opgave no.		naam	
		studienummer	
		vak AE3-914	code
		datum 2-2-6	fac.

Gebruik voor elke opgave een afzonderlijk vel papier !

1 At equilibrium points the potential energy is stationary, i.e., $V' = 0$

At <u>stable</u> equilibrium points the potential energy attains a strict minimum

$$V = 2mg\,2l\sin\theta + mgl\cos 2\theta$$

Equilibrium points:

$V' = 0$;

$$V' = mgl(4\cos\theta - 2\sin 2\theta)$$
$$= 4mgl(\cos\theta - \sin\theta\cos\theta) = 0$$
$$\cos\theta(1 - \sin\theta) = 0$$

$$\begin{cases} \cos\theta = 0 \Rightarrow \theta = \pm 90^\circ \\ \sin\theta = 1 \Rightarrow \theta = +90^\circ \end{cases}$$

So the equilibrium points are $\theta = \pm 90^\circ$

Stability:

$$V'' = mgl(-4\sin\theta - 4\cos 2\theta)$$
$$= -4mgl(\sin\theta + \cos 2\theta)$$

For $\theta = -90^\circ$ one has:

$$V''(-90) = -4mgl(-1-1) = 8mgl > 0$$

so $\theta = -90^\circ$ is <u>stable</u>

For $\theta = +90°$ one has:

$V''(90) = -4\,mgl\,(1-1) = 0 \Rightarrow$ no decision yet

$V''' = -4\,mgl\,(\cos\theta - 2\sin 2\theta)$

$V'''(90) = -4\,mgl\,(0-0) = 0 \Rightarrow$ no decision yet

$V^{IV} = -4\,mgl\,(-\sin\theta - 4\cos 2\theta)$

$\quad = 4\,mgl\,(\sin\theta + 4\cos 2\theta)$

$V^{IV}(90) = 4\,mgl\,(1-4) = -12\,mgl < 0$

for the 4th-order derivative

$\Rightarrow$ $\theta = 90°$ is a maximum

$\Rightarrow$ unstable

(2) $\underline{r}_P = \underline{r}_B + \underline{r}_{rel}$; $\underline{v}_P = \frac{d\underline{r}_P}{dt} = \underbrace{\frac{d\underline{r}_B}{dt}}_{\mathrm{I}} + \underbrace{\frac{d\underline{r}_{rel}}{dt}}_{\mathrm{II}}$

(I) $\frac{d\underline{r}_B}{dt} = \underline{v}_B$

(II) $\frac{d\underline{r}_{rel}}{dt} = \frac{d}{dt}\left(x\underline{i} + y\underline{j} + z\underline{k}\right)$

$$= \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k}$$
$$+ x\frac{d\underline{i}}{dt} + y\frac{d\underline{j}}{dt} + z\frac{d\underline{k}}{dt}$$

$$= \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k} + x(\underline{\omega}\times\underline{i}) + y(\underline{\omega}\times\underline{j}) + z(\underline{\omega}\times\underline{k})$$
$$= \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k} + \underline{\omega}\times(x\underline{i} + y\underline{j} + z\underline{k})$$
$$= \underline{v}_{rel} + \underline{\omega}\times\underline{r}_{rel}$$

$$\underline{v}_P = \mathrm{I} + \mathrm{II} = \underline{\underline{v}_B + \underline{\omega}\times\underline{r}_{rel} + \underline{v}_{rel}}$$

(3) a. The system is described by two generalised coordinates and one constraint.

The number of degrees-of-freedom is, in consequence,

$$\text{ndof} = 2 - 1 = \underline{\underline{1}}$$

b.

$$L = T - V$$

$$T = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} I \dot{\theta}^2$$
$$= \frac{1}{2} m \dot{s}^2 + \frac{1}{4} m R^2 \dot{\theta}^2$$

$$V = \frac{1}{2} k s^2$$

$$L = \frac{1}{2} m \dot{s}^2 + \frac{1}{4} m R^2 \dot{\theta}^2 - \frac{1}{2} k s^2$$

c.

We have the constraint $f(s, \theta) = s - R\theta = 0$

Lagrange multiplier method:

for $q_1 \dots q_n$ generalised coordinates and $f_1 \dots f_m$ constraints we have

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} = \displaystyle\sum_{j=1}^{m} \lambda_j \dfrac{\partial f_j}{\partial q_i} & i = 1, \dots, n \\ f_k = 0 & k = 1, \dots, m \end{cases}$$

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = m\ddot{s} \qquad \frac{\partial L}{\partial s} = -ks$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m R^2 \dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{1}{2} m R^2 \ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial f}{\partial s} = 1 \qquad \frac{\partial f}{\partial \theta} = -R$$

E.M.:

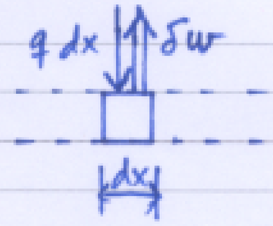
$$\left.\begin{aligned} m\ddot{s} + ks &= \lambda \\ \frac{1}{2} m R^2 \ddot{\theta} &= -\lambda R \\ s - R\theta &= 0 \end{aligned}\right\}$$

d.

The Lagrange multiplier λ represents the intensity of the generalised force induced by the constraint.

4

a.



$$d(\delta W) = \cdot \ - q\,dx\,\delta w$$

$$d(\delta V_g) = - d(\delta W)$$
$$= q\,dx\,\delta w$$

$$\Rightarrow \quad \underline{d V_g = q w\,dx}$$

b.

$$dV = dV_e + dV_g$$
$$= \left(\frac{1}{2} EI\, w''^2 + q w\right) dx$$

$$V = \int_0^L \left(\frac{1}{2} EI\, w''^2 + q w\right) dx + \underbrace{\frac{1}{2} k \left[w'(L)\right]^2}_{\text{potential energy of the spring}}$$

c.

Hamilton principle in static conditions: Principle of stationary potential energy

$$\delta V = 0$$

$$\delta V = \int_0^L EI\, w'' \delta w''\, dx + \int_0^L q\, \delta w\, dx + k\, w'(L)\, \delta w'(L)$$

$$\Longrightarrow$$

$$= EI\,w''\,\delta w'\Big|_0^L - \int_0^L EI\,w'''\,\delta w'\,dx + \int_0^L q\,\delta w\,dx$$

$$+ k\,w'(L)\,\delta w'(L)$$

$$= EI\,w''\,\delta w'\Big|_0^L - EI\,w'''\,\delta w\Big|_0^L + \int_0^L EI\,w^{IV}\,\delta w\,dx$$

$$+ \int_0^L q\,\delta w\,dx + k\,w'(L)\,\delta w'(L)$$

$$= EI\,w''(L)\,\delta w'(L) - EI\,w''(0)\,\delta w'(0)$$

$$- EI\,w'''(L)\,\delta w(L) + EI\,w'''(0)\,\delta w(0)$$

$$+ \int_0^L (EI\,w^{IV} + q)\,\delta w\,dx + k\,w'(L)\,\delta w'(L)$$

$$= \int_0^L (EI\,w^{IV} + q)\,\delta w\,dx \qquad \text{I}$$

$$+ \left(EI\,w''(L) + k\,w'(L)\right)\delta w'(L) \qquad \text{II}$$

$$- EI\,w''(0)\,\delta w'(0) \qquad \text{III}$$

$$- EI\,w'''(L)\,\delta w(L) \qquad \text{IV}$$

$$+ EI\,w'''(0)\,\delta w(0) \qquad \text{V}$$

$$= 0$$

(1) $= 0$ for any variation δw

$\Rightarrow$ $\underline{EI\, w^{IV} + q = 0}$ diff. equation for w.

(2) $= 0$ for any variation of the rotation $\delta w'(L)$

$\Rightarrow$ $\underline{EI\, w''(L) + k\, w'(L) = 0}$ natural B.C. at $x = L$

(3) $= 0$ Always, because $\delta w'(0) = 0$ at the fixed support $x = 0$

(4) $= 0$ Always, because $\delta w(L) = 0$ at the simple support $x = L$

(5) $= 0$ Always, because $\delta w(0) = 0$ at the fixed support $x = 0$

(5)

a.

Both thrust and gravity go through the mass centre and cause therefore no external moment.

b.

$$T = T(\dot{\varphi}, \theta, \dot{\theta}, \dot{\psi})$$

$\dot{\varphi}$ and $\dot{\psi}$ appear only in rate form $\Rightarrow$

$\underline{\varphi \text{ and } \psi}$ are ignorable

$$C_\varphi = \frac{\partial T}{\partial \dot{\varphi}} \qquad C_\psi = \frac{\partial T}{\partial \dot{\psi}}$$

$$\frac{\partial T}{\partial \dot{\varphi}} = I \dot{\varphi} \sin^2\theta + I_s (\dot{\varphi}\cos\theta + \dot{\psi})\cos\theta = C_\varphi$$

$$\frac{\partial T}{\partial \dot{\psi}} = I_s(\dot{\varphi}\cos\theta + \dot{\psi}) = C_\psi$$

c.

For $\theta = 0$ one has

$$C_\varphi = I_s(\dot{\varphi}\cos 0 + \dot{\psi})\cos 0 = I_s(\dot{\varphi} + \dot{\psi})$$
$$= 10\,000 \text{ kg m}^2/\text{s}$$

$$C_\psi = I_s(\dot{\varphi}\cos 0 + \dot{\psi}) = I_s(\dot{\varphi} + \dot{\psi})$$
$$= 10\,000 \text{ kg m}^2/\text{s}$$

For $\theta = 10^\circ$ C_φ and C_ψ remain the same, because they are integrals of motion.

$$C_\varphi = I\,\dot{\varphi}\sin^2 10 + I_s(\dot{\varphi}\cos 10 + \dot{\psi})\cos 10 = 10\,000 \quad \text{I}$$
$$C_\psi = I_s(\dot{\varphi}\cos 10 + \dot{\psi}) = 10\,000 \quad \text{II}$$

Substitute II into I:

$$50000 \sin^2 10\ \dot{\varphi} + 10000\cos 10 = 10\,000$$

$$\dot{\varphi} = \frac{10\,000(1-\cos 10)}{50000\sin^2 10}$$

$$\dot{\varphi} = 0{,}100765426626$$

$$\approx \underline{0{,}1\ \text{rad}/\text{s}}$$

Solve from II

$$\dot{\psi} = \frac{10\,000}{1000} - 0.100765426626\cos 10$$

$$= 9{,}900765426623 \approx \underline{9{,}9\ \text{rad}/\text{s}}$$