opgave no.		naam
		studienummer
		vak Dynamics & Stability — code AE3-914
		datum 21 - 8 - 2006 — fac.

Gebruik voor elke opgave een afzonderlijk vel papier !

1

a. $$I(y) = \int_a^b F(y, y')\,dx \qquad y(a) = y_a \quad y(b) = y_b$$

$$\delta I = \int_a^b \delta F(y, y')\,dx = \int_a^b \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]dx$$

$$= \int_a^b \frac{\partial F}{\partial y}\delta y\,dx + \frac{\partial F}{\partial y'}\delta y\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\,dx$$

$$= \underbrace{\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,dx}_{\mathrm{I}} + \underbrace{\frac{\partial F}{\partial y'}\Big|_{x=b}\delta y(b)}_{\mathrm{II}} - \underbrace{\frac{\partial F}{\partial y'}\Big|_{x=a}\delta y(a)}_{\mathrm{III}}$$

$$\delta I = 0 \Rightarrow \mathrm{I} = 0 \ \& \ \mathrm{II} = 0 \ \& \ \mathrm{III} = 0 \quad \forall \delta y$$

$$\mathrm{I} = 0: \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,dx = 0 \quad \forall \delta y \Rightarrow$$

Euler-Lagrange equation: $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

II $= 0$ & III $= 0$: Automatically fulfilled, because

$y(a) = y_a;\ y(b) = y_b \Rightarrow \delta y(b) = \delta y(a) = 0$, (ESSENTIAL b.c.)

There are no natural b.c., consequently.

b. Analogous to Jacobi integral for $L \neq L(t)$:

$F = F(y, y')$

$$\frac{d}{dx} F = \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'' = \underbrace{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)}_{\boxed{\text{due to E-L equation } \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}} y' + \frac{\partial F}{\partial y'} y''$$

$$= \frac{d}{dx}\left(\frac{\partial F}{\partial y'} y'\right)$$

$$\frac{d}{dx}\left(F - \frac{\partial F}{\partial y'} y'\right) = 0$$

$$F - \frac{\partial F}{\partial y'} y' = C$$

2

a. Polar coordinates:

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$V = k r^{\alpha}$$

$$L = T - V = \boxed{\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - k r^{\alpha}}$$

b. $L \neq L(\theta) \Rightarrow \theta$ is ignorable

$$\frac{\partial L}{\partial \dot{\theta}} = \boxed{m r^2 \dot{\theta} = C_\theta}$$ integral of motion

c. $$\frac{\partial L}{\partial \dot{r}} = m \dot{r} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m \ddot{r}$$

$$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 - \alpha k r^{\alpha - 1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = m \ddot{r} - m r \dot{\theta}^2 + \alpha k r^{\alpha-1} =$$

$$= m \ddot{r} - m r \frac{C_\theta^2}{m^2 r^4} + \alpha k r^{\alpha - 1} =$$

$$= \boxed{m \ddot{r} - \frac{C_\theta^2}{m r^3} + \alpha k r^{\alpha-1} = 0}$$

Steady motion: $\ddot{r} = \dot{r} = 0$

$$\frac{C_\theta^2}{m r^3} = \alpha k r^{\alpha-1}$$

$$\frac{m^2 r^4 \dot{\theta}^2}{m r^3} = \alpha k r^{\alpha-1}$$

$$m r \dot{\theta}^2 = \alpha k r^{\alpha-1}$$

$$\boxed{\dot{\theta}^2 = \frac{\alpha k}{m} r^{\alpha-2}}$$ condition for steady motion

$\alpha k > 0 \Rightarrow$ if $\alpha > 0 \Rightarrow k > 0$

if $\alpha < 0 \Rightarrow k < 0$

d.

$$\ddot{r} - \frac{C_\theta^2}{m^2 r^3} + \frac{\alpha k}{m} r^{\alpha-1} = 0$$

1st order system:

$$\left.\begin{aligned} \dot{r} &= s \\ \dot{s} &= \frac{C_\theta^2}{m^2 r^3} - \frac{\alpha k}{m} r^{\alpha-1} \end{aligned}\right\}$$

Linearization for steady motion:

$$\begin{bmatrix} \dot r \\ \dot s \end{bmatrix} = \left.\begin{bmatrix} 0 & 1 \\ -\dfrac{3 C_\theta^2}{m^2 r^4} - \alpha(\alpha-1)\dfrac{k}{m} r^{\alpha-2} & 0 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}\right|_{\dot\theta^2 = \frac{\alpha k}{m} r^{\alpha-2}}$$

$$= \left.\begin{bmatrix} 0 & 1 \\ -\dfrac{3 m^2 r^4 \dot\theta^2}{m^2 r^4} - \alpha(\alpha-1)\dfrac{k}{m} r^{\alpha-2} & 0 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}\right|_{\dot\theta^2 = \frac{\alpha k}{m} r^{\alpha-2}}$$

$$= \begin{bmatrix} 0 & 1 \\ -3\dot\theta^2 - (\alpha-1)\dot\theta^2 & 0 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -(2+\alpha)\dot\theta^2 & 0 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$$

Eigenvalues:

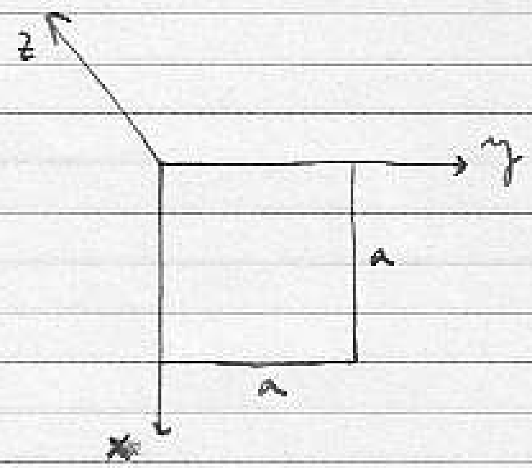
$$\begin{vmatrix} -\lambda & 1 \\ -(2+\alpha)\dot\theta^2 & -\lambda \end{vmatrix} = \lambda^2 + (2+\alpha)\dot\theta^2 = 0$$

For stability $\lambda \in \mathbb{C} - \mathbb{R} \Rightarrow \quad 2+\alpha > 0$

$$\boxed{\alpha > -2}$$

3

a.



$$I_{xx} = \frac{1}{3} m a^2$$

$$I_{yy} = \frac{1}{3} m a^2$$

$$I_{zz} = I_{xx} + I_{yy} \quad \text{(planar, flat body)}$$

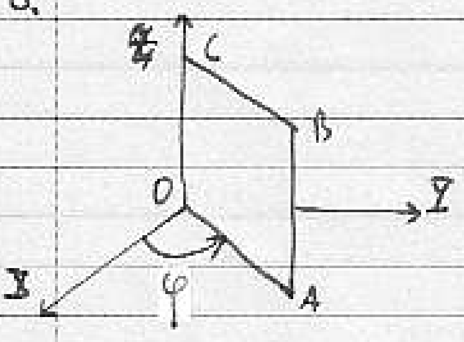
$$= \frac{2}{3} m a^2$$

$$I_{xy} = \underbrace{\bar{I}_{xy}}_{\substack{\shortparallel \\ 0 \text{ (symmetry)}}} + m \frac{a}{2} \frac{a}{2} = m \frac{a^2}{4}$$

$$I_{xz} = I_{yz} = 0 \quad \text{(flat body)}$$

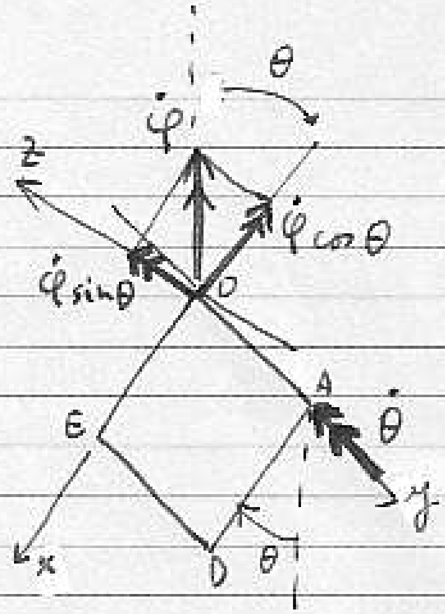
$$\underline{\underline{I}} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{3} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix} m a^2}$$

b.



$$T_{OABC} = \frac{1}{2} \frac{1}{3} m a^2 \dot{\varphi}^2$$

$$= \frac{1}{6} m a^2 \dot{\varphi}^2$$



$$\underline{\omega} = \begin{bmatrix} -\dot{\varphi}\cos\theta \\ -\dot{\theta} \\ \dot{\varphi}\sin\theta \end{bmatrix}$$

$$T_{OADE} = \frac{1}{2}\,\underline{\omega}^T \underline{\underline{I}}\,\underline{\omega} =$$

$$= \frac{1}{2}\begin{bmatrix} -\dot{\varphi}\cos\theta & -\dot{\theta} & \dot{\varphi}\sin\theta \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -\dot{\varphi}\cos\theta \\ -\dot{\theta} \\ \dot{\varphi}\sin\theta \end{bmatrix} ma^2$$

$$= \frac{1}{2}ma^2 \begin{bmatrix} -\frac{1}{3}\dot{\varphi}\cos\theta + \frac{1}{4}\dot{\theta} & \frac{1}{4}\dot{\varphi}\cos\theta - \frac{1}{3}\dot{\theta} & \frac{2}{3}\dot{\varphi}\sin\theta \end{bmatrix} \begin{bmatrix} -\dot{\varphi}\cos\theta \\ -\dot{\theta} \\ \dot{\varphi}\sin\theta \end{bmatrix}$$

$$= \frac{1}{2}ma^2 \left[\frac{1}{3}\dot{\varphi}^2\cos^2\theta - \frac{1}{4}\dot{\varphi}\dot{\theta}\cos\theta - \frac{1}{4}\dot{\varphi}\dot{\theta}\cos\theta + \frac{1}{3}\dot{\theta}^2 + \frac{2}{3}\dot{\varphi}^2\sin^2\theta \right]$$

$$= \frac{1}{2}ma^2 \left[\frac{1}{3}\dot{\varphi}^2(1+\sin^2\theta) + \frac{1}{3}\dot{\theta}^2 - \frac{1}{2}\dot{\varphi}\dot{\theta}\cos\theta \right]$$

$$T = T_{OABC} + T_{OADE} =$$

$$= \frac{1}{2}ma^2 \left[\frac{1}{3}\dot{\varphi}^2(2+\sin^2\theta) + \frac{1}{3}\dot{\theta}^2 - \frac{1}{2}\dot{\varphi}\dot{\theta}\cos\theta \right]$$

$$V_{OABC} = \frac{1}{2} mga = \text{constant}$$

$$V_{OADE} = -\frac{1}{2} mga \cos\theta$$

$$V = V_{OABC} + V_{OADE} = \cancelto{\text{constant}}{\frac{1}{2} mga} - \frac{1}{2} mga \cos\theta$$

$$L = T - V =$$

$$= \frac{1}{2} m a^2 \left[\frac{1}{3} \dot{\varphi}^2 (2 + \sin^2\theta) + \frac{1}{3} \dot{\theta}^2 - \frac{1}{2} \dot{\varphi} \dot{\theta} \cos\theta \right]$$

$$+ \frac{1}{2} mga \cos\theta$$

c. $m = 1\,\text{kg}$ $\quad a = 6\,\text{m}$ $\quad g = 10\,\text{m}/\text{s}^2$

$$L = 6\left[\dot{\varphi}^2 (2 + \sin^2\theta) + \dot{\theta}^2 \right] + 3(10 - 3\dot{\varphi}\dot{\theta}) \cos\theta$$

$$\frac{\partial L}{\partial \dot{\varphi}} = 12 \dot{\varphi} (2 + \sin^2\theta) - 9\dot{\theta}\cos\theta; \quad \frac{\partial L}{\partial \varphi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 12 \ddot{\varphi}(2 + \sin^2\theta) + 12\dot{\varphi} \cdot 2 \sin\theta \cos\theta\, \dot{\theta}$$

$$- 9\ddot{\theta}\cos\theta + 9\dot{\theta} \sin\theta\, \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 12\ddot{\varphi}(2 + \sin^2\theta) + 12\dot{\varphi}\dot{\theta} \sin 2\theta$$

$$- 9\ddot{\theta}\cos\theta + 9\dot{\theta}^2 \sin\theta = 0$$

$$\boxed{4\ddot{\varphi}(2 + \sin^2\theta) - 3\ddot{\theta}\cos\theta + 4\dot{\varphi}\dot{\theta}\sin 2\theta + 3\dot{\theta}^2 \sin\theta = 0}$$

$$\frac{\partial L}{\partial \dot\theta} = 12\dot\theta - 9\dot\varphi\cos\theta$$

$$\frac{\partial L}{\partial \theta} = 12\dot\varphi^2 \sin\theta\cos\theta - 3(10 - 3\dot\varphi\dot\theta)\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 12\ddot\theta - 9\ddot\varphi\cos\theta + 9\dot\varphi\sin\theta\,\dot\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 12\ddot\theta - 9\ddot\varphi\cos\theta + 9\dot\varphi\dot\theta\sin\theta - 12\dot\varphi^2\sin\theta\cos\theta + 3(10 - 3\dot\varphi\dot\theta)\sin\theta = 0$$

$$\boxed{4\ddot\theta - 3\ddot\varphi\cos\theta + 10\sin\theta - 2\dot\varphi^2\sin 2\theta = 0}$$

d.

φ is ignorable $\Rightarrow \frac{\partial L}{\partial \dot\varphi}$ is an integral of motion

$$\frac{\partial L}{\partial \dot\varphi} = \boxed{12\dot\varphi(2 + \sin^2\theta) - 9\dot\theta\cos\theta = C_\varphi}$$

$L \neq L(t) \Rightarrow h$ is an integral of motion

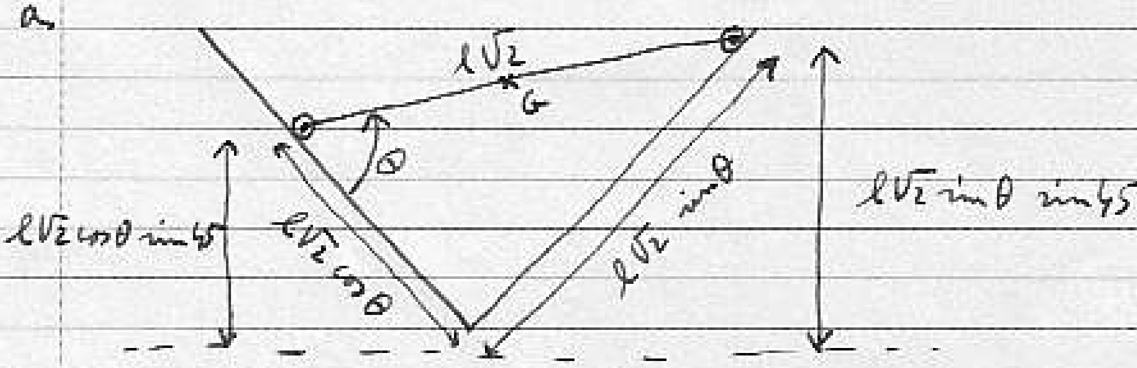
$$h = \dot\varphi\frac{\partial L}{\partial \dot\varphi} + \dot\theta\frac{\partial L}{\partial \dot\theta} - L =$$

$$= \dot\varphi\left[12\dot\varphi(2+\sin^2\theta) - 9\dot\theta\cos\theta\right] + \dot\theta\left[12\dot\theta - 9\dot\varphi\cos\theta\right] - 6\left[\dot\varphi^2(2+\sin^2\theta) + \dot\theta^2\right] - 30\cos\theta + 9\dot\varphi\dot\theta\cos\theta$$

$$= \boxed{6\left[\dot\varphi^2(2+\sin^2\theta) + \dot\theta^2\right] - 3(10 + 3\dot\varphi\dot\theta)\cos\theta = C_h}$$

4

a



$$y_G = \frac{1}{2}\left[l\sqrt{2}\cos\theta \sin 45 + l\sqrt{2}\sin\theta \sin 45 \right]$$

$$= \frac{l}{2}(\sin\theta + \cos\theta)$$

$$V_g = mg\frac{l}{2}(\sin\theta + \cos\theta)$$

$$V_e = \frac{1}{2}k(l\sqrt{2}\sin\theta - l)^2$$

natural length

$$V = mg\frac{l}{2}(\sin\theta + \cos\theta) + \frac{1}{2}k(l\sqrt{2}\sin\theta - l)^2$$

$$V' = mg\frac{l}{2}(\cos\theta - \sin\theta) + \frac{1}{2}k\,2(l\sqrt{2}\sin\theta - l)\,l\sqrt{2}\cos\theta$$

$$V'(45°) = mg\frac{l}{2}\underbrace{\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)}_{0} + \frac{1}{2}k\,2\underbrace{\left(l\sqrt{2}.\frac{\sqrt{2}}{2} - l\right)}_{0}\,l\sqrt{2}\cdot\frac{\sqrt{2}}{2}$$

$= 0 \Rightarrow$ indeed, equilibrium!

b.
$$V'' = -mg\frac{l}{2}(\sin\theta + \cos\theta) + k(l\sqrt{2}\cos\theta)^2$$
$$-k(l\sqrt{2}\sin\theta - l)\,l\sqrt{2}\sin\theta$$

$$V''(45°) = -mg\frac{l}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) + k\left(l\sqrt{2}\cdot\frac{\sqrt{2}}{2}\right)^2$$
$$-k\underbrace{\left(l\sqrt{2}\cdot\frac{\sqrt{2}}{2} - l\right)}_{=0}l\sqrt{2}\cdot\frac{\sqrt{2}}{2}$$

$$= -mgl\frac{\sqrt{2}}{2} + kl^2$$

$V'' > 0 \Rightarrow$ stable

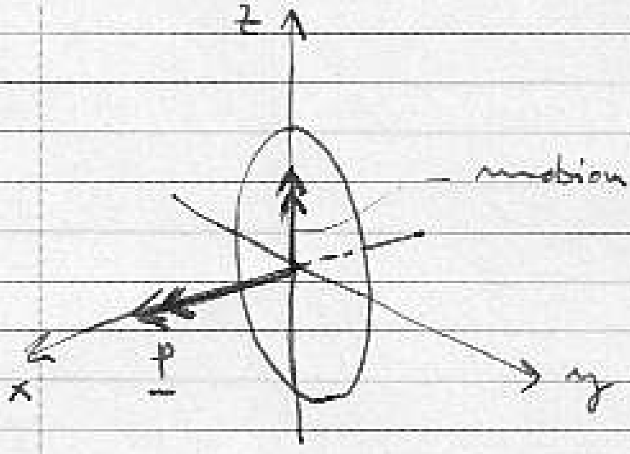
$$-mgl\frac{\sqrt{2}}{2} + kl^2 > 0$$

$$\boxed{k > \frac{mg\sqrt{2}}{2l}}$$

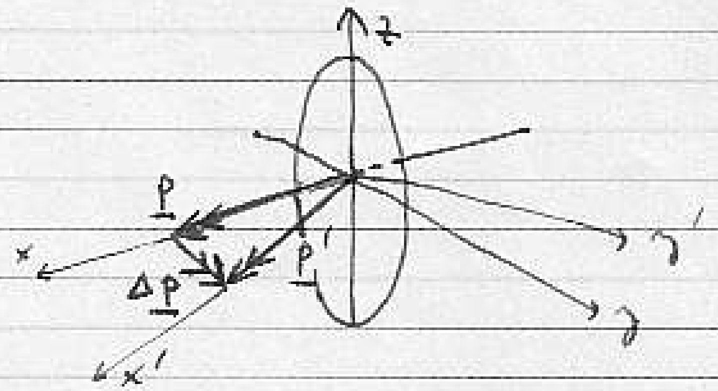
for stability at $\theta = 45°$

5

Angular momentum diagram:

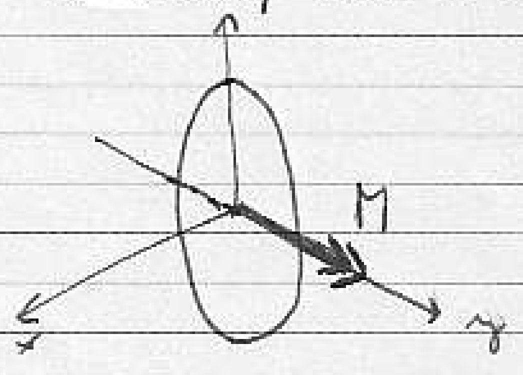


Change in angular momentum due to motion of the shaft



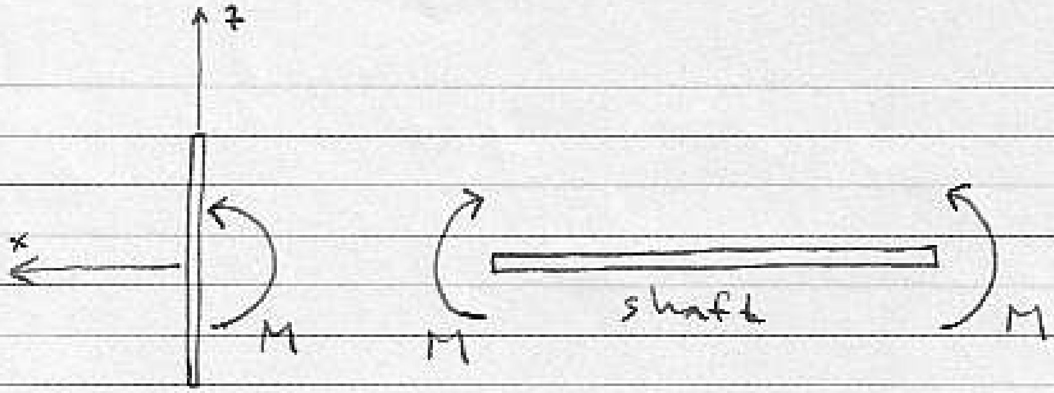
When $\delta t \to 0$ $\Delta \underline{p}$ is parallel to the y-axis,

a moment about the y-axis is thus needed to satisfy the change in angular momentum:



The moment is pointing in the positive y-direction

see overleaf



The moment in the shaft is then as represented above, which corresponds to a positive bending moment

M-diagram for the shaft: