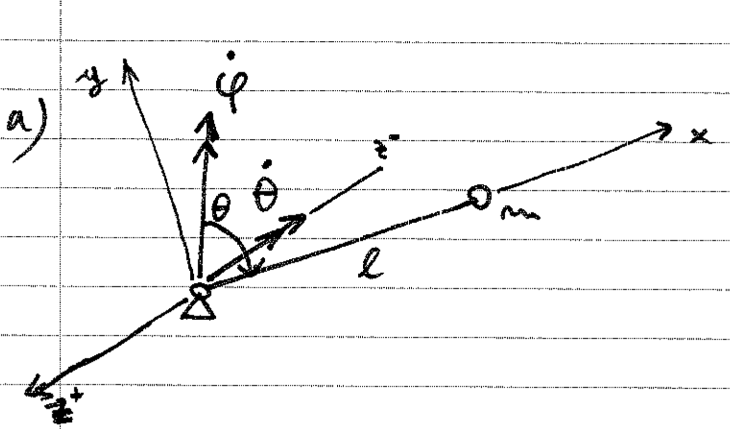
opgave no.		naam
		studienummer
		vak / code
		datum / fac.

Gebruik voor elke opgave een afzonderlijk vel papier !

(1)

a)



$$\underline{r} = l\,\underline{i}$$

$$\underline{\omega} = \dot\varphi\cos\theta\,\underline{i} + \dot\varphi\sin\theta\,\underline{j} - \dot\theta\,\underline{k}$$

$$\underline{v} = \underline{\omega}\times\underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dot\varphi\cos\theta & \dot\varphi\sin\theta & -\dot\theta \\ l & 0 & 0 \end{vmatrix}$$

$$= -\dot\theta l\,\underline{j} - \dot\varphi\sin\theta\, l\,\underline{k}$$

5 $$T = 2\cdot\left[\frac{1}{2} m v^2\right] = 2\cdot\frac{1}{2} m\left(\dot\theta^2 l^2 + \dot\varphi^2\sin^2\theta\, l^2\right)$$

$$= m l^2\left(\dot\theta^2 + \dot\varphi^2\sin^2\theta\right)$$

3 $$V = \frac{1}{2} k\,(2l\sin\theta)^2 = 2kl^2\sin^2\theta$$

$$L = ml^2(\dot\theta^2 + \dot\varphi^2\sin^2\theta) - 2kl^2\sin^2\theta$$

b)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot\theta} = 2ml^2\dot\theta \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 2ml^2\ddot\theta$$

$$\frac{\partial L}{\partial \theta} = 2ml^2\dot\varphi^2\sin\theta\cos\theta - 4kl^2\sin\theta\cos\theta$$

$$2ml^2\ddot\theta - 2ml^2\dot\varphi^2\sin\theta\cos\theta + 4kl^2\sin\theta\cos\theta = 0$$

2 $$\boxed{2ml^2\ddot\theta - ml^2\dot\varphi^2\sin 2\theta + 2kl^2\sin 2\theta = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = 0$$

$$\frac{\partial L}{\partial \dot\varphi} = 2ml^2\dot\varphi\sin^2\theta \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) = 2ml^2\left(\ddot\varphi\cdot\sin^2\theta + \dot\varphi\cdot 2\sin\theta\cdot\cos\theta\cdot\dot\theta\right)$$

$$\frac{\partial L}{\partial \varphi} = 0$$

$$\ddot\varphi\sin^2\theta + 2\dot\varphi\sin\theta\cos\theta\,\dot\theta = 0$$

2 $$\boxed{\ddot\varphi\sin^2\theta + \dot\varphi\sin 2\theta\,\dot\theta = 0}$$

c) φ is ignorable, as $L \neq L(\varphi)$

$\frac{\partial L}{\partial \dot{\varphi}}$ is an integral of motion.

3 $$\frac{\partial L}{\partial \dot{\varphi}} = \boxed{2 m l^2 \dot{\varphi} \sin^2\theta = C_\varphi}$$

$L \neq L(t)$, then h is an integral of motion.

$$h = \frac{\partial L}{\partial \dot{\theta}} \dot{\theta} + \frac{\partial L}{\partial \dot{\varphi}} \dot{\varphi} - L$$

$$= 2 m l^2 \dot{\theta}^2 + 2 m l^2 \dot{\varphi}^2 \sin^2\theta - m l^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + 2 k l^2 \sin^2\theta$$

3 $$= \boxed{m l^2 \dot{\theta}^2 + m l^2 \dot{\varphi}^2 \sin^2\theta + 2 k l^2 \sin^2\theta = C_h}$$

d) $$R = C_\varphi \dot{\varphi} - L$$

$$= C_\varphi \dot{\varphi} - m l^2 \dot{\theta}^2 - m l^2 \dot{\varphi}^2 \sin^2\theta + 2 k l^2 \sin^2\theta$$

$$\dot{\varphi} = \frac{C_\varphi}{2 m l^2 \sin^2\theta}$$

$$R = \frac{C_\varphi^2}{2 m l^2 \sin^2\theta} - m l^2 \dot{\theta}^2 - m l^2 \frac{C_\varphi^2 \sin^2\theta}{4 (m l^2)^2 \sin^4\theta} + 2 k l^2 \sin^2\theta$$

3 $$\boxed{R = \frac{C_\varphi^2}{4 m l^2 \sin^2\theta} - m l^2 \dot\theta^2 + 2 k l^2 \sin^2\theta}$$

e) $$h = R - \frac{\partial R}{\partial \dot\theta}\dot\theta$$

$$\frac{\partial R}{\partial \dot\theta} = -2 m l^2 \dot\theta$$

$$h = \frac{C_\varphi^2}{4 m l^2 \sin^2\theta} - m l^2 \dot\theta^2 + 2 k l^2 \sin^2\theta + 2 m l^2 \dot\theta^2$$

3 $$= \boxed{m l^2 \dot\theta^2 + 2 k l^2 \sin^2\theta + \frac{C_\varphi^2}{4 m l^2 \sin^2\theta}}$$

3 f) $$V_{eff} = h - T_2 = 2 k l^2 \sin^2\theta + \frac{C_\varphi^2}{4 m l^2 \sin^2\theta}$$

g) $$\frac{\partial R}{\partial \theta} = 0 \quad \text{or} \quad \frac{\partial V_{eff}}{\partial \theta} = 0$$

$$\frac{\partial R}{\partial \theta} = 4 k l^2 \sin\theta\cos\theta - \frac{2 C_\varphi^2}{4 m l^2 \sin^3\theta}\cos\theta = 0$$

$$\cos\theta \left(4kl^2 \sin\theta - \frac{C_\varphi^2}{2ml^2 \sin^3\theta} \right) = 0$$

4 $$\cos\theta = 0 \implies \boxed{\theta = 90^\circ}$$

$$4kl^2 \sin\theta - \frac{C_\varphi^2}{2ml^2 \sin^3\theta} = 0$$

$$C_\varphi = 2ml^2 \dot{\varphi} \sin^2\theta$$

$$4kl^2 \sin\theta - \frac{4(ml^2)^2 \dot{\varphi}^2 \sin^4\theta}{2ml^2 \sin^3\theta} = 0$$

$$4kl^2 \sin\theta - 2ml^2 \dot{\varphi}^2 \sin\theta = 0$$

$$(2k - m\dot{\varphi}^2) \sin\theta = 0$$

$$\sin\theta = 0 \implies \boxed{\theta = 0}$$

$$2k - m\dot{\varphi}^2 = 0$$

4 $$\boxed{\dot{\varphi}^2 = \frac{2k}{m}}$$

h) There is an error in the statement, it was meant to be calculated for $\theta = 90°$

0

(2)

$$I(y) = \int_0^1 \left(y^2 + x y'^2 + x^2 y''^2 \right) dx$$

$$y(0) = y'(0) = 0 \Rightarrow \delta y(0) = \delta y'(0) = 0$$

$$\delta I = \int_0^1 \delta \Phi(x, y, y', y'')\, dx$$

$$= \int_0^1 \left(\frac{\partial \Phi}{\partial y} \delta y + \frac{\partial \Phi}{\partial y'} \delta y' + \frac{\partial \Phi}{\partial y''} \delta y'' \right) dx$$

$$= \int_0^1 \frac{\partial \Phi}{\partial y} \delta y\, dx + \frac{\partial \Phi}{\partial y'} \delta y \Big|_0^1 - \int_0^1 \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y'}\right) \delta y\, dx$$

$$+ \frac{\partial \Phi}{\partial y''} \delta y' \Big|_0^1 - \int_0^1 \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y''}\right) \delta y'\, dx$$

$$= \int_0^1 \left[\frac{\partial \Phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y'}\right) \right] \delta y\, dx + \frac{\partial \Phi}{\partial y'} \delta y \Big|_0^1$$

$$+ \frac{\partial \Phi}{\partial y''} \delta y' \Big|_0^1 - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y''}\right) \delta y \Big|_0^1 + \int_0^1 \frac{d^2}{dx^2}\left(\frac{\partial \Phi}{\partial y''}\right) \delta y\, dx$$

$$= \int_0^1 \left[\frac{\partial \Phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \Phi}{\partial y''}\right) \right] \delta y \, dx \qquad \text{(I)}$$

$$+ \left[\frac{\partial \Phi}{\partial y'} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y''}\right) \right]_{x=1} \delta y(1) \qquad \text{(II)}$$

10 $$- \left[\frac{\partial \Phi}{\partial y'} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y''}\right) \right]_{x=0} \delta y(0) \qquad \text{(III)}$$

$$+ \left.\frac{\partial \Phi}{\partial y''}\right|_{x=1} \delta y'(1) \qquad \text{(IV)}$$

$$- \left.\frac{\partial \Phi}{\partial y''}\right|_{x=0} \delta y'(0) \qquad \text{(V)}$$

$$\text{(I)} = 0 \Rightarrow \frac{\partial \Phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \Phi}{\partial y''}\right) = 0$$

$$\frac{\partial \Phi}{\partial y} = 2y \quad \Bigg| \quad \frac{\partial \Phi}{\partial y'} = 2xy' \quad \Bigg| \quad \frac{\partial \Phi}{\partial y''} = 2x^2 y''$$

$$\frac{d}{dx}\left(\frac{\partial \Phi}{\partial y'}\right) = 2y' + 2xy'' \quad \Bigg| \quad \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y''}\right) = 4xy'' + 2x^2 y'''$$

$$\frac{d^2}{dx^2}\left(\frac{\partial \Phi}{\partial y''}\right) = 4y'' + 4xy''' + 4xy''' + 2x^2 y^{IV}$$

$$2y - 2y' - 2xy'' + 4y'' + 8xy''' + 2x^2 y^{IV} = 0$$

5 $$\boxed{y - y' + (2-x)y'' + 4xy''' + x^2 y^{IV} = 0}$$

$$II = 0 \Rightarrow \left[\frac{\partial \Phi}{\partial y'} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial y''}\right)\right]_{x=1} = 0$$

$$2\cdot 1\cdot y'(1) - 4\cdot 1\cdot y''(1) - 2\cdot 1\cdot y'''(1) = 0$$

5 $$\boxed{y'(1) - 2y''(1) - y'''(1) = 0}$$ NBC

$III = 0 \Rightarrow$ always, as $\delta y(0) = 0$ (EBC)

$$IV = 0 \Rightarrow \left.\frac{\partial \phi}{\partial y''}\right|_{x=1} = 0 \Rightarrow 2\cdot 1\cdot y''(1) = 0$$

5 $$\boxed{y''(1) = 0}$$ NBC

$V = 0 \Rightarrow$ always, as $\delta y'(0) = 0$ (EBC)

③

a) $$\underline{L} = I_1 \omega_1 \underline{i} + I_2 \omega_2 \underline{j} + I_3 \omega_3 \underline{k}$$

$$\underline{M} = \frac{d\underline{L}}{dt}$$

$$\frac{d\underline{L}}{dt} = I_1 \dot{\omega}_1 \underline{i} + I_2 \dot{\omega}_2 \underline{j} + I_3 \dot{\omega}_3 \underline{k} + \underline{\omega} \times \left(I_1 \omega_1 \underline{i} + I_2 \omega_2 \underline{j} + I_3 \omega_3 \underline{k} \right)$$

$$\underline{\omega} \times \left(I_1 \omega_1 \underline{i} + I_2 \omega_2 \underline{j} + I_3 \omega_3 \underline{k} \right) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ I_1 \omega_1 & I_2 \omega_2 & I_3 \omega_3 \end{vmatrix}$$

$$= \left(I_3 - I_2 \right) \omega_2 \omega_3 \underline{i} + \left(I_1 - I_3 \right) \omega_1 \omega_3 \underline{j} + \left(I_2 - I_1 \right) \omega_1 \omega_2 \underline{k}$$

$$\underline{M} = M_1 \underline{i} + M_2 \underline{j} + M_3 \underline{k}$$

10

$$\left. \begin{aligned} M_1 &= I_1 \dot{\omega}_1 - \left(I_2 - I_3 \right) \omega_2 \omega_3 \\ M_2 &= I_2 \dot{\omega}_2 - \left(I_3 - I_1 \right) \omega_3 \omega_1 \\ M_3 &= I_3 \dot{\omega}_3 - \left(I_1 - I_2 \right) \omega_1 \omega_2 \end{aligned} \right\}$$

b)

$$M_1 = M_2 = M_3 = 0$$

$$\dot{\omega}_3 = 0; \quad \omega_1 \simeq 0; \quad \omega_2 \simeq 0$$

$$\left.\begin{aligned} I_1 \dot{\omega}_1 &= (I_2 - I_3)\,\omega_2 \omega_3 \\ I_2 \dot{\omega}_2 &= (I_3 - I_1)\,\omega_3 \omega_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} \dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\,\omega_2 \omega_3 \\ \dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\,\omega_3 \omega_1 \end{aligned}\right\}$$

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{I_2 - I_3}{I_1} \\ \frac{I_3 - I_1}{I_2} & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

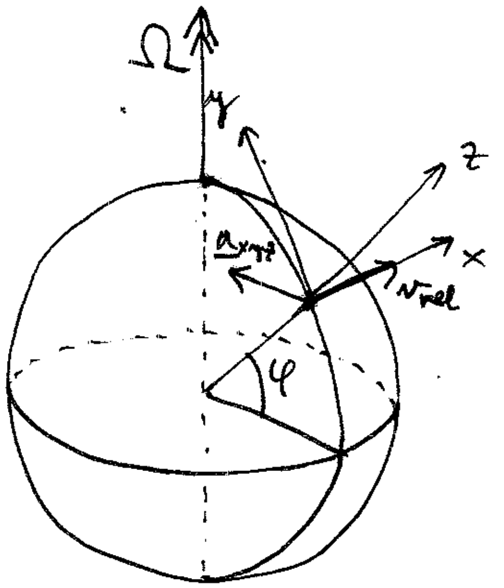
$\omega_1 = \omega_2 = 0 \Rightarrow \dot{\omega}_1 = \dot{\omega}_2 = 0 \rightarrow$ Equilibrium point

Stability:

10 $$\begin{vmatrix} -\lambda & \frac{I_2 - I_3}{I_1} \\ \frac{I_3 - I_1}{I_2} & -\lambda \end{vmatrix} = \lambda^2 - \frac{(I_3 - I_1)(I_2 - I_3)}{I_1 I_2} = 0$$

$(I_3 - I_1)(I_2 - I_3) < 0$ for stability $\Rightarrow$ $I_3 > I_1, I_2$ or $I_3 < I_1, I_2$

(4)



$$\underline{v}_{rel} = v\,\underline{i}$$

$$\underline{a}_{xyz} = \Omega^2\left(-R\cos\varphi\sin\varphi\,\underline{j} - R\cos\varphi\cos\varphi\,\underline{k}\right)$$

involves Ω^2 — involves Ω^2 — 0

$$\underline{F}_{fict} = -m\left(2\underline{\Omega}\times\underline{v}_{rel} + \cancel{\underline{a}_{xyz}} + \cancel{\dot{\underline{\Omega}}\times\underline{v}_{rel}} + \cancel{\underline{\Omega}\times(\underline{\Omega}\times\underline{v}_{rel})}\right)$$

$$\underline{\Omega} = \Omega\cos\varphi\,\underline{j} + \Omega\sin\varphi\,\underline{k}$$

$$\underline{\Omega}\times\underline{v}_{rel} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & \Omega\cos\varphi & \Omega\sin\varphi \\ v & 0 & 0 \end{vmatrix}$$

$$= v\Omega\sin\varphi\,\underline{j} - v\Omega\cos\varphi\,\underline{k}$$

$$F_{fict} = -2mv\Omega\sin\varphi\,\underline{j} + 2mv\Omega\cos\varphi\,\underline{k}$$

10

$\underline{F}_{fict}$ on tyres is the reaction of $\underline{F}_{fict}$, so

5 $$\underline{F}_{fict\ tyres} = -\underline{F}_{fict} = 2 m v \Omega \sin\varphi\ \underline{j} - 2 m v \Omega \cos\varphi\ \underline{k}$$

$m = 4\,000$ kg

$v = 200\ km/h \cdot \frac{1000\ m}{1\ km} \cdot \frac{1\ h}{3600\ s} = 55{,}6\ m/s$

$\Omega = 2\pi\ rad/day = 2\pi \cdot \quad \cdot \frac{1\ day}{24\ h} \cdot \frac{1\ h}{3600\ s} = 7{,}272 \times 10^{-5}\ rad/s$

$$\boxed{\underline{F}_{fict\ tyres} = 20{,}7\ \underline{j} - 24{,}8\ \underline{k}\quad [N]}$$

5