Elaboration of the exam "Dynamics & Stability" of April 16, 2010. (AE3-914).

Question ①

a.) $\bar{V} = \bar{V}_{xyz} + \bar{\omega}_{xyz} \times \bar{r}_{rel} + \bar{V}_{rel}$;

$\bar{r}_{rel} = r\bar{i}$;

$\bar{V}_{rel} = \dot{r}\bar{i}$;

$\bar{\omega}_{xyz} = -\cos\theta\,\dot{\varphi}\,\bar{i} + \sin\theta\,\dot{\varphi}\,\bar{j} + \dot{\theta}\,\bar{k}$;

$\bar{V}_{xyz} = \bar{0}$;

$$\bar{V} = \bar{0} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\cos\theta\,\dot{\varphi} & \sin\theta\,\dot{\varphi} & \dot{\theta} \\ r & 0 & 0 \end{vmatrix} + \dot{r}\bar{i}$$

$\rightarrow \bar{V} = \bar{j}\,\dot{\theta} r + \bar{k}\,(-\sin\theta\,\dot{\varphi}\,r) + \dot{r}\bar{i}$

$\rightarrow \boxed{\bar{V} = \dot{r}\bar{i} + \dot{\theta} r \bar{j} - \dot{\varphi}\sin\theta\, r\bar{k}}$

b.) $T = \frac{1}{2}\cdot m \cdot (\bar{V}\cdot\bar{V})$

$\rightarrow \boxed{T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)}$

c.) $L = T - V$;

$V = -mgr\cos\theta + \frac{1}{2}kr^2$;

(the spring is unstretched for $r=0$)

$$\rightarrow \boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2) + mgr\cos\theta - \frac{1}{2}kr^2}$$

d.) There are 3 degrees of freedom, $\theta(t)$, $\varphi(t)$ and $r(t)$, which correspond to 3 eqs. of motion.

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \; ; \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = mr^2\sin\theta\cos\theta\,\dot{\varphi}^2 - mgr\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \; ;$$

$$\rightarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\sin\theta\cos\theta\,\dot{\varphi}^2 + mgr\sin\theta = 0$$

$$\rightarrow \boxed{\ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r} - \frac{1}{2}\sin 2\theta \cdot \dot{\varphi}^2 + \frac{g}{r}\sin\theta = 0}$$

$$\frac{\partial L}{\partial \dot{\varphi}} = m r^2 \sin^2\theta \, \dot{\varphi} \; ;$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = m r^2 \sin^2\theta \, \ddot{\varphi} + 2 m r^2 \sin\theta \cos\theta \, \dot{\theta} \, \dot{\varphi}$$

$$+ \; 2 m r \dot{r} \sin^2\theta \, \dot{\varphi} \; ;$$

$$\frac{\partial L}{\partial \varphi} = 0 \; ; \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \; ;$$

$$\rightarrow \quad m r^2 \sin^2\theta \, \ddot{\varphi} + 2 m r^2 \sin\theta \cos\theta \, \dot{\theta} \, \dot{\varphi} + 2 m r \dot{r} \sin^2\theta \, \dot{\varphi} = 0$$

$$\rightarrow \quad \boxed{\ddot{\varphi} + 2 \tan^{-1}\theta \, \theta \, \dot{\varphi} + \frac{2 \dot{r} \dot{\varphi}}{r} = 0}$$

$$\frac{\partial L}{\partial \dot{r}} = m \dot{r} \; ; \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m \ddot{r} \; ;$$

$$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 + m r \sin^2\theta \, \dot{\varphi}^2 + m g \cos\theta - k r \; ;$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\rightarrow \quad m \ddot{r} - m r \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\varphi}^2\right) - m g \cos\theta + k r = 0$$

$$\rightarrow \quad \boxed{\ddot{r} - r\left(\dot{\theta}^2 + \sin^2\theta \, \dot{\varphi}^2\right) + \frac{k}{m} r = g \cos\theta}$$

e.) ①: φ is an ignorable coordinate, since $L \neq L(\varphi)$.

Integral of motion:

$$\frac{\partial L}{\partial \dot\varphi} = \boxed{m r^2 \sin^2\theta \, \dot\varphi = C_\varphi}$$

i.e "Conservation of Angular momentum"

②: Since $L \neq L(t)$, the Jacobi energy integral h is an integral of motion

$$h = \dot\theta \frac{\partial L}{\partial \dot\theta} + \dot\varphi \frac{\partial L}{\partial \dot\varphi} + \dot r \frac{\partial L}{\partial \dot r} - L = C_h$$

$$\rightarrow \quad h = m r^2 \dot\theta^2 + m r^2 \sin^2\theta \, \dot\varphi^2 + m \dot r^2 - \tfrac{1}{2} m (\dot r^2 + r^2 \dot\theta^2 + r^2 \sin^2\theta \, \dot\varphi^2) - m g r \cos\theta + \tfrac{1}{2} k r^2 = C_h$$

$$\rightarrow \quad \boxed{\tfrac{1}{2} m (\dot r^2 + r^2 \dot\theta^2 + r^2 \sin^2\theta \, \dot\varphi^2) - m g r \cos\theta + \tfrac{1}{2} k r^2 = C_h}$$

Since the Jacobi energy integral equals the total energy, $E = T + V$, the total energy is conserved!

f.) Routhian: Since φ is an ignorable coordinate:

$$R = \frac{\partial L}{\partial \dot\varphi} \dot\varphi - L$$

$$\rightarrow \quad R = m r^2 \sin^2\theta \, \dot\varphi^2 - \tfrac{1}{2} m (\dot r^2 + r^2 \dot\theta^2 + r^2 \sin^2\theta \, \dot\varphi^2) - m g r \cos\theta + \tfrac{1}{2} k r^2$$

Since $\dot{\varphi} = \dfrac{C_\varphi}{mr^2 \sin^2\theta}$, this expression turns into

$$\boxed{R = \frac{C_\varphi^2}{2mr^2\sin^2\theta} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta + \frac{1}{2}kr^2}$$

g.) The effective potential V_{eff} can be obtained from the Routhian by putting the velocity terms equal to ZERO, $\dot{r}=0$; $\dot{\theta}=0$; which leads to

$$\boxed{V_{eff} = \frac{C_\varphi^2}{2mr^2\sin^2\theta} - mgr\cos\theta + \frac{1}{2}kr^2}$$

h.) Steady motion corresponds to :

1: $\dfrac{\partial R}{\partial \theta} = 0$, which equals $\dfrac{\partial V_{eff}}{\partial \theta} = 0$.

$$\rightarrow \frac{-C_\varphi^2 \, 4mr^2\sin\theta\cos\theta}{4m^2r^4\sin^4\theta} + mgr\sin\theta = 0;$$

$$\rightarrow \sin\theta\left(mgr - \frac{C_\varphi^2\cos\theta}{mr^2\sin^4\theta}\right) = 0$$

$$\rightarrow \sin\theta = 0 \Rightarrow \theta = 0^\circ \quad \text{or} \quad mgr - \frac{C_\varphi^2\cos\theta}{mr^2\sin^4\theta} = 0$$

With $C_\varphi = mr^2 \sin^2\theta\, \dot\varphi$ the second condition turns into

$$mgr - mr^2 \cos\dot\theta\, \dot\varphi^2 = 0$$

$$\rightarrow \quad \dot\varphi^2 = \frac{g}{r\cos\theta} \;;$$

which, with the initial conditions $\dot\varphi(0) = \dot\varphi_0$ and $\theta(0) = \theta_0$ and $r(0) = r_0$, results into

$$\boxed{\dot\varphi_0^2 = \frac{g}{r_0 \cos\theta_0}}$$

(2): $\frac{\partial R}{\partial r} = 0$, which equals $\frac{\partial V_{eff}}{\partial r} = 0$

$$\rightarrow \quad \frac{-C_\varphi^2 \cdot 4mr\sin^2\theta}{4m^2 r^4 \sin^4\theta} - mg\cos\theta + kr = 0$$

With $C_\varphi = mr^2\sin^2\theta\,\dot\varphi$, this expression becomes

$$-mr\sin^2\theta\,\dot\varphi^2 - mg\cos\theta + kr = 0$$

$$\rightarrow \quad \dot\varphi^2 = \frac{mg\cos\theta - kr}{-mr\sin^2\theta}$$

With the above initial conditions, this expression becomes:

$$\dot\varphi_0^2 = \frac{mg\cos\theta_0 - kr_0}{-mr_0\sin^2\theta_0}$$

Equating this expression with the other expression for steady motion yields

$$\frac{mg\cos\theta_0 - kr_0}{-mr_0\sin^2\theta_0} = \frac{g}{r_0\cos\theta_0}$$

$$\rightarrow \quad mgr_0\cos^2\theta_0 - kr_0^2\cos\theta_0 = -mgr_0\sin^2\theta_0$$

$$\rightarrow \quad \boxed{\cos\theta_0 = \frac{mg}{kr_0}}$$

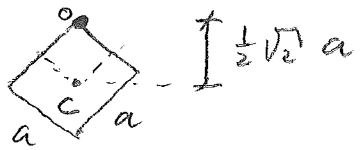
Hence, to obtain steady motion, for a given value of r_0, the above condition gives the value of θ_0, and subsequently, the condition on page 6 gives the corresponding value of $\dot{\varphi}_0$ (Substituting the 2nd condition into the first gives $\dot{\varphi}_0 = \sqrt{\frac{k}{m}}$!)

Note 1: Since $-1 \leq \cos\theta_0 \leq 1$, steady motion only can occur if $\boxed{0 \leq \frac{mg}{kr_0} \leq 1}$

Note 2: For large values of the stiffness k, i.e. $kr_0 >> mg$, the above condition indicates that $\theta_0 \rightarrow \frac{\pi}{2}$ in this limit.
If θ_0 is smaller than this limit value, the motion in a strict sense will not be steady, as oscillations will take place in the r-direction.

However, these oscillations will be (very) small, due to the high stiffness value k, and therefore hardly observable. Accordingly the motion will be quasi-steady, with the coordinate r almost being constant.

Question (2)



a.) $\bar{I}_{z_c z_c} = \frac{1}{12} m a^2$

$$I_{zz} = I_O = \bar{I}_{z_c z_c} + \left(\tfrac{1}{2}\sqrt{2}\, a\right)^2 \cdot m \qquad \text{(Steiner)}$$

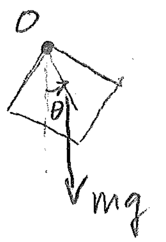
$$= \frac{1}{12} m a^2 + \frac{1}{2} m a^2$$

$$= \frac{7}{12} m a^2$$

$$\bar{L}_o = I_O \cdot \omega \bar{k}$$

$$\boxed{\bar{L}_o = \frac{7}{12} m a^2 \cdot \dot{\theta} \cdot \bar{k}}$$

b.) $\overset{+}{\circlearrowleft} \sum M_z = -mg \frac{1}{2}\sqrt{2}\, a \sin\theta$



$$\Rightarrow \dot{L}_o = \sum M_z$$

$$\rightarrow \frac{7}{12} m a^2 \ddot{\theta} = -\frac{1}{2}\sqrt{2}\, m g a \sin\theta$$

$$\rightarrow \quad \frac{7}{12} m a^2 \ddot{\theta} + \tfrac{1}{2}\sqrt{2}\, m g a \sin\theta = 0$$

$$\rightarrow \quad \boxed{\ddot{\theta} + \frac{6}{7}\sqrt{2}\,\frac{g}{a}\sin\theta = 0}$$

c.) Now, introducing $\dot{\theta} = \varphi$, the above differential equation can be written as

$$\begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \varphi \\ -\frac{6}{7}\sqrt{2}\,\frac{g}{a}\sin\theta \end{bmatrix}$$

Linearizing the above relations about $\theta = 0$, such that it is valid only for small angles $0 \leq |\theta| \ll \frac{\pi}{2}$, (where $\sin\theta \approx \theta$) gives

$$\begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \varphi \\ -\frac{6}{7}\sqrt{2}\,\frac{g}{a}\,\theta \end{bmatrix}$$

OR, in matrix-vector form:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{6}{7}\sqrt{2}\,\frac{g}{a} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \varphi \end{bmatrix}$$

which formally can be written as

$$\dot{\underline{x}} = \bar{A}\,\bar{x}$$

Substituting a solution of the form $\bar{x}(t) = \hat{\bar{x}}\, e^{\lambda t}$ (where $\dot{\bar{x}} = \lambda \hat{\bar{x}}\, e^{\lambda t}$)

leads to

$$\lambda \hat{\bar{x}}\, e^{\lambda t} = \bar{\bar{A}}\, \hat{\bar{x}}\, e^{\lambda t}$$

or $$\left(\bar{\bar{A}} - \lambda \bar{\bar{I}}\right) \hat{\bar{x}}\, e^{\lambda t} = 0.$$

The non-trivial solution to this problem is

$$|\bar{\bar{A}} - \lambda \bar{\bar{I}}| = 0,$$

or, more specifically:

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{6}{7}\sqrt{2}\,\frac{g}{a} & -\lambda \end{vmatrix} = 0$$

$$\rightarrow \quad \lambda^2 + \frac{6}{7}\sqrt{2}\,\frac{g}{a} = 0$$

$$\rightarrow \quad \lambda_{1,2} = \pm\sqrt{-\frac{6}{7}\sqrt{2}\,\frac{g}{a}}$$

$$= \pm i \sqrt{\frac{6}{7}\sqrt{2}\,\frac{g}{a}} = \pm i\,\omega_n \; ^*$$

The eigen frequency now is obtained as

$$\boxed{\omega_n = \mathrm{Im}\{\lambda_1\} = \sqrt{\frac{6}{7}\sqrt{2}\,\frac{g}{a}}}$$

*: Remember that (purely) imaginary eigen values are representative of harmonic motion at a specific eigenfrequency ω_n.

An alternative approach is to linearize the second-order differential equation directly about $\theta = 0$:

$$\ddot{\theta} + \frac{6}{7}\sqrt{2}\,\frac{g}{a}\,\theta = 0.$$

Then, the general solution is substituted $\theta(t) = \hat{\theta}\, e^{\lambda t}$, which leads to

$$\left(\lambda^2 + \frac{6}{7}\sqrt{2}\,\frac{g}{a}\right)\hat{\theta}\, e^{\lambda t} = 0$$

Again, the non-trivial solution is obtained by

$$\lambda^2 + \frac{6}{7}\sqrt{2}\,\frac{g}{a} = 0$$

$$\rightarrow \quad \lambda_{1,2} = \pm i \sqrt{\frac{6}{7}\sqrt{2}\,\frac{g}{a}}$$

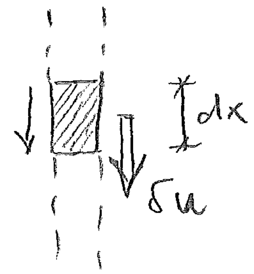
$$= \pm i\, \omega_n$$

$$\rightarrow \quad \boxed{\omega_n = \sqrt{\frac{6}{7}\sqrt{2}\,\frac{g}{a}}},$$

which is the same answer as for the other elaboration method.

Question 3

a.) $$d(\delta W) = q\,dx\,\delta u$$



$$d(\delta V_q) = -d(\delta W)$$
$$= -q\,dx\,\delta u$$

$$\rightarrow \boxed{dV_q = -q u\,dx}$$

$$dV = dV_e + dV_q$$

With $dV_e = \frac{1}{2} EA\,u_x^2\,dx$, this expression becomes:

$$dV = \left(\frac{1}{2} EA u_x^2 - q u\right) dx$$

The generalised potential energy density thus is

$$\mathcal{V} = \frac{1}{2} EA u_x^2 - q u$$

Since the kinetic energy density is given by

$$\mathcal{K} = \frac{1}{2} \rho A u_t^2$$

the generalised Lagrangian density becomes

$$\boxed{\begin{aligned}\mathcal{L} &= \mathcal{K} - \mathcal{V} \\ &= \tfrac{1}{2} \rho A u_t^2 - \tfrac{1}{2} EA u_x^2 + q u\end{aligned}}$$

b.) The action integral can be formulated as

$$I = \int_{t_0}^{t_1} \int_0^{l} \mathcal{L} \, dx \, dt \; ;$$

$$\rightarrow I = \int_{t_0}^{t_1} \int_0^{l} \left(\tfrac{1}{2} \rho A u_t^2 - \tfrac{1}{2} EA u_x^2 + q u \right) dx \, dt \; ;$$

Hamilton's principle states that

$$\delta I = 0 \quad ;$$

where the action I obtains an extremum (minimum). Accordingly, we have:

$$\delta I = \int_{t_0}^{t_1} \int_0^{l} \left(\rho A u_t \, \delta u_t - EA u_x \, \delta u_x + q \, \delta u \right) dx \, dt$$

$$= 0 \quad ;$$

Using integration by parts, this expression turns into:

$$\int_0^{l} \left[\rho A u_t \, \delta u \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \rho A u_{tt} \, \delta u \, dt \right] dx$$

$$- \int_{t_0}^{t_1} \left[EA u_x \, \delta u \Big|_0^{l} - \int_0^{l} EA u_{xx} \, \delta u \, dx \right] dt$$

$$+ \int_{t_0}^{t_1} \int_0^{l} q \, \delta u \, dx \, dt \quad = 0$$

$$\rightarrow \int_{t_0}^{t_1}\int_0^l \underbrace{(-\rho A u_{tt} + EAu_{xx} + q)\,\delta u\; dx\,dt}_{1}$$

$$-\underbrace{\int_0^l \rho A u_t\,\delta u \Big|_{t_0}^{t_1} dx}_{2} - \underbrace{\int_{t_0}^{t_1} EAu_x\,\delta u \Big|_0^l dt}_{3} = 0$$

1 $= 0$: for any variation δu

gives : $-\rho A u_{tt} + EAu_{xx} + q = 0$

$\rightarrow \boxed{\rho A u_{tt} = EAu_{xx} + q}$ = Equation of motion

2 $= 0$: $\delta u(t_0) = \delta u(t_1) = 0$

since the initial and final states of the system, $u(t_0)$ and $u(t_1)$, are assumed to be known (specified).

3 $= 0$: $\boxed{\delta u\big|_{x=0} = 0}$ since the displacement at the top of the bar is specified : $u(x=0) = 0$. (= essential B.C.)

Further, $\boxed{EAu_x\big|_{x=l} = 0}$, which essentially means that the normal force at $x = l$ equals zero (= natural B.C.).

Question 4

a.)
$$\delta W = -F \, \delta x$$

Since $x = 2l\cos\theta$

$\delta x = -2l\sin\theta\,\delta\theta$

$\rightarrow \delta W = 2Fl\sin\theta\,\delta\theta$

$= Q_\theta\,\delta\theta$

So, $\boxed{Q_\theta = 2Fl\sin\theta}$ is the generalized force related to coordinate θ.

The physical meaning of this generalized force is that it represents a bending moment.

b.)
$$\delta W = -\delta V_Q$$

$\rightarrow \delta V_Q = -2Fl\sin\theta\,\delta\theta$

$V_Q = 2Fl\cos\theta$

The internal elastic energy of the spring is

$$V_e = \frac{1}{2} K_r \theta^2$$

The generalized potential then becomes

$$\boxed{V = V_Q + V_e = 2Fl\cos\theta + \frac{1}{2}K_r\theta^2}$$

c.)

$$\frac{dV}{d\theta} = -2F\ell \sin\theta + k_r\theta$$

Equilibrium : $\frac{dV}{d\theta} = 0$

$$\rightarrow -2F\ell\sin\theta + k_r\theta = 0$$

$\theta = 0°$ is indeed an equilibrium state.

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0} = -2F\ell\cos\theta + k_r \Big|_{\theta=0}$$

$$= -2F\ell + k_r$$

For stable equilibrium, $\frac{d^2V}{d\theta^2} > 0$,

which means that

$$k_r - 2F\ell > 0$$

$$\boxed{-\infty < F < \frac{k_r}{2\ell}}$$