\sin\alpha}{s}}$$ as $s > 0$ always

so $\lambda \in \mathbb{C} - \mathbb{R}$

so the steady motion is stable

Problem IV

a)



z = rotor shaft

$$I_3 = I_{zz} = 3\, I_b$$

In xy-plane there are three symmetry axes, consequently all directions are principal in that plane, so:

$$I_{xx} = I_{yy} = I_1 = I_2 \qquad (1)$$

Moreover the object is contined in the xy-plane, consequently

$$I_{zz} = \int (x^2 + y^2)\, dm = \int x^2 dm + \int y^2 dm$$

$$= I_{yy} + I_{xx} \qquad (2)$$

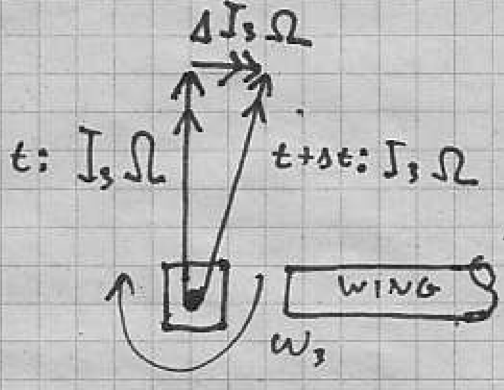
(1) and (2) together provide

$$I_{xx} = I_{yy} = \frac{I_{zz}}{2}\ ; \quad I_1 = I_2 = \frac{I_3}{2}$$

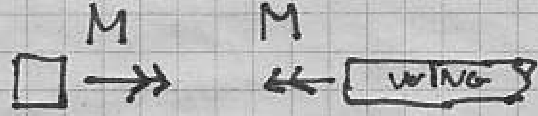
$$\boxed{I_1 = \frac{3}{2} I_b\ ; \quad I_2 = \frac{3}{2} I_b\ ; \quad I_3 = 3\, I_b}$$

b) Stairboard rotor:

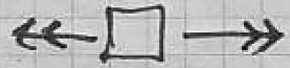
KD:



FBD:



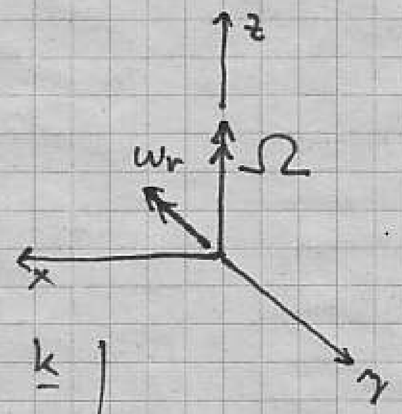
So the roll causes a positive torsional moment



c)

$w_1 = 0 \qquad w_2 = -w_r \qquad w_3 = \Omega$

$\dot{w}_3 = 0$



$\underline{w}_{12} = -w_r \underline{j}$

$$\frac{d\underline{w}_{12}}{dt} = \dot{w}_1 \underline{i} + \dot{w}_2 \underline{j} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ w_1 & w_2 & w_3 \\ w_1 & w_2 & 0 \end{vmatrix} = 0$$

$$\dot{w}_1 = w_2 w_3$$

$$\dot{w}_2 = -w_1 w_3$$

Euler equations:

$$\left.\begin{aligned} M_1 &= I_1 \dot{\omega}_1 - (I_2 - I_3)\,\omega_2 \omega_3 \\ M_2 &= I_2 \dot{\omega}_2 - (I_3 - I_1)\,\omega_1 \omega_3 \\ M_3 &= I_3 \dot{\omega}_3 - (I_1 - I_2)\,\omega_1 \omega_2 \end{aligned}\right\}$$

$$M_1 = \frac{3I_b}{2}(-w_r)\Omega - \left(\frac{3I_b}{2} - 3I_b\right)(-w_r)\Omega$$

$$= \boxed{-3 I_b w_r \Omega = M_1}$$

$$M_2 = \frac{3I_b}{2}\cdot 0 \cdot \Omega - \left(3I_b - \frac{3}{2}I_b\right)\cdot 0 \cdot \Omega = 0$$

$$M_3 = 3I_b \cdot 0 - \left(\frac{3I_b}{2} - \frac{3I_b}{2}\right)\cdot 0 \cdot (-w_r) = 0$$

So $M_1 = -3 I_b w_r \Omega$, which also is in the sense predicted in b)